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Multi-loop Twist Field Correlation Functions for Z_N Orbifolds*

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ABSTRACT

We calculate the correlation functions of bosonic twist fields for Z_N orbifolds on Riemann surfaces of arbitrary genus.

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1. Introduction

The recent interest in superstