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On Four-Dimensional Gauge Theories from Type II Superstrings

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ABSTRACT

We study the four-dimensional gauge theories which are associated with classical vacua of the type II superstring, *i.e.* which correspond to a superconformal field theory on the world sheet. Using the fact that gauge symmetry arises from a supersymmetric affine Kac-Moody algebra, and demanding unitarity of the underlying world-sheet field theory, we show that no such vacua can yield the particle spectrum of the standard model. Of the gauge theories which are permitted by unitarity, we find that many can be constructed explicitly as orbifolds which twist the left- and right-moving degrees of freedom of the string asymmetrically; among these are three $N = 4$ supersymmetric models — which have previously been constructed in a quite different fashion — and two $N = 1$ supersymmetric models with chiral gauge representations for the massless fermions.

1. Introduction

Recent attempts to extract realistic physics from superstring theories [1] have largely focused on the heterotic string.[2] Particular compactification schemes for the $E_8 \otimes E_8$ heterotic string (*e.g.*, on Calabi-Yau manifolds [3] or orbifolds [4]) give rise to a large variety of four-dimensional (4d) models with $N = 1$ supersymmetry, chiral fermions and realistic gauge groups. From the phenomenological point of view, the principal problem with this situation is the vast number of quasi-realistic models, and the lack at present of any dynamical reason for preferring one of them over the others. Without a better understanding of the dynamics of compactification, finding a single model which correctly describes the four-dimensional world (if such a model exists) is akin to finding a needle in a haystack. Loosely speaking, the abundance of 4d models with realistic gauge groups is related to the existence of a very large Yang-Mills multiplet in the massless spectrum of the ten-dimensional heterotic string. This embarrassment of riches has prompted us to consider in this paper the opposite extreme: the type II superstring theory,[5] which has no Yang-Mills fields in ten dimensions. Under this circumstance, one might expect that expect that any four-dimensional gauge symmetry must arise via a Kaluza-Klein mechanism; unfortunately, it has been shown [6] that symmetries generated by this mechanism act the same way on left-handed and right-handed fermions. In addition, any space-time supersymmetric background for the type II superstring which treats the left- and right-moving degrees of freedom symmetrically clearly has $N > 1$ supersymmetry and thus yields a non-chiral model, regardless of how the gauge symmetry may arise. (We will see subsequently that in this situation the gauge group is in fact always abelian.)

For these reasons type II superstrings were not considered to be suitable for realistic model building until the recent construction by Bluhm *et al.* [7] and by Kawai *et al.* [8] of several type II superstring models that have four flat space-time dimensions and gauge groups large enough to contain the standard $SU(3) \otimes$

$SU(2) \otimes U(1)$ model. The three models in ref. [7] all have $N = 4$ space-time supersymmetry in four dimensions, so they contain gauginos — massless fermions in the adjoint representation of the gauge group. Additional models constructed in ref. [8] also contain massless fermions in non-trivial gauge representations. The arguments of the previous paragraph do not apply to these models for two reasons: First, the gauge groups do not come entirely from a Kaluza-Klein mechanism, but also utilize inherently stringy states (soliton or winding-type configurations of the string). (Although the models were constructed out of free fermions in [7,8], they will be given a somewhat more geometric interpretation later in this paper.) Second, the models treat the left- and right-moving degrees of freedom of the string asymmetrically. None of the models in refs. [7,8] contain chiral (4d) fermions; however, they leave open the questions as to whether other left-right asymmetric constructions can give rise to chiral models and, more importantly, whether models with massless fermions in realistic gauge group representations can be constructed.

The aim of this paper is to investigate further the possible gauge groups and fermion representations for 4d models based on the type II superstring. We will give a complete list of 4d gauge groups that can appear in such models, and will show that most of them can be generated by compactifying the type II superstring on an asymmetric orbifold,[9] that is, an orbifold in which the left- and right-moving modes of the bosonic fields coordinatizing the internal space are twisted differently. Included in the list of groups are several containing the standard model gauge group. We will also show that the three $N = 4$ supersymmetric models in ref. [7] (which also appear in ref. [8]), with gauge groups $SU(2)^6$ (this group was first noted by Castellani *et al.*[10]), $SU(4) \otimes SU(2)$ and $SO(5) \otimes SU(3)$ can be constructed as asymmetric orbifolds, although their original construction was quite different. In particular, the models were first constructed entirely from free fermions (replacing the internal bosonic coordinates); the last two of the three models used an unconventional form for the world-sheet supersymmetry generator $T_F(z)$, which is the dimension 3/2 superpartner of the stress-energy tensor $T_B(z)$. The orbifold construction of the models uses the 'standard' supersymmetry generator for the internal dimensions, $T_F^{\rm int} = -\frac{1}{2}\psi^i \partial X^i$. However, the Neveu-Schwarz-Ramond fermions ψ^i and the bosonic fields ∂X^i appearing in $T_F^{\rm int}$ are twisted. The equivalence between the two types of constructions also demonstrates the relation between the two forms of T_F. The various other models found in [8] may be constructed in turn as asymmetric orbifolds of the three $N=4$ supersymmetric models (although we will not do this explicitly). On the other hand, asymmetric orbifolds also yield models which cannot be described using free fermions alone, for example $N=4$ supersymmetric models with gauge group $SU(3)^2$ or $SO(5) \otimes SU(2)^2$.

The existence of a number of four-dimensional $N=4$ supersymmetric models with gauge groups containing the standard model's suggests that models with chiral fermions (having at most $N=1$ supersymmetry) might also exist, and in fact construction of such models is particularly straightforward from the orbifold point of view. In the $N=4$ models mentioned above all four supersymmetries arise from modes with the same two-dimensional chirality, say right-moving. The gauge symmetries, on the other hand, are due to the presence of left-moving currents on the world-sheet. If one twists an $N=4$ model by a symmetry which breaks all but one of the right-moving supersymmetries, and which simultaneously acts on the left-moving gauge currents, one can build chiral 4d models. However, it is difficult to build models whose fermion content approximates that of the standard model. In fact, the main result of this paper is to show, on very general grounds, that *no models based on the type II superstring can possibly contain the standard model gauge group and its fermion content!*

Given the strength of this result, it is important to outline the assumptions leading to it. In particular, we need to define what we mean by a 'model based on the type II superstring.' First of all, we define the type II superstring theory itself as a theory of closed orientable strings that has local (1,1) world-sheet supersymmetry; this definition involves only the geometrical properties of the world sheet. (We consider 2d supersymmetry to be a geometrical feature since it makes the world sheet a superspace.) Classical space-time backgrounds in which the string might propagate are described in terms of a world-sheet action, *i.e.* in terms of a two-dimensional quantum field theory; we will refer to different backgrounds as being different string models (based on the same string theory). For example, the flat ten-dimensional space-time background corresponds to the world-sheet action of ref. [11] that describes ten free superfields $\mathbf{X}^\mu$, $\mu = 0, 1, \ldots, 9$. One of these superfields, namely $\mathbf{X}^0$, has negative metric; however, it can be shown that negative norm states decouple from the physical amplitudes provided that the 2d theory has (1,1) superconformal invariance.[12] In this paper we study models with four-dimensional Minkowski space-time left intact; this (apparently) requires four of the above superfields — $\mathbf{X}^{0,1,2,3}$ — to be present in the 2d field theory describing a given model. We therefore insist on (1,1) superconformal invariance of the 2d theory in order to decouple negative norm states created by $\mathbf{X}^0$.

A better way to justify our insistence on (1,1) superconformal invariance is via the equivalence of this condition to the classical equations of motion of the type II superstring.[3,13,14] Thus classical superstring vacua correspond to superconformally invariant 2d field theories while solutions of the quantum superstring theory that need quantum effects for their stability do not. Hence the arguments of this paper apply to classical superstring models only; however, quantum string vacua have yet to be constructed. On the other hand, we would like to stress that *superconformal invariance and existence of 4d Minkowski space-time are the only requirements we impose on the models*; in particular, we do *not* assume any particular compactification scheme, orbifold or otherwise, space-time supersymmetric or not. In fact the superconformal theory which replaces the six 'missing' spatial dimensions need not have any geometric interpretation at all.

This paper is organized as follows. In section 2 we discuss relations between 4d gauge symmetries and properties of the underlying superconformal field theory. We find that the Lie algebra associated with the gauge group extends to a supersymmetric affine Kac-Moody algebra,[15] which is generated by purely left-moving (or purely right-moving) supercurrents — world-sheet superfields of dimension $\frac{1}{2}$. Other sources of gauge bosons, such as the Ramond-Ramond sector of the type II superstrings, or both left- and right-moving supercurrents, are excluded. The requirement that the superconformal theory replacing the six internal coordinates is unitary and has super Virasoro central charge $\hat{c} = 6$ constrains the possible super Kac-Moody algebras and thus allows us to classify the possible gauge groups for 4d models based on the type II superstring. In section 3 we detour from the general approach to construct asymmetric orbifolds which give rise to many of the 4d gauge groups permitted by the results of section 2, including the three models of ref. [7]. In addition we describe two chiral $N = 1$ supersymmetric models. In section 4 we return to general considerations, focusing now on gauge quantum numbers of massless particles. We find that the representations of the 4d gauge group which can appear in the massless spectrum of a type II superstring model are severely restricted by the requirement that all physical states belong to unitary representations of the entire super Kac-Moody algebra. In particular, a 4d model whose gauge group contains $SU(3) \otimes SU(2) \otimes U(1)$ can have either massless $SU(3)$ triplets or massless $SU(2)$ doublets, but never both; this rules out the existence of realistic models based on the type II superstring. Our conclusions are presented in section 5.

2. Four-dimensional Gauge Groups

2.1. Superconformal Invariance

We begin this section by reviewing the general properties of a model based on the type II superstring which satisfies the classical string equations of motion. As emphasized in refs. [3,13,14], the latter condition is equivalent to superconformal invariance of the 2d field theory describing the model. Together with super-reparameterization invariance, this allows us to describe the 2d theory in a locally flat superspace parametrized by holomorphic and anti-holomorphic coordinates (z, θ) and $(\bar{z}, \bar{\theta})$. Whatever the matter superfields making up the theory, they must give rise to a holomorphic super-stress tensor $T(z,\theta) = T_F(z) + \theta T_B(z)$ and its anti-holomorphic counterpart $\overline{T}(\bar{z},\bar{\theta}) = \overline{T}_F(\bar{z}) + \bar{\theta}\overline{T}_B(\bar{z})$. The operator product expansion (OPE) of $T(z,\theta)$ is given by

$$T(z_1,\theta_1)\cdot T(z_2,\theta_2) = \frac{\hat{c}/4}{z_{12}^3} + \frac{3\theta_{12}}{2z_{12}^2}T(z_2,\theta_2) + \frac{1/2}{z_{12}}D_2T + \frac{\theta_{12}}{z_{12}}\partial_2 T + \cdots , \quad (2.1)$$

where $z_{12} = z_1 - z_2 - \theta_1\theta_2$ and $\theta_{12} = \theta_1 - \theta_2$. In addition to the matter fields, there are ghosts for the local super-reparameterization invariance giving rise[16] to a stress tensor $T^{\rm gh}$ with central charge $\hat{c}^{\rm gh} = -10$; hence $\hat{c} = 10$ is required in (2.1) in order for the total superconformal anomaly to cancel ($\hat{c}^{\rm tot} = 0$).

Expanding $T(z,\theta)$ in a Laurent series, one has

$$T(z,\theta) = \tfrac{1}{2}\sum_{r\in\mathbf{Z}+\kappa} G_r z^{-r-3/2} + \theta\sum_{n\in\mathbf{Z}} L_n z^{-n-2} , \quad (2.2)$$

where κ is either $1/2$ or 0 according to whether T_F and other world-sheet fermions satisfy anti-periodic or periodic boundary conditions. These two cases give rise respectively to the Neveu-Schwarz (NS) and Ramond (R) versions of the super Virasoro algebra, which is the algebra obeyed by the Laurent coefficients L_m and G_r as a consequence of the OPE (2.1):

$$\begin{aligned}
[L_m, L_n] &= \tfrac{\hat{c}}{8}(m^3 - m)\delta_{m+n,0} + (m-n)L_{m+n} , \\
\{G_r, G_s\} &= \tfrac{\hat{c}}{2}(r^2 - \tfrac{1}{4})\delta_{r+s,0} + 2L_{r+s} , \\
[L_m, G_r] &= (\tfrac{m}{2} - r)G_{m+r} .
\end{aligned} \quad (2.3)$$

Superconformal theories of physical interest are not completely arbitrary. In particular, four free superfields $\mathbf{X}^\mu(z,\theta,\bar{z},\bar{\theta}) = X^\mu(z,\bar{z}) + \theta\psi^\mu(z) + \bar{\theta}\bar{\psi}^\mu(\bar{z})$, $\mu = 0,1,2,3$ are needed to make four-dimensional Minkowski space-time; other, 'internal' fields should commute with $\mathbf{X}^\mu$. Thus we have a direct product of two superconformal theories, space-time (4d) and internal, whose respective super-stress tensors $T^{(4d)}(z,\theta) = -\frac{1}{2}D\mathbf{X}^\mu D^2\mathbf{X}_\mu$ and $T^{\rm int}$ anticommute with each other. It follows from (2.1) that $T(z,\theta) \equiv T^{(4d)}(z,\theta) + T^{\rm int}(z,\theta)$ gives rise to a super Virasoro algebra with central charge $\hat{c} = \hat{c}^{(4d)} + \hat{c}^{\rm int}$; since $\hat{c}$ is normalized to be 1 for a free superfield, $\hat{c}^{(4d)} = 4$ and we need $\hat{c}^{\rm int} = 6$ in order to cancel the total superconformal anomaly. Note that while local (on the world sheet) properties of the internal and the space-time components of the theory need not be related to each other, the boundary conditions obeyed by $T_F^{\rm int}$ and by $T_F^{(4d)} = -\frac{1}{2}\psi_\mu \partial X^\mu$ should be the same.[17] (This follows from the fact that the world-sheet gravitino couples to the total T_F of all matter fields.) Thus we can refer to sectors of the Hilbert space as being Neveu-Schwarz (NS) or Ramond (R) even when the superconformal theory describing the internal degrees of freedom is highly non-trivial.

2.2. Ramond-Ramond Vector Bosons

The type II superstring has both a left- and a right-moving superconformal algebra, so there are four sectors altogether. Clearly states belonging to (R,NS) and (NS,R) sectors are space-time fermions while (NS,NS) and (R,R) sectors contain space-time bosons. (R,R) bosons are peculiar to the type II superstring, so let us take a closer look at them. The first question we would like to ask is whether any *massless* scalar or vector particles could be of this type. In general, the answer to this question is yes; for example, consider a type IIA (non-chiral) ten-dimensional superstring compactified on a Calabi-Yau manifold or even on a torus. In ten dimensions, massless (R,R) bosons form a vector and a three-index anti-symmetric tensor of the (10d) Lorentz group. After this compactification, they give rise to $B_2 + 2$ massless (4d) vectors and $B_3 + 2B_1$ massless scalars. ($B_{1,2,3}$ are the Betti numbers of the manifold.)

Our next question concerns the couplings of massless (R,R) bosons to other massless particles. Consider a vertex operator $V_s(k)$ for an (R,R) scalar. Unlike the (NS,NS) case, this operator must behave like a (4d) spinor with respect to both left- and right-moving superconformal algebras. Moreover, these two spinors should have opposite helicities and thus opposite chiralities since they combine into a scalar rather than into a vector. Thus the vertex operator should look like $V^{\alpha,\dot\beta}(k)$ or $V^{\dot\alpha,\beta}(k)$. Due to the absence of any polarization vector, spinor indices have to be contracted with $k_\mu\gamma^\mu$: $V_s(k) \equiv k_\mu\sigma^\mu_{\alpha\dot\beta}V^{\alpha,\dot\beta}(k)$ or $k_\mu\sigma^\mu_{\dot\alpha\beta}V^{\dot\alpha,\beta}(k)$. Massless (R,R) vectors also have vertex operators of spinor-spinor type, but now the two spinors have the same chirality. Such bispinors cannot be contracted with a polarization vector ς^ν alone, so again the momentum vector k^μ is required: $V_v(k,\varsigma) \equiv k_\mu\varsigma_\nu(\sigma^{\mu\nu}_{\alpha\beta}V^{\alpha,\beta} + \sigma^{\mu\nu}_{\dot\alpha\dot\beta}V^{\dot\alpha,\dot\beta})$.

How does an explicit momentum factor in the vertex of an (R,R) boson affect its couplings? In a three-point function involving massless particles only, external momenta cannot be contracted with themselves or with each other ($k_i \cdot k_j = 0$ on shell, $i,j = 1,2,3$). Therefore, all cubic couplings of a massless (R,R) field A to other massless fields cannot involve A itself, but only its derivatives. If the other two fields are space-time fermions, then we have a coupling of dimension five (or more, if other derivatives are present), and thus non-renormalizable. If the other two fields are space-time bosons, then one of them must also be of the (R,R) type; this gives us a three-boson coupling with at least two derivatives. Again, the dimension of such a coupling is at least five, so it is also non-renormalizable. Therefore, massless (R,R) bosons do not have any renormalizable cubic couplings, such as gauge couplings, to any massless particles. We immediately conclude that *massless particles are neutral with respect to any gauge groups generated by (R,R) gauge bosons*; and, as a corollary, *all such gauge groups must be abelian.*

2.3. Super Kac-Moody Algebras

Phenomenologically, only gauge symmetries that act non-trivially on massless particles are relevant. As we have just seen, such symmetries must be generated by gauge bosons coming from the (NS,NS) sector, to which we will henceforth confine our attention. In the (0,0) picture for the superconformal ghost system[14] vertex operators for these bosons are superfields of conformal dimension $(\frac{1}{2},\frac{1}{2})$:

$$V^a(k,\varsigma) \;=\; \varsigma_\mu \overline{D}\mathbf{X}^\mu(\bar z,\bar\theta)\,\mathbf{J}^a(z,\theta)\,e^{ik\cdot\mathbf{X}}. \tag{2.4}$$

The superfield $\overline{D}\mathbf{X}^\mu = \bar{\psi}^\mu + \bar{\theta}\bar{\partial}X^\mu$ is required here in order to make a four-dimensional vector; the conformal dimension of $\overline{D}\mathbf{X}^\mu$ is $(0, \frac{1}{2})$. (Of course the dimension $(\frac{1}{2}, 0)$ superfield $D\mathbf{X}^\mu$ would have served equally well; this would give us the complex conjugate of (2.4).) For $k^2 = 0$ the dimension of $e^{ik\cdot\mathbf{X}}$ is $(0,0)$; thus the dimension of $\mathbf{J}^a$ must be $(\frac{1}{2}, 0)$. This means that $\mathbf{J}^a$ is a holomorphic superfield, *i.e.* a function of (z, θ) and not $(\bar{z}, \bar{\theta})$, and that the operator product expansion of $\mathbf{J}(z,\theta)$ with $T(z,\theta)$ is given by:

$$T(z_1,\theta_1)\cdot\mathbf{J}^a(z_2,\theta_2) = \frac{\theta_{12}}{z_{12}^2}\mathbf{J}^a(z_2,\theta_2) + \frac{1/2}{z_{12}}D_2\mathbf{J}^a(z_2,\theta_2) + \frac{\theta_{12}}{z_{12}}\partial_2\mathbf{J}^a(z_2,\theta_2) + \cdots . \tag{2.5}$$

In terms of Laurent series

$$\begin{aligned} \mathbf{J}^a(z,\theta) &= J^a(z) + \theta J^a(z) \\ &= \sum_{r\in\mathbf{Z}+\kappa} J^a_r\, z^{-r-1/2} + \theta \sum_{n\in\mathbf{Z}} J^a_n\, z^{-n-1}, \end{aligned} \tag{2.6}$$

where κ is $\frac{1}{2}$ (0) in the NS (R) sector, (2.5) becomes

$$\begin{aligned} [J^a_m, L_n] &= m\, J^a_{m+n} , \\ [J^a_r, L_n] &= (r + \tfrac{n}{2})\, J^a_{r+n} , \\ [J^a_m, G_s] &= m\, J^a_{m+s} , \\ \{J^a_r, G_s\} &= J^a_{r+s} . \end{aligned} \tag{2.7}$$

Let us consider the algebra generated by the supercurrents $\mathbf{J}^a$. The product of two supercurrents $\mathbf{J}^a(z_1,\theta_1)$ and $\mathbf{J}^b(z_2,\theta_2)$ must be single-valued as $z_1 \to z_2$, so only integral powers of $1/z_{12}$ are allowed in the operator product expansion. Since the conformal dimension of $\mathbf{J}^a$ is $\frac{1}{2}$, the OPE is limited on dimensional grounds to

$$\mathbf{J}^a(z_1,\theta_1)\cdot\mathbf{J}^b(z_2,\theta_2) = \frac{1}{2z_{12}}\cdot k^{ab}\mathbf{1} + \frac{\theta_{12}}{z_{12}}\cdot i f^{abc}\mathbf{J}^c(z_2,\theta_2) + \cdots , \tag{2.8}$$

which means that the Laurent coefficients of the supercurrents — J^a_n and J^a_r — obey a supersymmetrized affine Kac-Moody algebra[15] (henceforth called an

SKM algebra):

$$
\begin{aligned}
[J^a_m, J^b_n] &= \tfrac{m}{2}\, k^{ab}\delta_{m+n,0} \;+\; if^{abc}J^c_{m+n}\ , \\
[J^a_m, J^b_r] &= if^{abc}J^c_{m+r}\ , \\
\{J^a_r, J^b_s\} &= \tfrac{1}{2}\, k^{ab}\delta_{r+s,0}\ .
\end{aligned}
\tag{2.9}
$$

As with the super Virasoro algebra, there is a Neveu-Schwarz ($\kappa = \frac{1}{2}$) and a Ramond ($\kappa = 0$) version of the SKM algebra, whose respective representations describe states in the NS and R sectors of the superconformal theory. (The boundary conditions on $J^a(z)$ must match those on $T_F(z)$.) In either sector, zero modes of the bosonic currents — J^a_0 — form a closed Lie algebra with structure constants f^{abc}; it is this algebra that generates the 4d gauge symmetry.

Consider the case of a simple gauge group G. Note that although we have written (2.8) with an arbitrary matrix k^{ab} of Kac-Moody central charges, G-invariance requires $k^{ab} = k \cdot \delta^{ab}$. (From now on we assume that supercurrents are normalized such that f^{abc} have their conventional values, which means $f^{abc}f^{dbc} = C_G(\mathrm{adj})\delta^{ad}$, where $C_G(\mathrm{adj})$ is the quadratic Casimir of the adjoint representation of the group: $C_{SU(N)}(\mathrm{adj}) = N$, *etc.*) Moreover, the existence of unitary representations of the SKM algebra requires the Kac-Moody central charge k to be an integer and

$$
\hat{k} \equiv k - C_G(\mathrm{adj}) \tag{2.10}
$$

to be non-negative.[15] (We will further discuss the restrictions imposed by unitarity in section 4.)

The super-stress tensor T^{SKM} for an SKM algebra can be written as a normal-ordered product of the supercurrents[15,18]:

$$
T^{\mathrm{SKM}}(z,\theta) \;=\; \tfrac{-1}{k}\, {:}D\mathbf{J}^a(z,\theta)\cdot\mathbf{J}^a(z,\theta){:} \;+\; \tfrac{2}{3k^2}\, {:}f^{abc}\mathbf{J}^a(z,\theta)\mathbf{J}^b(z,\theta)\mathbf{J}^c(z,\theta){:}\,. \tag{2.11}
$$

(The relation (2.11) is analogous to the construction[19] of the stress tensor $T^{\mathrm{KM}}_B(z)$ for an ordinary, *i.e.* non-supersymmetric, affine Kac-Moody algebra.)

The super-stress tensor $T^{\rm SKM}$ generates a super Virasoro algebra with central charge $\hat{c}$ given by[15,18]

$$\hat{c}(G) \;=\; \frac{d(G)}{3} \;+\; \frac{2d(G)}{3}\cdot\frac{k - C_G(\rm adj)}{k}\;, \tag{2.12}$$

where $d(G)$ is the dimension of G.

In general the 4d gauge group generated by J_0^a is not simple, but rather a direct product of several simple or abelian subgroups. It is easy to see that in this case the SKM algebra is also a direct product of several subalgebras (supercurrents belonging to different subalgebras anticommute with each other); the value of k may differ from subalgebra to subalgebra. For each subalgebra we may use equations (2.11) and (2.12) to compute its share of T and $\hat{c}$; for the combined SKM algebra these shares add up. Note that for any non-abelian group G, (2.12) gives $d(G)/3 \leq \hat{c}(G) < d(G)$ since $C_G(\rm adj) \leq k < \infty$. (In the limit of infinite k, rescaling the currents in (2.9) and (2.11) shows that one actually has an abelian group $U(1)^{d(G)}$ instead of G.) On the other hand, for an abelian group $\hat{c}(G) = d(G)$ regardless of k; this follows from the fact that abelian supercurrents are free superfields ($\mathbf{J} \sim D\mathbf{X}$), whose normalization can absorb any positive k.

The SKM algebra, whether simple or not, does not have to constitute the entire internal superconformal theory. In fact, if $\hat{c}^{\rm SKM} < 6$, there should be a 'left-over' piece, whose super-stress tensor $T^L = T^{\rm int} - T^{\rm SKM}$ anticommutes with $\mathbf{J}^a$ and $T^{\rm SKM}$ and has central charge $\hat{c}^L = 6 - \hat{c}^{\rm SKM}$. This left-over piece is rather arbitrary; however, the representations of its super Virasoro algebra must be unitary. The latter restriction is actually rather useful since for $\hat{c}^L < 1$ all unitary representations of the super Virasoro algebra have been classified,[20] and the only allowed values of $\hat{c}^L$ are

$$\hat{c}_m \;=\; 1 \;-\; \frac{8}{m(m+2)}\;, \quad m = 2,3,4,\ldots\;. \tag{2.13}$$

Thus, *either* $\hat{c}^{\rm SKM} \leq 5$, *or* $\hat{c}^{\rm SKM} = 5 + 8/m(m+2)$ *for some integer* $m \geq 2$ ($m = 2$ corresponds to $\hat{c}^L = 0$ and $\hat{c}^{\rm SKM} = 6$, *i.e.* $T^{\rm int} = T^{\rm SKM}$). Combining this restriction on $\hat{c}^{\mathrm{SKM}}$ with (2.12), we can write down all physically allowed SKM algebras and thus give the **complete list of 4d gauge groups allowed in type II superstrings:**

$$
\begin{array}{ll}
\text{(A)} & SU(2)^6, \\
\text{(B)} & SU(4) \otimes SU(2), \\
\text{(C)} & SO(5) \otimes SU(3), \\
\text{(D)} & SO(5) \otimes SU(2) \otimes SU(2), \\
\text{(E)} & SU(3) \otimes SU(3), \\
\text{(F)} & G_2, \\
\text{(G)} & \text{all proper subgroups of (A)–(F).}
\end{array}
\tag{2.14}
$$

Central charges of the allowed SKM algebras are also restricted: algebras (A), (B), (C) and (D) must have $\hat{k} = 0$ for each of their simple factors; however, $\hat{k} > 0$ is allowed in cases (D) and (G).

2.4. Right-moving Supercurrents

There are two remarks to be made with respect to the above result. First, we have not shown that all the groups in (2.14) can actually be realized as 4d gauge groups of consistent string models; this will be the subject of the next section. Second, we have concentrated on the holomorphic SKM algebras, but have ignored the anti-holomorphic ones. If both kinds of SKM algebras were present in the same string model, we could have doubled the gauge group. We are now going to show that this never happens in models with massless fermions.

In the type II superstring theory fermions belong to the (R,NS) and (NS,R) sectors of the Hilbert space; for the sake of definiteness, we will demand the presence of massless (R,NS) fermions and concentrate on the consequences for the left-handed currents. The key fact about massless states $|\psi\rangle$ in the Ramond sector is that they satisfy

$$G_0^{\text{int}}\,|\psi\rangle \;=\; 0\,. \tag{2.15}$$

The easiest way to deduce (2.15) is to use the light-cone relation between masses of states and eigenvalues of L_0: $(L_0^{l.c.} - \frac{1}{2})\,|\psi\rangle = \frac{\alpha'}{4}\text{mass}^2\,|\psi\rangle$.* In the Ramond sector, $L_0^{l.c.} - \frac{1}{2} = (G_0^{l.c.})^2$, so massless Ramond states are annihilated by $(G_0^{l.c.})^2$. Now $G_0^{l.c.}$ is a sum of G_0^{int} and the transverse part of G_0^{4d}, which are both hermitian and anticommute with each other. By squaring this decomposition, we immediately see that any state annihilated by $(G_0^{l.c.})^2$ is also annihilated by $(G_0^{\text{int}})^2$ — and thus by G_0^{int} itself. This verifies (2.15). Alternatively, we could have proved (2.15) in the covariant Neveu-Schwarz-Ramond formalism with the help of the Dirac-Ramond equation $G_0^{\text{tot}}\,|\psi\rangle = 0$ and the fact that massless states obey $G_0^{4d}\,|\psi\rangle = 0$.

Now consider the SKM algebra for the non-abelian (simple) group G. As shown by Kac and Todorov,[15] eigenvalues of L_0^{SKM} in the Ramond sector are given by $h^{\text{SKM}} = d(G)/16 +$ positive terms. Using $G_0^2 = L_0 - \hat{c}/16$ and (2.12), we find that for any non-null (physical) state $|\psi\rangle$:

$$\langle\psi|\,(G_0^{\text{int}})^2\,|\psi\rangle \;\geq\; \langle\psi|\,(G_0^{\text{SKM}})^2\,|\psi\rangle \;\geq\; \frac{d(G)C_G(\text{adj})}{24k}\,\langle\psi|\psi\rangle \;>\; 0 \tag{2.16}$$

and equation (2.15) cannot be satisfied. Thus, if a string model has *any* non-abelian factor in the gauge group coming from a holomorphic, *i.e.* left-moving, SKM algebra, then there are no massless (R,NS) fermions in that model; there are also no massless (R,R) bosons. Similarly, if the source of a non-abelian gauge group is a right-moving (anti-holomorphic) SKM algebra, then there are no massless fermions of the (NS,R) type. Consequently, if both left-moving and right-moving non-abelian SKM algebras are present, then there are no massless

* The definition of $L_0^{l.c.}$ used here was chosen for compatibility with the super Virasoro algebra (2.3); it differs from the usual light-cone convention for the Ramond sector by an additive $\frac{1}{2}$.

fermions at all![20] In order to avoid such a calamity, only left-moving SKM algebras should be allowed in string models of phenomenological interest; then all massless bosons come from the (NS,NS) sector and all massless fermions come from the (NS,R) sector. (We choose this convention rather than its opposite by analogy with the heterotic string convention.) As a corollary, *all type II superstring models that treat left- and right-moving degrees of freedom symmetrically are excluded.*

The problem with the above argument is that it does not apply to abelian gauge groups. And in fact there do exist string models with massless fermions and with abelian gauge bosons coming from both left- and right-moving internal algebras. For example, the ten-dimensional superstring compactified on a torus has all of these features. However, one can show that massless (NS,R) fermions are always neutral with respect to charges arising from any right-handed currents.[21] (Similarly any massless (R,NS) fermions are neutral with respect to left-handed charges.) The argument is based on (2.15) and on commutation relations (2.7). Let $|\psi\rangle$ be a massless (NS,R) state and let $\overline{\mathbf{J}}^a(\bar{z},\bar{\theta})$ be a right-moving supercurrent. The charge operator is $\overline{J}_0^a$ — the zero mode of the bosonic part of $\overline{\mathbf{J}}^a$ — so the charge of $|\psi\rangle$ is given by

$$\langle\psi|\,\overline{J}_0^a\,|\psi\rangle \;=\; \langle\psi|\,\{\bar{\mathcal{J}}_0^a,\overline{G}_0^{\mathrm{int}}\}\,|\psi\rangle\,.$$

But the latter expression vanishes because a massless state $|\psi\rangle$ satisfies $\langle\psi|\,\overline{G}_0^{\mathrm{int}} = 0 = \overline{G}_0^{\mathrm{int}}\,|\psi\rangle$ (this is just the right-moving version of (2.15) and its hermitian conjugate).

While decoupling of the abelian gauge groups generated by right-moving supercurrents from massless fermions is already sufficient to render such groups phenomenologically worthless, their very presence also causes phenomenological problems. Specifically, the existence of a free right-moving superfield $\bar{\mathbf{J}}^a$ is incompatible with having a chiral 4d gauge theory. The easiest way to see this is to notice that the contribution of the fermionic current $\bar{\mathcal{J}}^a$ to the stress tensor $\overline{T}_B$

and to the GSO projector $(-1)^{\overline{F}}$ is in no way distinguishable from contributions of fermionic partners $\bar{\psi}^\mu$ of space-time dimensions X^μ. So, as far as right-moving fermions are concerned, there are five (or more) flat space-time dimensions, just like in the case of a toroidal compactification. Consequently, Ramond states form spinor representations not just of the ordinary Lorentz group $SO(1,3)$, but of $SO(1,4)$ (or an even bigger Lorentz-like group, if there are several supercurrents $\bar{\mathbf{J}}^a$). With respect to $SO(1,3)$, these bigger spinors transform as non-chiral pairs $(\mathbf{2}+\bar{\mathbf{2}})$ of Lorentz spinors. Thus, for every left-handed 4d fermion there is a right-handed one with the same internal quantum numbers and *vice versa*; in particular, all gauge interaction are vector-like.

Let us now summarize the results of this section. We have considered the most general classical solutions of the type II superstring that incorporate flat four-dimensional Minkowski space-time and thus lead to effective 4d field theories at sufficiently low energies. If the effective 4d theory contains both a non-abelian gauge group and massless fermions that are chiral (these are basic features of any realistic model), then the gauge group is one of the groups listed in (2.14). Moreover, only left-right asymmetric solutions can have these features: The gauge group should be generated by the left-moving SKM algebra, while massless fermions come just from the (NS,R) sector of the superstring; also, only the (NS,NS) sector contains massless bosons.

3. Explicit Models via Asymmetric Orbifolds

3.1. General Considerations

In this section we show how to construct various 4d type II models which explicitly realize many of the gauge groups permitted by the considerations of the previous section. Those considerations involve only classical (tree-level) string physics; they rely only on local properties of a 2d superconformal theory as formulated on the two-sphere, without regard to the theory's consistent extension to

Riemann surfaces of higher genus. We are now interested in constructing models which are also consistent at the quantum level, *i.e.* for which higher-loop string amplitudes can be defined. Thus we require the 2d theory to be invariant under modular transformations — diffeomorphisms of higher-genus surfaces which are not continuously connected to the identity.

We could try to construct modular-invariant models directly in terms of the representations of the super Kac-Moody algebra which they contain. At one loop this *might* be feasible, using the known modular transformation properties of partition functions for representations of Kac-Moody algebras.[22] However at higher loops this approach appears difficult. On the other hand, orbifold techniques provide an efficient way to construct string models which are modular-invariant to all orders in the loop expansion.[23] The general orbifold construction starts with some initial modular-invariant string model with a first-quantized action and Hilbert space which both have some discrete symmetry P; then one *twists* the initial model by the group P in order to construct a new model. The procedure of twisting a model by P has two steps. One first projects out those states in the original Hilbert space which are not invariant under P. The surviving states form the *untwisted sector* of the new model. Then one adds states in *twisted sectors*, where the boundary conditions on the fields in the original action are modified – the string need only close modulo an element of P. If the group P is non-abelian the twisting procedure is slightly more complicated; for details see reference [4]. We will be using only abelian (in fact cyclic) groups P in this paper.

The orbifolds we consider here start from a generalized toroidal compactification[24,25] of the type II superstring. The metric of the internal six-dimensional space is that of a torus, but we also allow for constant background values $\langle B^{ij} \rangle$ of the anti-symmetric tensor field. In any compactification of this kind, the eigenvalues p_L^i and p_R^i of the left- and right-moving zero-modes of the internal coordinates $X^i(z,\bar{z})$ $(i = 1, \ldots, 6)$ lie on an even, self-dual Lorentzian lattice of signature $(6,6)$, $\Gamma^{6,6} = \{(p_L^i, p_R^i)\}$.* These models have $N=8$ supersymmetry in four dimensions, with four supersymmetries each coming from left- and right-moving fermion zero-modes. The gauge group is $U(1)^6 \otimes U(1)^6$, with 6 $U(1)$'s each from left- and right-moving supercurrents (the internal free superfields $D\mathbf{X}^i$ and $\overline{D}\mathbf{X}^i$).

To construct models with more interesting gauge groups, we will twist these $N=8$ models by rotations of the lattice $\Gamma^{6,6}$ taking lattice points to lattice points, *i.e.* automorphisms of $\Gamma^{6,6}$. We will accompany these rotations by translations or *shifts* (v_L^i, v_R^i) on the lattice. The general transformation we consider therefore takes

$$
\begin{aligned}
X_L^i(z) &\to \omega_L^{ij} X_L^j(z) + 2\pi v_L^i \ , \\
X_R^i(\bar{z}) &\to \omega_R^{ij} X_R^j(\bar{z}) + 2\pi v_R^i \ ,
\end{aligned}
\tag{3.1}
$$

where ω_L and ω_R are $SO(6)$ rotations and the combined rotation (ω_L, ω_R) is an automorphism of $\Gamma^{6,6}$. Equation (3.1) is a somewhat schematic representation of an asymmetric twist, because there is only one center-of-mass coordinate for the field $X^i(z,\bar{z}) = X_L^i(z) + X_R^i(\bar{z})$. The proper definition is as an action on the bosonic Hilbert space, which takes $|p_L; p_R\rangle \to e^{2\pi i(p_L\cdot v_L - p_R\cdot v_R)}\,|\omega_L p_L; \omega_R p_R\rangle$, *etc.*[9] In order to preserve the world-sheet supersymmetry generators $T_F(z) = -\frac{1}{2}\psi^i\partial X^i$ and $\overline{T}_F(\bar{z}) = -\frac{1}{2}\bar{\psi}^i\bar{\partial}X^i$ (see the previous section), the fermions ψ^i and $\bar{\psi}^i$ must transform in the same way as ∂X^i and $\bar{\partial}X^i$ respectively:

$$
\begin{aligned}
\psi^i(z) &\to \omega_L^{ij}\psi^j(z) \ , \\
\bar{\psi}^i(\bar{z}) &\to \omega_R^{ij}\bar{\psi}^j(\bar{z}) \ .
\end{aligned}
\tag{3.2}
$$

So we can denote an arbitrary element of P by $(\omega_L, v_L; \omega_R, v_R)$; this completely specifies its action on the fields X^i, ψ^i and $\bar{\psi}^i$.

---

* Strictly speaking, all lattices $\Gamma^{6,6}$ are equivalent under $SO(6,6)$ rotations; but the generic $SO(6,6)$ rotation does not commute with the left- and right-moving Hamiltonians L_0 and $\overline{L}_0$, so different embeddings of $\Gamma^{6,6}$ with respect to the division into left- and right-moving zero-modes lead to different physics. When we refer to *a* lattice $\Gamma^{6,6}$ we really mean an embedding of *the* lattice $\Gamma^{6,6}$ into the momentum space of left- and right-moving zero modes.

In the previous section we saw that if there are to be any massless fermions in a 4d model then all gauge bosons that couple to them must arise from, say, left-moving supercurrents. Now we would like to show that in an orbifold model these gauge bosons come only from the sectors of the Hilbert space that are twisted by elements of P of the form $(\omega_L, v_L; 1, 0)$, that is, from the sectors in which the right-moving fields X^i_R and $\bar{\psi}^i$ are completely untwisted. Consider the vector index μ on a gauge boson whose gauge index is given by left-moving supercurrents, as in eq. (2.4). The vector index requires the use of a right-moving Minkowski oscillator such as $\bar{\psi}^{\mu}_{-1/2}$, which raises the $\overline{L}_0$ eigenvalue (the 'right-moving energy') by at least one-half unit. (Note that the Minkowski field $\bar{\psi}^{\mu}$ is necessarily untwisted.) We can ignore the 'left-moving energy' (L_0 eigenvalue) in this discussion. Given the energy cost of the vector index, a massless vector particle requires the vacuum energy to be $-\frac{1}{2}$ or lower. The untwisted NS sector indeed has vacuum energy $-\frac{1}{2}$; in order to complete the proof we have to show that all twisted sectors (with respect to the right-moving fields) have higher vacuum energies. This follows from the fact that a complex free boson with boundary conditions twisted by $e^{2\pi i\beta}$ with $0 \leq \beta < 1$ (*i.e.* $\partial X(e^{2\pi i}z) = e^{2\pi i\beta}\partial X(z)$) contributes

$$\tfrac{1}{2}\beta(1-\beta) - \tfrac{1}{12} \tag{3.3}$$

to the vacuum energy,[4] while the contribution of a complex fermion with the same boundary conditions has the same absolute value but the opposite sign. The expression (3.3) is minimized for periodic bosons ($\beta = 0$) and anti-periodic fermions ($\beta = 1/2$), just the combinations which occur in the untwisted NS sector; any other combination will thus have higher vacuum energy. This argument excludes any non-trivial rotation $\omega_R \neq 1$,* and the case against a non-zero shift

---

* We must be a little more careful in the exceptional case of the twist which rotates one of the right-handed superfields by 2π ($\beta = 1$). This twist does not actually change the boundary conditions on any of the fields, and hence does not change the vacuum energy. However, it does alter the GSO projection (see below), so that the state $\bar{\psi}^{\mu}_{-1/2}$ is projected out by GSO. Hence there are no twisted massless vectors in this case either.

v_R is even simpler: Sectors with $v_R \neq 0$ do not contain states with $p_R = 0$, and $p_R \neq 0$ costs $\frac{1}{2}p_R^2 > 0$ energy units.

Consider now the model obtained by twisting the initial toroidal compactification, not by P but by the (normal) subgroup P_L of P consisting of all elements of P of the form $(\omega_L, v_L; 1, 0)$. Then the part of the gauge group of the P model coming from left-handed currents is contained in the gauge group of the P_L model. This is because, as we have just shown, all such P gauge bosons come from sectors twisted by elements of P_L. If they survive the projection onto P-invariant states, then clearly they also survive the projection onto P_L-invariant states, since $P_L \subseteq P$.† So if we are interested only in the maximal gauge symmetries of 4d type II models, we can confine ourselves to studying the $P = P_L$ models, in which the right-handed fields have been left completely untwisted — thus they are examples of *asymmetric orbifolds*.[9]

If we are to generate models with non-abelian gauge symmetry, we know from section 2 that there must be no massless (R,NS) states. In particular we must break all four left-moving supersymmetries, by projecting out the four (R,NS) gravitinos of the toroidal compactification. This can be done by choosing rotations ω_L which do not sit in an $SU(3)$ subgroup of $SO(6)$. However, we must also ensure that no massless (R,NS) states appear in any of the *twisted* sectors. But in the Ramond sector, whether twisted or not, the vacuum energy contributions from X_L^i and from ψ^i always cancel. One way to avoid massless twisted (R,NS) states is by having a nonzero shift v_L in each twisted sector. Then the lowest energy (R,NS) state has nonzero momentum p_L and therefore positive energy, $\frac{1}{2}p_L^2$. Since the P_L models of interest all have $N = 4$ space-time supersymmetry, we will refer to them as $N = 4$ models, and to the generalized toroidal compactifications as $N = 8$ models.

The right-moving $U(1)^6$ gauge symmetry of the $N = 8$ models survives in

† If P is non-abelian, the projections are not in general onto P-invariant states, but one can show that the same conclusion still holds, using the fact that P_L is a normal subgroup of P.

the $N = 4$ models, but the massless (NS,R) fermions are neutral with respect to this symmetry (see section 2), which will in any case be broken if one breaks the space-time supersymmetry to $N = 1$ by further twisting. So we will omit the $U(1)^6$ factor when discussing the gauge group of an $N = 4$ model. Starting from a few different lattices $\Gamma^{6,6}$ and twisting by discrete groups P_L such as Z_2 or Z_3, we can generate a variety of $N = 4$ models, whose gauge groups include all even rank groups in the list (2.14). (Some of our constructions use a succession of cyclic twists.) We would like to give some details of the orbifold construction of these models, but first we need to describe a special class of lattices $\Gamma^{6,6}$ which have automorphisms acting only on the left-moving zero-modes p_L^i.

The generic toroidal compactification does not admit purely left-moving symmetries. However, suppose we choose[24] p_L and p_R to lie on the weight lattice $\Lambda_W(\mathcal{G})$ of a semi-simple, simply-laced Lie algebra $\mathcal{G}$ of rank six, and restrict the difference $p_L - p_R$ to lie on the root lattice $\Lambda_R(\mathcal{G})$. If $\Lambda_R(\mathcal{G})$ is normalized so that all roots α of $\mathcal{G}$ have length $\alpha^2 = 2$, then $\Lambda_R(\mathcal{G})$ is even and is the dual lattice of $\Lambda_W(\mathcal{G})$, and it is easily verified that the lattice

$$\Gamma^{6,6}(\mathcal{G}) \equiv \{(p_L, p_R)\} \ , \quad p_L, p_R \in \Lambda_W(\mathcal{G}) \ , \quad p_L - p_R \in \Lambda_R(\mathcal{G}) \ ,$$

is even and self-dual. The Lie algebra $\mathcal{G}$ should not be confused with the gauge group G of a 4d model constructed using $\Gamma^{6,6}(\mathcal{G})$; the two groups bear no direct relation to each other.

Consider now which automorphisms of $\Gamma^{6,6}(\mathcal{G})$ can act only on the p_L. Clearly they must take $(\alpha, 0)$ to $(\alpha', 0)$, where α and α' are roots of $\mathcal{G}$, so they must be elements of the automorphism group of $\Lambda_R(\mathcal{G})$. But the automorphisms must also take a weight λ in $\Lambda_W(\mathcal{G})$ to weights λ' differing from λ by a root vector; this means they must be in the Weyl group of $\mathcal{G}$ — the group of automorphisms generated by reflections in the hyperplanes orthogonal to the simple roots of $\mathcal{G}$. Indeed, a reflection in the hyperplane orthogonal to α takes $\lambda \to \lambda - 2(\alpha \cdot \lambda)\alpha$, which differs from λ by a root vector, since $\alpha \cdot \lambda$ is an integer. The remaining (outer) automorphisms of the root lattice correspond to symmetries of the

Dynkin diagram for $\mathcal{G}$; they exchange weights of representations which transform differently under the center of the Lie group for $\mathcal{G}$ and hence do not differ by root vectors — for example $N \leftrightarrow \overline{N}$ for $SU(N)$. Thus the group of automorphisms acting only on the left-moving modes p_L of the lattice $\Gamma^{6,6}(\mathcal{G})$ is precisely the Weyl group $\mathcal{W}(\mathcal{G})$.

As far as the bosons X^i are concerned, any automorphism of the lattice $\Gamma^{6,6}$ is an allowed rotation. However, the fermions ψ^i impose an additional condition, namely that ω_L (and ω_R) should preserve parity, *i.e.*, be an element of $SO(6)$ rather then $O(6)$. Note that this restriction is important for selecting an appropriate subgroup $P_L \subset \mathcal{W}(\mathcal{G})$ since the Weyl group $\mathcal{W}(\mathcal{G})$ does contain parity-violating reflections. The reason parity must be preserved is that a Ramond sector twisted by a parity-violating rotation has an odd number of (real) fermionic zero modes and the GSO projection cannot be defined. On the other hand, given a well-defined GSO projection one can distinguish between two twists that differ by a 2π rotation around some axis and thus act on the bosons in exactly the same way. Therefore, the proper way to describe a rotation ω_L is to treat it as an element of $\mathrm{Spin}(6) \equiv SU(4)$ rather then $SO(6)$ or $O(6)$.

The only physical condition on P_L other than $P_L \subset \mathrm{Spin}(6) \cap \mathcal{W}(\mathcal{G})$ is that the orbifold model generated by P_L should be modular invariant. Modular invariance at one loop, for the case of an abelian group P_L, reduces to a simple 'level-matching' condition:[26] In any twisted sector there should exist physical states, which of course must satisfy $L_0 = \overline{L}_0$. In a sector twisted by an element of order ℓ, applying twisted oscillators to the vacuum changes the $L_0, \overline{L}_0$ eigenvalues by multiples of $1/\ell$. Therefore the left- and right-moving vacuum energies in that sector should differ by multiples of $1/\ell$. This condition constrains the lengths of shifts v_L which can accompany a given rotation ω_L. One-loop modular invariance for the explicit models we construct below is easily verified just by noting that there are physical states in every twisted sector. It appears that the level-matching conditions are sufficient for an abelian orbifold to be modular invariant to all orders; this has been proved[23] when P acts arbitrarily on any free world-sheet fermions but acts symmetrically on the bosons. The models constructed below involve asymmetric twists of bosons, so the arguments of ref. [23] do not apply directly. However, some of the models can alternatively be described entirely in terms of free fermions; then level-matching does suffice for all-loop modular invariance.

3.2. $SU(2)^6$ MODEL

The first $N=4$ model we will construct is the $SU(2)^6$ model of references [10,7,8], which is also a useful intermediate step in the orbifold construction of the two other models of [7,8], with gauge groups $SU(4)\otimes SU(2)$ and $SO(5)\otimes SU(3)$. The starting point for these three $N=4$ models is the $N=8$ model which uses the Lorentzian lattice $\Gamma^{6,6}(D_6)$. The most convenient way to write this lattice is as a union of four conjugacy classes of D_6 weight vectors. That is,

$$\Gamma^{6,6}(D_6) \;=\; (A_6;A_6)+(V_6;V_6)+(S_6;S_6)+(C_6;C_6)\,, \tag{3.4}$$

where in an orthonormal basis

$$\begin{aligned}
A_{2l} &= \{(n_1,\ldots,n_{2l})\}\,, \quad n_i\in \mathbf{Z}, \quad \textstyle\sum n_i \in 2\mathbf{Z}\;,\\
V_{2l} &= \{(n_1,\ldots,n_{2l})\}\,, \quad n_i\in \mathbf{Z}, \quad \textstyle\sum n_i \in 2\mathbf{Z}+1\;,\\
S_{2l} &= \{(n_1,\ldots,n_{2l})\}\,, \quad n_i\in \mathbf{Z}+\tfrac12, \quad \textstyle\sum n_i \in 2\mathbf{Z}\;,\\
C_{2l} &= \{(n_1,\ldots,n_{2l})\}\,, \quad n_i\in \mathbf{Z}+\tfrac12, \quad \textstyle\sum n_i \in 2\mathbf{Z}+1\;,
\end{aligned} \tag{3.5}$$

define the four classes of representations of the Lie algebra D_{2l}: adjoint, vector, spinor and conjugate spinor. Two weight vectors within a given class differ from each other by a root vector.

The $N=8$ model using the lattice $\Gamma^{6,6}(D_6)$ can alternatively be described in terms of free fermions. One replaces the six bosonic coordinates $X^i(z,\bar z)$ by twelve left-moving and twelve right-moving Majorana-Weyl fermions, denoted by

λ^i, μ^i and $\bar{\lambda}^i, \bar{\mu}^i$ respectively. Then one sums over the 2^{2g} spin structures for the Riemann surface of genus g, treating all twenty-four fermions as one group. The sum over spin structures at genus one is a sum over periodic and anti-periodic boundary conditions in the σ and τ directions on the torus. States in the sector of the Hilbert space with anti-periodic (periodic) boundary conditions in σ correspond to the entries in (3.4) all being integers (half-integers); the sum over τ boundary conditions is a GSO projection which discards the $(A_6; V_6)$, $(V_6; A_6)$, $(S_6; C_6)$ and $(C_6; S_6)$ classes. The free-fermion interpretation of this $N = 8$ model will allow us to show explicitly an equivalence between the three $N = 4$ models described in [7,8] and our orbifold construction of them. (We will use other lattices later in the section which do not have as simple an interpretation in terms of free fermions.)

The $SU(2)^6$ model is obtained from the above $N = 8$ model by twisting by the group $P = P_L = Z_2$ generated by

$$\omega_L = (-1)^{F_L} = \left(e^{2\pi i J_{12}}\right)_L , \quad v_L = (1, 0^5) , \quad \omega_R = 1 , \quad v_R = 0 .$$

Here $\left(e^{2\pi i J_{12}}\right)_L$ denotes a rotation of a pair of internal left-moving free superfields, say $D\mathbf{X}^1$ and $D\mathbf{X}^2$, by an angle of 2π in the 1-2 plane. The shift v_L is written in the orthonormal basis for $\Gamma^{6,6}(D_6)$ with exponents standing for repeated entries, so v_L is a weight in the vector representation of D_6.* Let us concentrate first on the massless spectrum of this model, using light-cone gauge for simplicity, and denoting the eight left- (right-) moving NSR fermions by $\psi^{\mu,i}$ ($\bar{\psi}^{\mu,i}$), where $\mu = 1, 2$ is a light-cone Minkowski index and $i = 1, 2, \ldots, 6$ is an internal index. The 256 massless states of the initial $N = 8$ model are the tensor product of a left-moving and a right-moving $N = 4$ supermultiplet. Of the eight massless left-moving NS states, the pair $\psi^{\mu}_{-1/2} |0\rangle$ has helicity ± 1, and

* Precisely the same group element can be used to obtain the non-supersymmetric $SO(32)$ heterotic string from the supersymmetric $Spin(32)/Z_2$ heterotic string in ten dimensions,[27] if v_L is taken to be a vector weight of D_{16} rather than D_6, and of course $(-1)^{F_L}$ becomes $(-1)^{F_R}$ since the heterotic string NSR fermions are right-moving.

the remaining six states $\psi^i_{-1/2}\,|0\rangle$ have helicity 0. The eight Ramond states, $\psi_0^{i_1}\dots\psi_0^{i_k}\,|0\rangle$, form four helicity $\pm\frac{1}{2}$ pairs. The right-movers have the same set of helicities. States in the left-moving Neveu-Schwarz sector are unaffected by the rotation ω_L; however, left-moving Ramond states acquire a minus sign. The shift v_L does not affect the massless states because they carry no momentum on the lattice. So the projection onto P-invariant states simply removes the 128 (R,NS) and (R,R) states from the spectrum, leaving an $N=4$ supergravity multiplet and a $U(1)^6$ $N=4$ Yang-Mills multiplet in the untwisted sector.

In the twisted sector the Fock space for the ψ modes is the same as in the untwisted sector, but the GSO projection is now onto states with an *odd* number of ψ^μ, ψ^i excitations (see below). The bosonic zero-modes now lie on the *shifted* lattice

$$\Gamma^{6,6}(D_6)+v_L \;=\; (V_6;A_6)+(A_6;V_6)+(C_6;S_6)+(S_6;C_6)\,. \qquad (3.6)$$

The NS vacuum $|0\rangle$ with energy $-1/2$ can be raised up to the massless level using the 12 momentum states $(\underline{\pm1,\dots,0};0,\dots,0)$ (the underline denotes permutations of the six left-moving coordinates). These 12 modes (with zero space-time helicity), when tensored with the right-moving $N=4$ supermultiplet, complete the gauge symmetry from $U(1)^6$ to $SU(2)^6$. To verify that the gauge group is $SU(2)^6$ we should compute the charges of the additional 12 vector bosons under the $U(1)^6$ Cartan subalgebra; we will perform this check shortly using an alternative formulation of the same model.

The $N=4$ $SU(2)^6$ model, like the $N=8$ $\Gamma^{6,6}(\mathcal{G})$ model, can be described entirely in terms of free fermions. The only difference is that the spin structures for the eight left-moving NSR fermions $\psi^{\mu,i}$ and for the twenty-four fermions $\lambda^i,\mu^i,\bar\lambda^i,\bar\mu^i$ are now summed over together, whereas in the $N=8$ model they were summed over independently. To see the equivalence between these two descriptions, we first need to express the translation $v_L=(1,0^5)$ on the lattice $\Gamma^{6,6}(\mathcal{G})$ as an action on the fermions. It is convenient to group the fermions into complex pairs, defining

$$\chi^i \equiv \tfrac{1}{\sqrt{2}}(\lambda^i + i\mu^i),\ \phi^i \equiv \tfrac{1}{\sqrt{2}}(\lambda^i - i\mu^i),\ \bar{\chi}^i \equiv \tfrac{1}{\sqrt{2}}(\bar{\lambda}^i + i\bar{\mu}^i),\ \bar{\phi}^i \equiv \tfrac{1}{\sqrt{2}}(\bar{\lambda}^i - i\bar{\mu}^i).$$

(Note that the bars refer to complex conjugation on the world-sheet rather than on the space-time torus.) In general, the translation of a boson by β, $X \to X + 2\pi\beta$, is equivalent to the rotations $\chi \to e^{2\pi i\beta}\chi$, $\phi \to e^{-2\pi i\beta}\phi$ acting on the corresponding complex fermions. This is apparent from the bosonization dictionary*:

$$\chi = -i\, e^{iX}, \quad \phi = -i\, e^{-iX}, \quad \partial X = i\,\chi\phi\,. \tag{3.7}$$

In the case at hand, $\beta_1 = 1$ and $\beta_i = 0$ for $i = 2, \ldots, 6$. For a single fermion pair χ and ϕ with $\beta = 1$, the boundary conditions are unchanged, so it might seem that the fermion twist corresponding to v_L acts trivially. However, if one continuously varies β from 0 to 1, one finds that the fermion number of the vacuum state for $\beta = 1$ has been changed by one unit from that for $\beta = 0$. Therefore the GSO projection onto even fermion states in the twisted sector selects precisely those states which were projected out by GSO in the untwisted sector. (See the first example in the appendix of the second reference in [4].) In order to project onto P-invariant states we also need the eigenvalues of states in the untwisted sector under the twist, remembering that there are two subsectors, with χ and ϕ either both anti-periodic or both periodic. For $\beta = 1$ the modes of χ and ϕ have eigenvalue 1, so all states in the anti-periodic (periodic) subsector of the untwisted sector have the same eigenvalue: $+1(-1)$. Note that the action of v_L on the fermions χ^i and ϕ^i is identical to the action of $\omega_L = (-1)^{F_L}$ on the NSR fermions $\psi^{\mu,i}$, so the same results hold for the ψ system. In the twisted sector the condition $L_0 = \overline{L}_0$ requires the states in the anti-periodic (periodic) subsector with respect to χ and ϕ to be tensored with the NS (R) states of the ψ system.

* Our conventions for operator products of free bosons ∂X and free Majorana fermions λ are $\partial_z X \partial_w X \sim -1/(z-w)^2 + \cdots$ and $\lambda(z)\lambda(w) \sim -1/(z-w)$.

Combining the results for the two systems, one finds that in both the untwisted and twisted NS (R) sectors — *i.e.*, $\psi^{\mu,i}$ anti-periodic (periodic) — χ^i and ϕ^i are also anti-periodic (periodic), and the projection is onto states with even *total* fermion number $F_\psi + F_{\chi,\phi}$. This verifies that the spin structures for $\psi^{\mu,i}$ and χ^i, ϕ^i are now summed over together (along with that for $\bar{\chi}^i, \bar{\phi}^i$).

Thus the 18 'internal' fermions ψ^i, χ^i, ϕ^i are on an equal footing now, and in fact they are just the fermionic components $\mathcal{J}^a(z)$ of the supercurrents $\mathbf{J}^a(z,\theta)$ for the $SU(2)^6$ SKM algebra, up to a normalization factor. To check that the group is $SU(2)^6$, one needs the bosonic currents $J^a(z)$, because the $\mathcal{J}^a_r$ commutation relations are just those of free fermions, for any SKM algebra. The $J^a(z)$ are obtained from the $\mathcal{J}^a(z)$ by a world-sheet supersymmetry transformation, generated by

$$T_F^{\rm int} \;=\; -\tfrac{1}{2}\psi^i \partial X^i \;=\; -\tfrac{i}{2}\sum_i \psi^i \chi^i \phi^i = \;-\tfrac{1}{2}\sum_i \psi^i \lambda^i \mu^i \;. \tag{3.8}$$

Let us consider a supersymmetry generator of the general form

$$T_F(z) \;=\; \frac{f_G^{abc}}{3\sqrt{2C_G(\mathrm{adj})}}\,\eta^a(z)\eta^b(z)\eta^c(z)\,, \tag{3.9}$$

where $\eta^a(z)$ are free Majorana fermions, and f_G^{abc} are the structure constants for a (simple) Lie algebra G. One can verify[15,28] that $T_F(z)$ in (3.9) has the appropriate super-Virasoro operator product relations. It also generates from the fermions $\eta^a(z)$ the bosonic currents

$$J^a(z) = \tfrac{i}{2} f_G^{abc} \eta^b(z)\eta^c(z),$$

which obey an ordinary Kac-Moody algebra with $k = C_G(\mathrm{adj})$, corresponding to the minimal value $\hat{k} = 0$ for the SKM algebra. Note that $\mathcal{J}^a(z) = -i(C_G(\mathrm{adj})/2)^{1/2}\,\eta^a(z)$ are the correctly normalized fermionic currents, so that we can also write

$$J^a(z) = \tfrac{-i}{k} f_G^{abc} \mathcal{J}^b(z) \mathcal{J}^c(z) \tag{3.10}$$

for this minimal case.

Obviously the six terms of $T_F^{\rm int}$ in (3.8) are of the form (3.9) with $G = SU(2)$ ($f^{abc} = \epsilon^{abc}$ and $C_G(\text{adj}) = 2$), so all we need to complete the identification of the gauge group as $G = SU(2)^6$ is to check that all of the supercurrents correspond to physical states, *i.e.* are not projected out. This is indeed the case here: the tensor products of the 18 left-moving states $\psi^i_{-1/2}|0\rangle$, $\chi^i_{-1/2}|0\rangle$, $\phi^i_{-1/2}|0\rangle$ with the right-moving helicity $\pm\frac{1}{2}$ pair $\bar\psi^\mu_{-1/2}|0\rangle$ are all physical. (Note that the same 18 free fermions were present in the original $N = 8$ model as here, but 12 of the above 18 states were projected out by the sum over spin structures, leaving only an abelian gauge group.)

We conclude the discussion of the $SU(2)^6$ model with the remark that just as we have fermionized X^i, we can now bosonize the entire system of 20 left-moving fermions ψ^i, χ^i, ϕ^i, ψ^μ and 12 right-moving fermions $\bar\chi^i$, $\bar\phi^i$. The spin structures of all these fermions are summed over together, so the zero-mode eigenvalues for the respective bosons belong to the set

$$(V_{10}; A_6) + (A_{10}; V_6) + (C_{10}; S_6) + (S_{10}; C_6)\,, \tag{3.11}$$

where A_{2l}, V_{2l}, S_{2l}, C_{2l} are defined in (3.5). In our convention the last left-moving boson (H^{10}) corresponds to the two light-cone Minkowski fields ψ^μ, $\mu = 1, 2$. The set (3.11) is not a lattice because we are using light-cone gauge here; however, had we used the covariant formulation and incorporated the bosonized superconformal ghost zero-modes, we would have found an odd self-dual Lorentzian lattice of signature $(11, 7)$.[29,30]

3.3. $SU(4) \otimes SU(2)$ AND $SO(5) \otimes SU(3)$ MODELS

In this subsection we will construct the other two $N = 4$ models of references [7,8] as orbifolds. The $SU(4) \otimes SU(2)$ model can be obtained by twisting the $N = 8$ $\Gamma^{6,6}(D_6)$ model by the group $P = P_L = Z_4$ generated by

$$\omega_L = \text{diag}(C_2, C_2, 1, 1), \quad v_L = (0^4, \tfrac{1}{2}, 0), \quad \omega_R = 1, \quad v_R = 0. \tag{3.12}$$

Here ω_L and v_L are written in the orthonormal basis for $\Gamma^{6,6}(D_6)$ described above, and $C_2 = \binom{0\ 1}{1\ 0}$ exchanges two bosonic coordinates. Note that ω_L appears to have order two, but the actual order of the twist (3.12) acting on the Hilbert space is four, due to cocycle factors which appear in twisted vertex operators.[31] The square of this transformation is precisely the Z_2 transformation used in the previous subsection to construct the $SU(2)^6$ model. Therefore we can alternatively describe the $SU(4) \otimes SU(2)$ model as a Z_2 orbifold of the $SU(2)^6$ model by re-interpreting (3.12) as a Z_2 transformation of the 2d conformal fields of the latter model. Since the $SU(2)^6$ model involves 18 *real* left-moving fermions ψ^i, λ^i, μ^i all on the same footing, it is convenient to write the Z_2 action directly on these fermions. Clearly ω_L just exchanges $\psi^1 \leftrightarrow \psi^2$, $\psi^3 \leftrightarrow \psi^4$, $\lambda^1 \leftrightarrow \lambda^2$, $\lambda^3 \leftrightarrow \lambda^4$, $\mu^1 \leftrightarrow \mu^2$, $\mu^3 \leftrightarrow \mu^4$; whereas v_L takes $\psi^5 \rightarrow \psi^5$, $\lambda^5 \rightarrow -\lambda^5$, $\mu^5 \rightarrow -\mu^5$.

We would like to show that this model is equivalent to the $SU(4) \otimes SU(2)$ model of [7,8]. The general strategy is to alternately bosonize and fermionize the model defined by (3.12), until it is rewritten in terms of fermions which are totally untwisted, *i.e.* have the same set of boundary conditions (spin structure) as the original $SU(2)^6$ model. Then one writes the world-sheet supersymmetry generator $T_F(z)$ in terms of the new, untwisted fermions, and one discovers that $T_F^{\rm int}$ takes the form (3.9) with group $G = SU(4) \otimes SU(2)$ — the form in which the model has been described in refs. [7,8].

The first step is to diagonalize the Z_2 action of P on the fermions by defining

$$\begin{aligned} \tilde\psi^1 &= \tfrac{1}{\sqrt{2}}(\psi^1 - \psi^2), \qquad & \tilde\psi^2 &= \tfrac{1}{\sqrt{2}}(\psi^1 + \psi^2), \\ \tilde\psi^3 &= \tfrac{1}{\sqrt{2}}(\psi^3 - \psi^4), \qquad & \tilde\psi^4 &= \tfrac{1}{\sqrt{2}}(\psi^3 + \psi^4), \end{aligned} \tag{3.13}$$

plus the same definitions with ψ replaced by λ and by μ. Thus the 8 fermions $\lambda^5, \mu^5, \tilde\psi^1, \tilde\psi^3, \tilde\lambda^1, \tilde\lambda^3, \tilde\mu^1, \tilde\mu^3$ are odd under P, whereas the remaining fermions are even. The next step is to re-bosonize the new set of 20 real fermions (*i.e.* including also the two fields ψ^μ) such that the action of P on the 10 bosons is a pure translation. That is, we let

$$
\begin{aligned}
\tfrac{i}{\sqrt{2}}\left(\lambda^5 \pm i\mu^5\right) &= e^{\pm iH^1},\\
\tfrac{i}{\sqrt{2}}\left(\tilde{\psi}^1 \pm i\tilde{\psi}^3\right) &= e^{\pm iH^2},\\
\tfrac{i}{\sqrt{2}}\left(\tilde{\lambda}^1 \pm i\tilde{\lambda}^3\right) &= e^{\pm iH^3},\\
\tfrac{i}{\sqrt{2}}\left(\tilde{\mu}^1 \pm i\tilde{\mu}^3\right) &= e^{\pm iH^4},\\
\tfrac{i}{\sqrt{2}}\left(\tilde{\psi}^2 \pm i\tilde{\psi}^4\right) &= e^{\pm iH^5},\\
\tfrac{i}{\sqrt{2}}\left(\tilde{\lambda}^2 \pm i\tilde{\lambda}^4\right) &= e^{\pm iH^6},\\
\tfrac{i}{\sqrt{2}}\left(\tilde{\mu}^2 \pm i\tilde{\mu}^4\right) &= e^{\pm iH^7},\\
\tfrac{i}{\sqrt{2}}\left(\psi^5 \pm i\psi^6\right) &= e^{\pm iH^8},
\end{aligned}
\tag{3.14}
$$

the remaining 2 bosons being the original coordinate $X^6 = H^9$ $(\partial X^6_L = \lambda^6\mu^6)$, plus one boson (H^{10}) made by combining the two transverse 4d fermions ψ^μ. Using the bosonization dictionary (3.7), one sees that the action of P on the new bosons is to shift by $\frac{1}{2}$ in the $H^{1,2,3,4}$ directions, and to leave the other 6 left-moving bosons alone. We also need to describe the set of zero-mode eigenvalues for the new bosons *before* P acts on the system, and here we must include the right-moving zero-modes as well. But the diagonalization (3.13) does not alter the coherent sum over spin structures for the 20 left-moving and 12 right-moving fermions, so clearly the zero-mode eigenvalues before shifting are given by the same set as for the $SU(2)^6$ model, namely eq. (3.11).

What effect does a shift of the form $\left((\frac{1}{2})^4, 0^6; 0^6\right)$ have on this system? The answer is that it gives rise to a new set of zero-mode eigenvalues which, like the $SU(2)^6$ set, would form an odd self-dual Lorentzian lattice of signature $(11,7)$ had we used the covariant formulation. But all such lattices are unique up to automorphism,[32] and in our case the automorphism turns out to be a matrix acting just on the first four of the ten left-moving coordinates H^i (the shifted directions): $M^{ij} = M^{ji} = (M^{ij})^{-1} = \delta^{ij} - \frac{1}{2}$. In other words, we define four new bosons

$$\tilde{H}^i(z) = M^{ij} H^j(z) \qquad i,j = 1,\ldots,4, \tag{3.15}$$

to replace the four shifted bosons $H^i(z)$, $i = 1,\ldots,4$; then the set of zero-mode eigenvalues for the newest set of bosons — $\tilde{H}^i$, $i = 1,\ldots,4$, H^i, $i = 5,\ldots,10$, plus the same six right-moving bosons as before — is just the $SU(2)^6$ model's set (3.11).

Finally we re-fermionize the bosons. The four new bosons give us 8 new real fermions

$$\eta^i \;=\; \tfrac{-i}{\sqrt{2}}(e^{i\tilde{H}^i} + e^{-i\tilde{H}^i}), \quad \xi^i \;=\; \tfrac{-1}{\sqrt{2}}(e^{i\tilde{H}^i} - e^{-i\tilde{H}^i}), \quad i = 1,\ldots,4, \tag{3.16}$$

while from the old bosons we recover the 12 left-moving fermions $\tilde{\psi}^2$, $\tilde{\psi}^4$, ψ^5, ψ^6, $\tilde{\lambda}^2$, $\tilde{\lambda}^4$, λ^6, $\tilde{\mu}^2$, $\tilde{\mu}^4$, μ^6, and ψ^μ ($\mu = 1,2$) which were left invariant by P, as well as all 12 right-moving fermions $\bar{\lambda}^i$ and $\bar{\mu}^i$. Again we have a set of $20 + 12$ real fermions that are free and untwisted, and just as in the original $SU(2)^6$ model we have a coherent sum over their spin structures. The *only* difference between the two models is that after the repeated bosonization and re-fermionization the world-sheet supersymmetry generator $T_F(z)$ no longer has the same form in terms of the new set of fermions as it had in terms of the old ($SU(2)^6$) set of fermions. Clearly the change in T_F is restricted to the piece $T_F^{(15)}$ that is composed of the 15 twisted fermions ψ^i, λ^i, μ^i, $i = 1,\ldots,5$, in terms of which $T_F^{(15)}$ takes the form (3.9) with group $G = SU(2)^5$. We will now see that in terms of the 15 untwisted fermions $\tilde{\psi}^2$, $\tilde{\psi}^4$, ψ^5, $\tilde{\lambda}^2$, $\tilde{\lambda}^4$, $\tilde{\mu}^2$, $\tilde{\mu}^4$, η^i and ξ^i $T_F^{(15)}$ is given by (3.9) with $G = SU(4)$ instead.

First we rewrite $T_F^{(15)}$ in terms of the intermediate set of fermions introduced in eq. (3.13):

$$\begin{aligned} T_F^{(15)} \;=\; -\tfrac{1}{2}\Big\{ & \tfrac{1}{\sqrt{2}}\left[\tilde{\psi}^2(\tilde{\lambda}^1\tilde{\mu}^1 + \tilde{\lambda}^2\tilde{\mu}^2) \;+\; \tilde{\psi}^1(\tilde{\lambda}^1\tilde{\mu}^2 + \tilde{\lambda}^2\tilde{\mu}^1)\right] \\ & + \tfrac{1}{\sqrt{2}}\left[\tilde{\psi}^4(\tilde{\lambda}^3\tilde{\mu}^3 + \tilde{\lambda}^4\tilde{\mu}^4) \;+\; \tilde{\psi}^3(\tilde{\lambda}^3\tilde{\mu}^4 + \tilde{\lambda}^4\tilde{\mu}^3)\right] \;+\; \psi^5\lambda^5\mu^5 \Big\}. \end{aligned} \tag{3.17}$$

Then we use the two bosonizations (3.14), (3.16) and the relation (3.15) between H^i and $\tilde{H}^i$ to rewrite bilinears of the fermions in (3.17) which are odd under P as bilinears in η^i and ξ^i:

$$\begin{aligned}
\tilde{\lambda}^{\binom{1}{3}}\tilde{\mu}^{\binom{1}{3}} &= \tfrac{1}{2}(\pm\eta^1\eta^2 \mp \xi^1\xi^2 + \eta^3\eta^4 + \xi^3\xi^4)\ , \\
\tilde{\psi}^{\binom{1}{3}}\tilde{\lambda}^{\binom{1}{3}} &= \tfrac{1}{2}(\pm\eta^1\eta^4 \mp \xi^1\xi^4 + \eta^2\eta^3 + \xi^2\xi^3)\ , \\
\tilde{\psi}^{\binom{1}{3}}\tilde{\mu}^{\binom{1}{3}} &= \tfrac{1}{2}(\pm\eta^1\eta^3 \mp \xi^1\xi^3 + \eta^2\eta^4 + \xi^2\xi^4)\ , \\
\lambda^5\mu^5 &= \tfrac{1}{2}(\eta^1\xi^1 - \eta^2\xi^2 - \eta^3\xi^3 - \eta^4\xi^4)\ .
\end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.17), one writes $T_F^{(15)}$ as a trilinear product of the untwisted fermions, and it is easy to check that it indeed has the form (3.9) for the group $SU(4)$. That the four-dimensional gauge group is $SU(4)\otimes SU(2)$ then follows from the SKM algebra construction [15,28] mentioned above, plus the fact that all 18 supercurrents correspond to physical states.

The remaining $N=4$ model of refs. [7,8] — the one with gauge group $SO(5)\otimes SU(3)$ — can also be constructed as an orbifold. The construction — and the proof of equivalence — will closely parallel those just presented for the $SU(4)\otimes SU(2)$ case. Once again we start from the $SU(2)^6$ model, but now we twist it by $P = P_L = Z_3$ generated by

$$\omega_L = \mathrm{diag}(C_3,1,1,1)\,, \quad v_L = \left((\tfrac{1}{3})^6\right), \quad \omega_R = 1, \quad v_R = 0\,.$$

As usual, ω_L and v_L are written in an orthonormal basis for $\Gamma^{6,6}(D_6)$, and C_3 cyclically permutes the first three bosonic coordinates. Our first step is to write the action of P on the fermions ψ^i, χ^i and ϕ^i of the $SU(2)^6$ model (we have to use complex fermions here). The rotation ω_L permutes the fermions: $\psi^1 \to \psi^2 \to \psi^3 \to \psi^1$, and similarly for χ^i and ϕ^i, $i=1,2,3$, and the shift v_L takes $\psi^i \to \psi^i$, $\chi^i \to \omega\chi^i$, $\phi^i \to \bar{\omega}\phi^i$, $i=1,\ldots,6$, where $\omega \equiv e^{2\pi i/3}$. Therefore the linear combinations

$$\tilde{\psi}^i = \frac{1}{\sqrt{3}}\sum_{j=1}^{3}\omega^{i\cdot j}\psi^j\ , \quad \tilde{\chi}^i = \frac{1}{\sqrt{3}}\sum_{j=1}^{3}\omega^{(i+1)\cdot j}\chi^j\ , \quad \tilde{\phi}^i = \frac{1}{\sqrt{3}}\sum_{j=1}^{3}\omega^{(i-1)\cdot j}\phi^j \quad (3.19)$$

for $i = 1,2,3$, together with $\psi^{4,5,6}$, $\chi^{4,5,6}$ and $\phi^{4,5,6}$, diagonalize the action of P: $\tilde{\psi}^2$, $\tilde{\chi}^2$, $\tilde{\phi}^2$ and $\chi^{4,5,6}$ acquire the phase ω under P, $\tilde{\psi}^1$, $\tilde{\chi}^1$, $\tilde{\phi}^1$ and $\phi^{4,5,6}$ acquire $\bar{\omega}$, and $\tilde{\psi}^3$, $\tilde{\chi}^3$, $\tilde{\phi}^3$ and $\psi^{4,5,6}$ are left invariant.

Next we bosonize:

$$\begin{array}{llll} \tilde{\psi}^2 = -i\,e^{iH^1}, & \tilde{\psi}^1 = -i\,e^{-iH^1}, & \chi^4 = -i\,e^{iH^4}, & \phi^4 = -i\,e^{-iH^4}, \\ \tilde{\chi}^2 = -i\,e^{iH^2}, & \tilde{\chi}^1 = -i\,e^{-iH^2}, & \chi^5 = -i\,e^{iH^5}, & \phi^5 = -i\,e^{-iH^5}, \\ \tilde{\phi}^2 = -i\,e^{iH^3}, & \tilde{\phi}^1 = -i\,e^{-iH^3}, & \chi^6 = -i\,e^{iH^6}, & \phi^6 = -i\,e^{-iH^6}, \end{array} \quad (3.20)$$

with the remaining, P-invariant, fermions combining into $H^7,\ldots,H^{10}$. With this choice of bosonization P acts like a shift on the H^i; in particular, $H^1,\ldots,H^6$ are shifted by $\frac{1}{3}$, whereas $H^7,\ldots,H^{10}$ are left invariant. Just as in the $SU(4)\otimes SU(2)$ case, one finds that the shift $\left((\frac{1}{3})^6, 0^4; 0^6\right)$ gives rise to a set of zero-mode eigenvalues for the H^i which becomes identical to the original $SU(2)^6$ set (3.11) after one performs an appropriate lattice automorphism. In this case the matrix for the automorphism affects only the first six H^i: $M^{ij} = M^{ji} = (M^{ij})^{-1} = \delta^{ij} - \frac{1}{3}$. So, we redefine the six shifted bosons, $H^i(z) = M^{ij}\tilde{H}^j(z)$ $(i = 1,\ldots,6)$, and then re-fermionize them:

$$\eta^i \ =\ \frac{-i}{\sqrt{2}}(e^{i\tilde{H}^i} + e^{-i\tilde{H}^i}), \quad \xi^i \ =\ \frac{-1}{\sqrt{2}}(e^{i\tilde{H}^i} - e^{-i\tilde{H}^i}), \quad i = 1,\ldots,6. \qquad (3.21)$$

(Re-fermionization also recovers the P-invariant fermions $\tilde{\psi}^3$, $\tilde{\chi}^2$, $\tilde{\phi}^1$, $\psi^{4,5,6}$ and ψ^μ as well as the right-moving fermions.) Once again the model is defined as a coherent sum over spin structures for all of these $20+12$ fermions, and only the form of the supersymmetry generator $T_F^{\rm int}$ has changed.

In order to reconstruct the new $T_F^{\rm int}$ we first write it in terms of the fermions that diagonalize P (eq. (3.19)):

$$T_F^{\rm int} = -\tfrac{1}{2}\Big\{\tfrac{1}{\sqrt{3}}\big[\tilde\psi^3(\tilde\chi^3\tilde\phi^3 + \tilde\chi^1\tilde\phi^2 + \tilde\chi^2\tilde\phi^1) + \tilde\psi^1(\tilde\chi^1\tilde\phi^1 + \tilde\chi^2\tilde\phi^3 + \tilde\chi^3\tilde\phi^2) + \tilde\psi^2(\tilde\chi^2\tilde\phi^2 + \tilde\chi^1\tilde\phi^3 + \tilde\chi^3\tilde\phi^1)\big] + \sum_{i=4}^{6}\psi^i\chi^i\phi^i\Big\}\,. \tag{3.22}$$

Then we rewrite $T_F^{\rm int}$ in terms of the final set of fermions $\tilde\psi^3$, $\tilde\chi^3$, $\tilde\phi^3$, $\psi^{4,5,6}$, η^i and ξ^i, using the two bosonizations (3.20), (3.21) and the relation (3.15) between H^i and $\tilde H^i$ to rewrite not only the fermion bilinears

$$\begin{aligned}
&\tilde\psi^1\tilde\chi^2 + \tilde\psi^2\tilde\chi^1 = \eta^1\eta^2 + \xi^1\xi^2,\\
&\tilde\chi^1\tilde\phi^2 + \tilde\chi^2\tilde\phi^1 = \eta^2\eta^3 + \xi^2\xi^3,\\
&\tilde\phi^1\tilde\psi^2 + \tilde\phi^2\tilde\psi^1 = \eta^3\eta^1 + \xi^3\xi^1\\
\text{and}\quad &\chi^i\psi^i = \eta^i\xi^i - \tfrac{1}{3}\sum_{j=1}^{6}\eta^j\xi^j\;, \quad i=4,5,6 \quad (\text{no sum over } i),
\end{aligned} \tag{3.23}$$

but also one trilinear pair

$$\tilde\psi^1\tilde\chi^1\tilde\phi^1 + \tilde\psi^2\tilde\chi^2\tilde\phi^2 = \tfrac{1}{\sqrt{2}}(\eta^4\eta^5\eta^6 - \eta^4\xi^5\xi^6 - \xi^4\eta^5\xi^6 - \xi^4\xi^5\eta^6). \tag{3.24}$$

Substituting (3.23) and (3.24) into (3.22) then gives $T_F^{\rm int}$ in the desired form (3.9) with $G = SO(5)\otimes SU(3)$. The 10 $SO(5)$ fermions are $\eta^{1,2,3}$, $\xi^{1,2,3}$, $\tilde\psi^3$, $\tilde\chi^3$, $\tilde\phi^3$ and $\frac{1}{\sqrt{3}}(\psi^4+\psi^5+\psi^6)$; the 8 $SU(3)$ fermions are $\eta^{4,5,6}$, $\xi^{4,5,6}$ and the remaining two linear combinations of $\psi^{4,5,6}$.

3.4. OTHER $N = 4$ MODELS

Let us now describe a new $N = 4$ orbifold model; its gauge group is $SU(3)^2$, which is not contained in the gauge groups of the previous models. Our starting point is the $N = 8$ model that has zero modes of X^i taking values on the lattice

$$\Gamma^{6,6}(E_6) \;\equiv\; (\mathbf{1};\mathbf{1}) + (\mathbf{27};\mathbf{27}) + (\overline{\mathbf{27}};\overline{\mathbf{27}}) \tag{3.25}$$

where $\mathbf{1}$, $\mathbf{27}$ and $\overline{\mathbf{27}}$ denote the three conjugacy classes of E_6 weights. Class $\mathbf{1}$ is the root lattice of E_6; it consists of weights of E_6 representations of zero triality. Weights of representations with triality ± 1 make classes $\mathbf{27}$ and $\overline{\mathbf{27}}$. The easiest way to visualize the weight lattice of E_6 is to describe it in terms of weights of the maximal subgroup $A_2 \otimes A_2 \otimes A_2$:

$$\begin{aligned}
\mathbf{1} &= (\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{3},\mathbf{3},\mathbf{3}) + (\overline{\mathbf{3}},\overline{\mathbf{3}},\overline{\mathbf{3}})\,,\\
\mathbf{3} &= (\mathbf{1},\mathbf{3},\overline{\mathbf{3}}) + (\overline{\mathbf{3}},\mathbf{1},\mathbf{3}) + (\mathbf{3},\overline{\mathbf{3}},\mathbf{1})\,,\\
\overline{\mathbf{3}} &= (\mathbf{1},\overline{\mathbf{3}},\mathbf{3}) + (\mathbf{3},\mathbf{1},\overline{\mathbf{3}}) + (\overline{\mathbf{3}},\mathbf{3},\mathbf{1})\,.
\end{aligned}$$

Each of the A_2 lattices* is a two-dimensional hexagonal lattice; the three 2-planes spanned by these sublattices are orthogonal to each other.

Unlike $\Gamma^{6,6}(D_6)$, the lattice $\Gamma^{6,6}(E_6)$ has no simple interpretation in terms of free fermions, so we are forced to describe this $N = 8$ model and its orbifolds in bosonic language. (The formalism for describing asymmetric orbifolds bosonically has been worked out in [9]; we will draw on the results of that paper when necessary.) The $N = 4$ model with gauge group $SU(3)^2$ is obtained by twisting the above $N = 8$ model by $P = P_L = Z_3$ which is generated by $(\omega_L, v_L; 1, 0)$, where ω_L rotates the plane of the first A_2 through an angle of $4\pi/3$, and v_L is a weight vector for the triplet representation of the second A_2. Note that ω_L^3 is a 4π rotation, and hence is trivial on space-time fermions as well as bosons, and $3v_L \in \Gamma^{6,6}(E_6)$, so that $(\omega_L, v_L; 1, 0)$ indeed has order 3.

* We use A_2 to refer to certain sublattices of $\Gamma^{6,6}(E_6)$ to avoid confusion with the $SU(3)$ gauge group factors.

Let us follow our bosonic description of the $SU(2)^6$ model and focus on the massless spectrum, beginning with the untwisted sector. Since the $SO(6)$ rotation ω_L does not belong to $SU(3)$, it breaks all four left-moving supersymmetries and in fact leaves no massless (R,NS) states. The surviving (NS,NS) and (NS,R) states again form an $N = 4$ supergravity multiplet, and also a $U(1)^4$ super-Yang-Mills multiplet. The rank is reduced from 6 to 4 because only four of the six internal directions — those tangent to the second and third A_2 root lattice planes — are left invariant by ω_L. The generators of $U(1)^4$ form the Cartan subalgebra of the $SU(3)^2$ gauge group; gauge bosons corresponding to root vectors come from the twisted sectors.

Following the formalism of ref. [9], we begin the description of the twisted sectors by defining I — the sublattice of $\Gamma^{6,6}(E_6)$ which is left invariant by ω_L — and its dual lattice I^*. Using $(A_2)^3$ notations for the left-moving bosonic zero modes and E_6 for the right-moving ones, we can write I as

$$I \;=\; (\vec{0},\mathbf{1},\mathbf{1};\mathbf{1}) \,+\, (\vec{0},\mathbf{3},\overline{\mathbf{3}};\mathbf{27}) \,+\, (\vec{0},\overline{\mathbf{3}},\mathbf{3};\overline{\mathbf{27}})\,,$$

The dual lattice I^* is obtained by adjoining the vector f which is a weight vector for the 3 representation on both the second and third A_2 lattices, and has zero components in the remaining directions. Hence the index of I in I^* is $|I^*/I| = 3$. This index is important because it affects the overall degeneracy factor D for states in the twisted sector, which is the generalization to an asymmetric orbifold of the number of fixed points (tori) of a symmetric twist (ω, v). The general formula is [9]

$$D = \sqrt{\frac{\det'(1-\omega)}{|I^*/I|}}\,, \tag{3.26}$$

where $\det'(1-\omega)$ omits the unit eigenvalues of $\omega = (\omega_L, \omega_R)$ and hence is equal to 3 in the present case, so that $D = 1$ here.

The formula (3.26) may be obtained by making the modular transformation $\tau \to -1/\tau$ on the trace of $(\omega_L, v_L; \omega_R, v_R)$ in the untwisted sector. The same transformation also shows that in the twisted sector the bosonic zero-mode eigenvalues (p_L, p_R) lie on the shifted lattice $I^* + v$. In $N = 4$ models massless states are constrained to have $p_R = 0$ while the value of $\frac{1}{2}p_L^2$ is fixed by the requirement that the left-moving energy of the state also vanishes. In the case at hand we find that massless states in the singly-twisted sector have

$$p_L \in (\vec{0}, \mathbf{3}, \mathbf{1}) + (\vec{0}, \mathbf{1}, \bar{\mathbf{3}}) + (\vec{0}, \bar{\mathbf{3}}, \mathbf{3})$$

and $p_L^2 = \frac{2}{3}$. There are precisely six such vectors p_L: $(\vec{0}, 3, \vec{0}) + (\vec{0}, \vec{0}, \bar{3})$. (Here 3 ($\bar{3}$) refers to the (anti)-triplet weights only rather than to the entire class of weights for the appropriate A_2 sublattice.) The negatives of these 6 vectors lie in $I^* + 2v_L$ and therefore give rise to the 6 massless states in the doubly-twisted sector; together the 12 states make up the roots of $SU(3)^2$.

To verify that the 4d gauge group really is $SU(3)^2$ we should construct the supercurrents for the 16 states and compare their operator product expansions with (2.8). The supercurrents for the 4 generators of the Cartan algebra are simply $i\,D\mathbf{X}^i$, $i = 3,4,5,6$ (coordinates of the second and third A_2 planes, which are left invariant by ω_L). The supercurrents for the 12 root vectors are the products of a dimension $\frac{1}{6}$ *twist superfield*,* which twists the supercoordinates $D\mathbf{X}^{1,2}$, and dimension $\frac{1}{3}$ soliton operators $e^{ip_L \cdot \mathbf{X}_L}$, where p_L is a triplet or an anti-triplet weight vector of the second or third A_2. Without going into the details of twist superfield operator products, it is easy to see that the supercurrent operator products have the right form. Clearly the supercurrents form two mutually anticommuting sets of eight, according to whether they involve the second or third A_2, because the soliton operator products vanish otherwise. Look at one of these two sets. The charges of the 6 twisted states under the Cartan currents ∂X^i ($i = 3,4$, say) are proportional to p_L^i. But these vectors (the $3 + \bar{3}$ on the A_2 weight lattice) form a regular hexagon about the origin,

* The dimension of the twist field equals the difference between vacuum energies of the twisted and the untwisted sectors. See ref. [33] for the theory of twist fields and superfields.

just as the roots of $SU(3)$ do, so the Cartan–root operator products are correct up to overall normalization. (The unusual normalization arises because we are constructing an $SU(3)$ Kac-Moody algebra with $k = 3$ rather than $k = 1$.) Finally the root–root operator products have the correct general form just as a consequence of 'momentum' conservation for the soliton operators; the twist superfields provide the appropriate fractional powers of $z - w$ to make the right singularity. In fact the representation of the $k = 3$ non-supersymmetric $SU(3)$ Kac-Moody algebra which is provided by the above $SU(3)$ supercurrents is identical to the parafermionic construction of ref. [34], which has also appeared in the mathematical literature in connection with Z algebras.[35] The soliton operators appearing in the construction for central charge k are $e^{i\alpha\cdot X/\sqrt{k}}$, with α a root; *i.e.*, the length^2's of the root vectors which appear are effectively rescaled by a factor of $\frac{1}{k}$ from the usual normalization (length^2 $= 2$ for long roots). In the $SU(3)$ case at hand, the parafermions are particular combinations of twist fields and the fermions $\psi^{3,4,5,6}$.

Another $N = 4$ orbifold model — with 4d gauge group $SO(5) \otimes SU(2) \otimes SU(2)$ — can be constructed by a different twisting of the same $N = 8$ $\Gamma^{6,6}(E_6)$ model as above. This time we have $P = P_L = Z_6$, generated by an ω_L which rotates the first A_2 plane by $2\pi/3$ rather than $4\pi/3$, so that it has order 6 acting on spacetime fermions. It is accompanied by a shift v_L which is half of a weight vector for the triplet (3) representation of the second A_2 and half of a $\bar{3}$ vector for the third A_2. Without going into the details of the construction, we will simply say which generators of the gauge group come from which sector. As usual, Cartan generators reside in the untwisted sector. Long roots of $SO(5)$ — which have the appropriate length^2 $= \frac{2}{3}$ for $k = 3$, *i.e.* $\hat{k} = 0$ — come from the doubly-twisted sector and its anti-sector. Two of the short roots of $SO(5)$ (length^2 $= \frac{1}{3}$) come from the singly-twisted sector and its anti-sector, while the other two come from the triply-twisted sector. The latter sector also contains all four roots of the two $SU(2)$ factors; these roots have length^2 $= 1$, which corresponds to $k = 2$, *i.e.*also $\hat{k} = 0$.

Both of these E_6-based $N = 4$ models have gauge groups of dimension 16 rather than the 18 of the first three models. Nevertheless, all simple SKM factors involved are minimal, *i.e.* have $\hat{k} = 0$ and thus can be fully described in terms of 16 free fermions (*cf.* the next section). Hence the super-stress tensor associated with the SKM algebras has central charge $\hat{c}^{\mathrm{SKM}} = 16/3$ in each case (*cf.* eq. (2.12)), and both models have to incorporate a left-over super Virasoro algebra with $\hat{c}^L = \frac{2}{3}$ — one in the discrete series of unitary algebras with $\hat{c} < 1$. In addition to $\hat{c}^L$, we can also determine which eigenvalues h^L of L_0^L occur in the spectra of the two models by using the physical state condition $L_0 = \overline{L}_0$, which implies that $h^L + h^{\mathrm{SKM}} = \bar{h}^{\mathrm{int}}$ (modulo $\frac{1}{2}$), or (modulo 1) in the (R,R) sector. Comparing the spectrum of the appropriate L_0^{SKM} with the spectrum of $\overline{L}_0$ — the latter is the same for both models and also for the $N = 8$ $\Gamma^{6,6}(E_6)$ model — yields the same result for both models: NS states have $h^L = 0$ or $\frac{1}{6}$ (modulo $\frac{1}{2}$) while Ramond states have $h^L = \frac{1}{24}$ or $\frac{3}{8}$ (modulo 1). Note that while all these values of h^L are allowed for unitary representations of the $\hat{c} = \frac{2}{3}$ algebra,[20] there are other allowed values of h^L, namely $h^L = \frac{1}{16}$ (modulo $\frac{1}{2}$), that do not occur here. However, the values of h^L that we found are the only ones allowed if the left-over algebra is extended to an (untwisted) $N = 2$ super Virasoro algebra.[36] And in fact there is a good reason for the existence of a global $N = 2$ supersymmetry on the world sheet: the $U(1)$ current $\sum_{i=1}^{3} \psi^{\bar{i}}\psi^{i}$ that generates it is preserved by any abelian twist.

The fact that both E_6 orbifolds yield the same left-over algebra may appear surprising, but it can easily be explained by comparing the respective twist groups $P = P_L$: The square of the Z_6 generator we have used to make the $SO(5) \otimes SU(2)^2$ model is precisely the Z_3 twist that generates the $SU(3)^2$ model. This means that we should be able to start directly from the $SU(3)^2$ model, twist it by a Z_2 group, and get the $SO(5) \otimes SU(2)^2$ gauge group. If the Z_2 element twists only the fermions that make up the SKM algebra, then the left-over algebra will indeed be the same. The required twist does exist: it acts on supercurrents of both $SU(3)$'s as the adjoint representation of the matrix diag$(-1, -1, +1)$.

Eight out of the sixteen currents are left invariant by this Z_2 twist and form the untwisted sector of the new model; the group they generate is $SU(2)^2 \otimes U(1)^2$. The twisted sector also contains eight states; their quantum numbers with respect to the untwisted group are $(\mathbf{2},\mathbf{2},0,0) + (\mathbf{1},\mathbf{1},\pm 1,\pm 1) + (\mathbf{1},\mathbf{1},\pm 1,\mp 1)$. It is easy to see that the first four twisted states enlarge the $SU(2)^2 \equiv SO(4)$ group to $SO(5)$ while the last four are the non-Cartan generators of the $SU(2)^2$ algebra.

Unlike the $SU(3)^2$ case, an $SO(5) \otimes SU(2)^2$ SKM algebra with $\hat{c} \leq 6$ does not have to be minimal: One can have $\hat{k} = 1$ for one of the $SU(2)$ factors, while keeping $\hat{k} = 0$ for the other one and for $SO(5)$. This combination would have $\hat{c}^{\mathrm{SKM}} = 6$ and thus would not need a left-over algebra. Such a model can in fact be constructed as an $N = 4$ orbifold. Again without going into the details, we can outline the construction as follows: We start with the $N = 8$ model whose lattice is $\Gamma^{6,6}(D_5 + A_1)$. We twist it by $P = P_L = Z_6$ generated by a $2\pi/3$ rotation ω_L that permutes the first three coordinates of the D_5 lattice, accompanied by a shift v_L which is one third of a vector weight for D_5 (along the fourth or fifth D_5 coordinate) plus one third of a root for A_1. It is a straightforward procedure to construct the properly normalized root diagram for the non-Cartan generators of the gauge group; we find that its geometry indeed fits $SO(5) \otimes SU(2)^2$. Moreover, the root lengths squared are $\frac{2}{3}$ and $\frac{1}{3}$ for the $SO(5)$ roots, 1 for the roots of one of the $SU(2)$'s and $\frac{2}{3}$ for the other; this translates into $k = (3,2,3)$ or $\hat{k} = (0,0,1)$ for the respective gauge group factors.

For our final example of a $P = P_L$ orbifold we present an $N = 4$ model with gauge group G_2. This time both bosonic and fermionic techniques will be useful. As usual, our starting point is an $N = 8$ model; its lattice is $\Gamma^{6,6}(A_2 + A_2 + D_2)$. First, we fermionize the two dimensions corresponding of the D_2 sublattice and make a Z_2 twist that unifies the spin structure of the resulting fermions with the spin structure of the ψ^i (*cf.* the $SU(2)^6$ model). This twist reduces the supersymmetry to $N = 4$ and enlarges the gauge group to $SU(2)^2 \otimes U(1)^4$; both $SU(2)$ factors are minimal. Second, we make a Z_3 twist that leaves the $SU(2)$'s alone, rotates the first A_2 sublattice by $4\pi/3$ and shifts the second A_2 by a weight vector in the **3** class. This time we use the bosonic arguments we've developed for the $SU(3)^2$ model and find that instead of the A_2^2 lattice we now have a minimal $SU(3)$ SKM algebra, plus a left-over algebra with $\hat{c} = \frac{4}{3}$ (presumably a sum of two unitary super Virasoro algebras with $\hat{c} = \frac{2}{3}$). So far we have built a model with gauge group $SU(3) \otimes SU(2)^2$, which is interesting by itself since it contains the standard model gauge group.

We have promised a G_2 model, however, so another twist is due. The $SU(3)$ Lie algebra has an outer Z_2 automorphism — complex conjugation in an anti-hermitian basis — that leaves an $SO(3)$ subalgebra invariant and changes the signs of the other five generators. The $SU(2)^2$ algebra also has an outer Z_2 automorphism — exchange of the two $SU(2)$'s. Combining these two automorphisms together, we leave six of the 14 generators of $SU(3) \otimes SU(2)^2$ invariant and change the signs of the other eight. Now we can use the fact that we have a minimal SKM algebra which can be described in terms of free fermions, since we already know how to treat a Z_2 twist which changes the signs of eight real fermions and leaves everything else invariant. The untwisted sector gives rise to an $SU(2) \otimes SU(2)$ SKM algebra, which is however highly non-minimal: the first $SU(2)$ (subalgebra of the $SU(3)$) has $k = 4 \cdot 3 = 12$ or $\hat{k} = 10$ while the second one has $k = 2 \cdot 2 = 4$ or $\hat{k} = 2$. The twisted sector yields another eight generators which transform with respect to the untwisted subalgebra as $(\mathbf{4}, \mathbf{2})$; this completes the adjoint representation of G_2. If we now compute the root lengths of G_2, we find that the short roots have length$^2 = \frac{2}{12}$ while the long roots have length$^2 = \frac{2}{4}$; this corresponds to $k = 4$ or $\hat{k} = 0$, which agrees with the requirement that $\hat{c}(G_2) = \frac{14}{3}$. (The remaining $\hat{c}^L = \frac{4}{3}$ is supplied by the same left-over algebra as in the $SU(3) \otimes SU(2)^2$ model — we have not twisted the left-over piece).

Our main reason for constructing a G_2 model was to show that all allowed 4d gauge groups that we have explicitly listed in lines (A)–(F) of (2.14) do appear in $N = 4$ orbifold models. Using the same techniques, but starting from different lattices $\Gamma^{6,6}(\mathcal{G})$ and twisting by other cyclic groups $P = P_L = Z_\ell$, we have constructed $N = 4$ models with all the other allowed gauge groups, except those having odd rank. The latter failure is due to the fact that abelian twists always preserve a Cartan subalgebra of an even dimension; presumably non-abelian twist groups $P = P_L$ can be used to make at least some of the allowed odd-rank gauge groups.

3.5. Chiral $N = 1$ Models

By now we have constructed several models with gauge groups large enough to contain the standard model, but they were non-chiral because of $N = 4$ supersymmetry. Our next question is whether there are chiral models in the type II superstring framework. The answer to this question is 'yes': we will now give an example of a chiral 4d model; this model has $(SU(2) \otimes U(1))^2$ gauge group and $N = 1$ space-time supersymmetry.

The starting point is the previous $N = 4$ $SO(5) \otimes SU(3)$ model, which is most conveniently described here in terms of untwisted free fermions, using the form (3.9) for the supersymmetry generator T_F^{SKM}. We twist this model by a group $P = Z_4$ which acts on both left- and right-movers, in order to correlate left-moving gauge quantum numbers with right-moving space-time helicities. In order to get a chiral theory, we need to break at least three of the four (right-moving) space-time supersymmetries. This is accomplished by the right-moving rotation

$$\omega_R = \text{diag}(C_2^-, C_2^-, -1, -1),$$

with $C_2^- = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and written in the usual orthonormal basis for the D_6 lattice. One can rewrite ω_R in terms of the linear combinations of the right-moving fermions $\bar{\psi}^i$, $\bar{\chi}^i$, $\bar{\phi}^i$ which diagonalize it, and then bosonize those fermions. One finds that ω_R is equivalent to the shift

$$\tilde{v}_R = (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0; \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{2}; 0)$$

acting on the new bosons. The semicolons separate the bosons arising from $\bar{\chi}^i$

and $\bar{\phi}^i$; $\bar{\psi}^i$; $\bar{\psi}^\mu$, respectively. The distinction between $\bar{\psi}$ and $\bar{\chi}, \bar{\phi}$ is important because the spin structures for $\bar{\psi}$ are summed over separately from the rest of the fermions.

The action of P on the left-moving fermions must be a group automorphism in order to preserve the world-sheet supersymmetry generator (3.9). In our case, we rotate the $SU(3)$ fermions by (the adjoint representation of) the matrix $\mathrm{diag}(i, i, -1)$, while the corresponding $SO(5)$ matrix is $\mathrm{diag}(C_2^-, C_2^-, 1)$. This twist leaves $G = SU(2) \otimes U(1) \otimes SU(2)' \otimes U(1)'$ as the unbroken 4d gauge group. If we bosonize the left-moving fermions in a particular way we can describe the action of P on them as a shift:

$$\tilde{v}_L = (\tfrac{1}{4}, \tfrac{1}{4}, 0, 0; \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, 0, 0; 0),$$

acting on bosons. Here the semicolons separate the bosons H^i; $\tilde{H}^i$; H^{10} arising from η^a; $\tilde{\eta}^a$; ψ^μ, respectively, where η^a ($\tilde{\eta}^a$) are the fermionic parts of the $SU(3)$ ($SO(5)$) supercurrents.

Since P acts as a pure shift on the 10 left-moving and 10 right-moving bosons we have defined, it is a straightforward exercise to list all the massless states, sector by sector, characterizing them by their 10+10 bosonic zero-mode eigenvalues. However, in order to compute the gauge charges of the states, we need to identify the generators $(I_3, Y; I_3', Y')$ of the Cartan subalgebra of G as linear combinations of the fields ∂H^i and $\partial \tilde{H}^i$. Defining I_3^i by $I_3(z) = \sum_{i=1}^4 I_3^i \partial H^i(z)$, *etc.*, and using the known charges of states in the adjoint representations, we find that

$$\begin{aligned} I_3^i &= (\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0)\,, & I_3^{i\prime} &= (\tfrac{1}{2}, -\tfrac{1}{2}, 0, 1, 0)\,, \\ Y^i &= (1, 1, 0, 0)\,, & Y^{i\prime} &= (1, 1, 2, 0, 0)\,. \end{aligned} \tag{3.27}$$

Gauge charges are given by inner products of these vectors with the vectors of H_0^i and $\tilde{H}_0^i$ eigenvalues for the massless states. Because of $N = 1$ supersymmetry, we need only list the spectrum of massless positive-helicity fermions other than gauginos; the list below gives their $SU(2) \otimes U(1) \otimes SU(2)' \otimes U(1)'$ charges, preceded by the number of times they appear in the spectrum.

Untwisted sector: $1(\mathbf{1},0;\mathbf{1},\pm 2)$, $2(\mathbf{2},1;\mathbf{1},0)$, $2(\mathbf{1},0;\mathbf{2},1)$.

Singly-twisted sector: $2(\mathbf{2},-\frac{1}{2};\mathbf{1},\frac{3}{2})$, $2(\mathbf{1},\frac{1}{2};\mathbf{2},\frac{1}{2})$, $2(\mathbf{2},-\frac{1}{2};\mathbf{1},-\frac{1}{2})$, $2(\mathbf{1},\frac{1}{2};\mathbf{2},-\frac{3}{2})$.

Doubly-twisted sector: $1(\mathbf{1},\pm 1;\mathbf{1},\pm 1)$, $1(\mathbf{2},0;\mathbf{2},0)$, $2(\mathbf{2},0;\mathbf{1},-1)$, $2(\mathbf{1},-1;\mathbf{2},0)$.

One can check that the spectrum is anomaly free.

A second chiral $N = 1$ model very similar to first one can be constructed from the $N = 4$ $SU(2)^6$ model by twisting by exactly the same group $P = Z_4$ as above. To be precise, P acts as the same shift in an appropriate bosonic picture; its action as an isomorphism of the SKM algebra is of course different. Specifically, P acts on the generators of four of the six $SU(2)$'s as $\exp(i\frac{\pi}{2}I_3)$, on the fifth $SU(2)$ as $\exp(i\pi I_3)$, and the sixth $SU(2)$ is left invariant by P. Thus the 4d gauge group of the second model is $SU(2) \otimes U(1)^5$, and all massless (non-gauge) fermions are $SU(2)$ singlets. Here we just list their $U(1)$ charges:

Untwisted sector: $1(\pm 1,0,0,0,0)$, $2(0,\underline{1,0,0,0})$.

Singly-twisted sector: $2(\pm\frac{1}{2},\underline{-\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}})$

Doubly-twisted sector: $1(0,\underline{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}})$, $1(0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $1(0,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$, $2(0,\underline{\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}})$.

An underline indicates that all distinct permutations of the charges appear. Again the spectrum is anomaly free. One can similarly twist the $N = 4$ $SU(4) \otimes SU(2)$ model by the same group P; however in this case the $N = 1$ model obtained turns out to be non-chiral.

The results of this section can be summarized as follows: Using asymmetric orbifolds, we have constructed several modular-invariant classical solutions of the type II superstring which lead to supersymmetric gauge theories in four dimensions. In particular, we have reproduced the three models of Bluhm *et al.*[7] The additional models of Kawai *et al.*[8] can be constructed as asymmetric orbifolds as well, using the three $N = 4$ models as intermediate steps.* We have also constructed other $N = 4$ models that are not equivalent to free fermions; one of these models has gauge group $SU(3)^2$ which contains the standard model group $SU(3) \otimes SU(2) \otimes U(1)$. Finally, we have given two *chiral* models, in which massless fermions transform as a non-self-conjugate representation of the gauge group.

4. Gauge Quantum Numbers of Massless Particles

In the previous section we have seen how to construct several models from the type II superstring which have realistic gauge groups, and also models with chiral fermions, but we have found no models with realistic group representations for the massless fermions. In this section we will see that this problem is a general feature of models based on the type II superstring, and is not due merely to our lack of imagination in constructing models via asymmetric orbifolds. In fact, **no classical vacuum of the type II superstring can be consistent with the particle spectrum of the standard model.** More specifically, we will show that in any four-dimensional model whose gauge group contains $SU(3) \otimes SU(2) \otimes U(1)$, massless triplets of $SU(3)$ (such as quarks) cannot coexist with massless $SU(2)$ doublets (such as left-handed leptons). To prove this general result we return to the algebraic approach used in section 2. In that section we classified the different super Kac-Moody (SKM) algebras which can be incorporated into a type II model in a way consistent with unitarity and the existence of massless chiral fermions. In this section we will see that unitarity restricts the possible *representations* of a given SKM algebra, and therefore also limits the possible gauge quantum numbers of massless particles, such that $SU(3)$ triplets are incompatible with $SU(2)$ doublets.

* While this paper was being typed we received [37], in which models similar to those of Kawai *et al.* are constructed using slightly different techniques.

Our starting point is the obvious remark that every state in the spectrum of a type II model belongs to some representation of the (left-moving) SKM algebra associated with the 4d gauge group. This just means that the supercurrent modes $\mathcal{J}_r^a$ and J_n^a act within the Hilbert space of states. The SKM representations of interest for string theory are those in which the spectrum of the 'left-moving energy' operator L_0 is bounded from below. Since the positive-frequency modes of $\mathcal{J}^a(z)$ and $J^a(z)$ when applied to a state lower its energy, by successively applying these operators to any state one must eventually arrive at a state $|\mathbf{r}\rangle$ which is annihilated by all of these operators:

$$\forall r, n > 0: \quad \mathcal{J}_r^a \,|\mathbf{r}\rangle = J_n^a \,|\mathbf{r}\rangle = 0\,; \tag{4.1}$$

$|\mathbf{r}\rangle$ is called a *highest weight* or *primary* state with respect to the SKM algebra. Also $|\mathbf{r}\rangle$ must represent the zero-mode subalgebra of the SKM algebra. In the NS sector this subalgebra is just the ordinary Lie algebra generated by J_0^a, so

$$J_0^a \,|\mathbf{r}\rangle \;\equiv\; T_{(\mathbf{r})}^a \,|\mathbf{r}\rangle \tag{4.2}$$

for some representation matrices $T_{(\mathbf{r})}^a$; eq. (4.2) defines the gauge group representation $\mathbf{r}$ under which the state $|\mathbf{r}\rangle$ transforms. In the R sector $|\mathbf{r}\rangle$ must also represent the Clifford algebra of fermionic zero-modes $\mathcal{J}_0^a$. We will discuss only NS sector SKM representations in this section, however — we know from section 2 that all R states are massive whenever the SKM algebra is non-abelian.

All states in a model can be constructed by applying the negative-frequency modes $\mathcal{J}_{-r}^a, J_{-n}^a$ $(r, n > 0)$ to the various highest weight states. The collection of states

$$\mathcal{J}_{-r_1}^{a_1} \cdots \mathcal{J}_{-r_k}^{a_k} J_{-n_1}^{a_1} \cdots J_{-n_l}^{a_l} \,|\mathbf{r}\rangle \tag{4.3}$$

for a particular highest weight state $|\mathbf{r}\rangle$ is called a *highest weight representation.* Thus each state belongs to some highest weight representation, or is a linear combination of states in different highest weight representations. Since the operators

J^a_{-r}, J^a_{-n} carry an adjoint index a, all states in a highest weight representation transform in the same way under the center of the ordinary Lie group. In particular, for the Lie group $SU(3)$ ($SU(2)$), states can only have nonzero triality (half-integer isospin) if some highest weight state does also.

In fact, as long as we are interested only in massless particles, we can ignore the non-primary states in (4.3), with the exception of gauge bosons and their superpartners. The reason is as follows: left-moving energies of states in (4.3) differ from each other by integers or half-integers. Thus if the superfield $\mathbf{r}(z,\theta)$ that makes a highest-weight state $|\mathbf{r}\rangle$ has positive conformal dimension $h_{\mathbf{r}}$ — which means that $|\mathbf{r}\rangle$ has energy greater than $\frac{-1}{2}$ — then all states in (4.3) other than $|\mathbf{r}\rangle$ itself have strictly positive energies, *i.e.* are massive. The only exception occurs when $\mathbf{r}(z,\theta)$ is the identity operator of dimension 0. In this case $|\mathbf{r}\rangle$ is just the usual NS vacuum state $|0\rangle$ (which transforms trivially under the Lie algebra), and the adjoint states $J^a_{-1/2}\,|0\rangle$ are massless. These states give rise to the 4d gauge bosons when tensored with the helicity ± 1 right-moving modes $\overline{\psi}^{\mu}_{-1/2}\,|\bar{0}\rangle$, but they can also yield massless scalars or fermions in the adjoint representation of the group if tensored with right-moving modes of appropriate helicity. For example, gauginos are made by tensoring $J^a_{-1/2}\,|0\rangle$ with $|\bar{\alpha}\rangle$, which is the same right-moving Ramond state that is tensored with $\psi^{\mu}_{-1/2}\,|0\rangle$ in order to make a gravitino. Actually this gaugino $\leftrightarrow$ gravitino relation works both ways: If there are massless fermions of type $J^a_{-1/2}\,|0\rangle \otimes |\bar{\alpha}\rangle$, then $\psi^{\mu}_{-1/2}\,|0\rangle \otimes |\bar{\alpha}\rangle$ is a gravitino, hence there is a space-time supersymmetry, and these adjoint fermions are in fact gauginos. If there is more than one $|\bar{\alpha}\rangle$, then there is $N > 1$ supersymmetry; such theories are necessarily non-chiral. On the other hand, if there is a massless scalar whose vertex contains a left-moving supercurrent $\mathbf{J}^a(z,\theta)$, then on dimensional grounds this vertex also contains a superfield of dimension $(0, \frac{1}{2})$, *i.e.* a right-moving supercurrent; as we have seen in section 2, such supercurrents also destroy 4d chirality. Therefore, in chiral models all massless states other than gauge bosons and gauginos should be primary with respect to the SKM algebra.

At this point we have reduced the question 'What gauge quantum numbers can a massless particle have?' to the task of classifying unitary highest weight representation of SKM algebras and describing how the highest weight states transform with respect to the Lie algebra. This has been done by Goddard and Olive[38] for the ordinary Kac-Moody algebras and generalized to SKM algebras by Kac and Todorov[15]; the following couple of paragraphs reproduce their results. Unitarity of a highest weight representation is the requirement that all the states (4.3) have non-negative norm. The hermitian conjugation needed to define the norm in a 2d field theory exchanges incoming and outgoing states on the 2d cylinder; therefore it exchanges $z \leftrightarrow 1/\bar{z}$ in the complex plane. If we choose a basis in which the generators $T^a = J_0^a$ of the Lie algebra G are hermitian, then the supercurrents $\mathbf{J}^a(z,\theta)$ are also self-adjoint under hermitian conjugation, which means that their modes satisfy

$$(J_r^a)^\dagger = J_{-r}^a\,, \qquad (J_n^a)^\dagger = J_{-n}^a\,. \tag{4.4}$$

Now consider the state $(J_{-1}^a + iJ_{-1}^b)\,|\mathbf{r}\rangle$ in the highest weight representation generated from a highest weight state $|\mathbf{r}\rangle$. Its norm can be computed as*

$$\left\|(J_{-1}^a + iJ_{-1}^b)\,|\mathbf{r}\rangle\right\|^2 = \langle\mathbf{r}|\, k_G - 2f_G^{abc}J_0^c\,|\mathbf{r}\rangle\,, \tag{4.5}$$

and the requirement that the norm is non-negative imposes an upper limit on the eigenvalues of $T_{\mathbf{r}}^c$. Since the latter eigenvalues are completely determined by the Lie algebra for any of its representations, the choice of allowed Lie representations for Kac-Moody primary states is severely limited for small k_G. For example, for $SU(2)$ non-negativity of (4.5) implies $|I_3| \leq k/2$, which means that only representations of isospin $I \leq k/2$ are allowed for the highest weight states. For other groups one replaces $2I$ with the so called representation level $\lambda_G(\mathbf{r})$ which is defined in terms of weight vectors of $\mathbf{r}$[38]; for $G = SU(N)$, $\lambda_G(\mathbf{r})$ is equal to

* It suffices to consider the case where G is simple.

the number of columns in the Young tableau of $\mathbf{r}$ (which for $N = 2$ indeed equals to $2I(\mathbf{r})$). To summarize, highest weight states of a unitary representation must have

$$\lambda_G(\mathbf{r}) \;\leq\; k_G\,. \tag{4.6}$$

On the other hand, representations satisfying (4.6) are *integrable* — they occur in the spectrum of a string propagating on the group manifold (with its radius quantized in terms of k) [39] — so they are indeed unitary.

The condition (4.6) was derived using bosonic currents only, so it applies to any Kac-Moody algebra, not necessarily supersymmetric. In fact, for the SKM case a stronger condition can be derived, namely

$$\lambda_G \;\leq\; \hat{k} \equiv k - C_G(\mathrm{adj}) \tag{4.7}$$

for all highest weight NS states; the condition for Ramond states is different. This stronger restriction follows from the fact that an SKM algebra contains a *second* non-supersymmetric Kac-Moody subalgebra. It is generated by the currents

$$\hat{J}^a(z) \;\equiv\; J^a(z) \;+\; \tfrac{i}{k} f^{abc}\, {:}\mathcal{J}^b(z)\,\mathcal{J}^c(z){:} \tag{4.8}$$

and has central charge equal to $\hat{k}$. Note that in order for this second Kac-Moody algebra to have *any* unitary representations one must have $\hat{k} \geq 0$. Moreover, in the minimal case $\hat{k} = 0$ this second algebra degenerates: there are no primary states other than $|0\rangle$ and all $\hat{J}^a_{-n}\,|0\rangle$ are null. What this means is that the currents $\hat{J}^a(z)$ actually vanish; hence the original bosonic currents $J^a(z)$ can be written as bilinears of $\mathcal{J}^a(z)$ (*cf.* (3.10)). But the $\mathcal{J}^a(z)$ are free fermions (up to normalization), so their algebra has a unique highest weight state in the NS sector, namely the usual Neveu-Schwarz vacuum, which is of course a gauge singlet.

In the general case one can rewrite the states (4.3) occurring in a highest weight representation of the SKM algebra using the negative-frequency modes of $\hat{J}^a(z)$ instead of $J^a(z)$:

$$J^{a_1}_{-r_1} \dots J^{a_k}_{-r_k} \hat{J}^{a_1}_{-n_1} \dots \hat{J}^{a_l}_{-n_l} |\mathbf{r}\rangle \, . \tag{4.9}$$

Because the $\hat{J}^a(a)$ commute with the fermionic currents $J^a(z)$, this shows that the Hilbert space is the tensor product of a free fermion part, $J^{a_1}_{-r_1} \dots J^{a_k}_{-r_k} |\mathbf{r}\rangle$, and an ordinary Kac-Moody part, $\hat{J}^{a_1}_{-n_1} \dots \hat{J}^{a_l}_{-n_l} |\mathbf{r}\rangle$. Again, the free fermion part has a unique highest weight state in the NS sector, so there is a one-to-one correspondence between unitary NS representations of the SKM algebra with central charge k and unitary representations of the ordinary Kac-Moody algebra with central charge $\hat{k}$, with identical gauge transformation properties for the corresponding primary states $|\mathbf{r}\rangle$. This is the main result of ref. [15].

Having learned the restrictions on unitary highest weight representations of SKM algebras, we can now apply them to the problem at hand — building realistic superstring models. As we have seen in section 2, existence of gluons implies an $SU(3)$ SKM algebra, but now eq. (4.7) tells us that the existence of quarks — massless color triplets — requires the SKM algebra to be non-minimal: $\hat{k}_3 \geq 1$, or $k_3 \geq 4$.* Similarly, existence of $W^\pm$ and Z^0 gauge bosons implies an $SU(2)$ algebra while existence of weak doublets requires $\hat{k}_2 \geq 1$, or $k \geq 3$. We will now see that while each of the conditions $\hat{k}_2 \geq 1$ and $\hat{k}_3 \geq 1$ is quite innocuous by itself, together they spell a disaster. Indeed, when substituted into eq. (2.12), they yield

$$\hat{c}\,(SU(3) \otimes SU(2) \otimes U(1)) \;\geq\; 6\tfrac{2}{3}\, , \tag{4.10}$$

which exceeds the $\hat{c}(\text{SKM}) \leq 6$ limit that any 4d models based on the type II superstring must obey. Thus we are left with the following unpleasant alternatives:

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* Note that if the $SU(3)$ is a subalgebra of a bigger SKM algebra, the latter algebra could be minimal. For example, a minimal $SU(4)$ algebra ($k = 4$ or $\hat{k}_4 = 0$) contains a non-minimal $SU(3)$ subalgebra (also $k = 4$ but $\hat{k}_3 = 1$).

♠ Giving up weak doublets. If $\hat{k}_2 = 0$, then one can have $\hat{k}_3 = 1$ (this allows quarks) in a model with $\hat{c}(\mathrm{SKM}) = 6$.

♣ Giving up quarks. If $\hat{k}_3 = 0$, then one can have $\hat{k}_2 = 1$ or $\hat{k}_2 = 4$. The former choice allows weak doublets and requires a $\hat{c} = \frac{2}{3}$ left-over algebra to achieve $\hat{c}^{\mathrm{int}} = 6$. The second alternative has no left-over algebra and allows massless states of higher weak isospin.

♢ Giving up the $U(1)$ gauge boson; then one can have $\hat{k}_3 = 1$ and $\hat{k}_2 = 2$ and get $\hat{c}(SU(3) \otimes SU(2)) = 6$. Curiously, $\hat{k}_2 = 2$ allows for massless scalars or fermions to be weak triplets as well as doublets, so perhaps a model containing, say, the Georgi-Glashow model could be constructed.

♡ Or, perhaps, giving up one of the space-time dimensions, instead of a $U(1)$ factor (since both contribute 1 to $\hat{c}$).

Obviously none of the above sacrifices is phenomenologically acceptable, and the only real alternative is to give up on the type II superstring itself. Despite its ultimate failure, the type II superstring theory comes tantalizingly close to producing a realistic model: any of the important features of the standard model, such as chiral fermions, gauge groups containing the $SU(3) \otimes SU(2) \otimes U(1)$, massless quarks or leptons, can be achieved. It is just that we can never have all these features at once, *i.e.* in the same model. Although limit (4.10) exceeds the critical value $\hat{c}^{\mathrm{int}} = 6$ by a rather modest amount $\delta\hat{c} = \frac{2}{3}$, this is enough to render any classical vacuum† of the type II superstring incapable of reproducing all the particles of the standard model in its massless spectrum.

† The smallness of $\delta\hat{c}$ makes one hope that perhaps quantum string effects can increase the critical dimension of the superstring so that (4.10) could be accommodated. At present this possibility is of course pure speculation, whose test will have to wait until we understand quantum string effects much better than we do now.

5. Conclusions

In this paper we have explored the phenomenological possibilities for classical vacua of the type II superstring and have found them lacking. Previous work [7,8] exhibited four-dimensional models with gauge groups containing the standard model's. We have shown explicitly that those models are in fact asymmetric orbifolds. In addition we have constructed asymmetric orbifold models with other gauge groups which contain the standard model's, but which are not subgroups of the previously realized groups, and also models with chiral fermions. The latter constructions are important because they provide an argument [9] that at least some asymmetric orbifolds cannot be interpreted as supersymmetric non-linear sigma models. The argument assumes that a Kaluza-Klein mechanism is responsible for the Cartan subalgebra of any 4d gauge group generated by sigma model compactification of the type II superstring. Then it can be shown [6] that all 4d fermions transform non-chirally under the gauge group. Hence constructions resulting in chiral models, such as those in section 3, must be inherently different from supersymmetric sigma models.

The main result of this paper, however, was to show, under some very mild assumptions as to what constitutes a model based on the type II superstring, that it is *not* possible to obtain a phenomenologically acceptable model. In particular, our proof did not depend on any particular compactification scheme, or even on whether the non-Minkowski degrees of freedom could be interpreted as internal dimensions at all, so long as they gave rise to a superconformal 2d field theory. In view of the physical importance of the question of whether realistic type II superstring vacua exist, and also the fact that one can construct classical vacua which are not too far from the standard model in their massless particle content, it might be worthwhile to see if any of the assumptions leading to the no-go result could be relaxed.

In any case, the general approach described here applies as well to the $(0,1)$ superconformal theory describing the compactified heterotic string, although restrictions from unitarity will be much less stringent. The most obvious application is to show that abelian dimension $(0, \frac{1}{2})$ supercurrents in a 4d heterotic string model destroy 4d chirality[21] — non-abelian supercurrents eliminate all massless fermions.[20] So all gauge symmetry arises from a left-moving non-supersymmetric Kac-Moody algebra with $c^{\mathrm{KM}} \leq 22$. Because so many quasi-realistic heterotic string models satisfy this constraint, one must ask subtler questions than just whether massless fermions exist with the gauge quantum numbers of quarks and leptons. One will undoubtedly need to apply to the problem more than just the string tree-level considerations of sections 2 and 4, for example by incorporating as well the constraints of modular invariance; even then one may be unable to progress without a better understanding of the dynamics of compactification.

Finally, some of the specific constructions we gave in section 3 may be of use in constructing heterotic string models. For example, the two orbifolds of the $N = 8$ model based on the lattice $\Gamma^{6,6}(E_6)$ demonstrate how to incorporate members of the $N = 1$ or $N = 2$ unitary discrete series into modular invariant string models in a rather non-trivial fashion (*i.e.* not just taking direct products of discrete systems which are individually modular invariant). They might therefore be useful prototypes for similar constructions in the heterotic string as well. In general, modular invariant systems involving members of the discrete series need not have any geometric interpretation at all. It may prove fruitful to study the exact circumstances under which such models *can* be interpreted in terms of orbifolds or perhaps other generalized geometries.

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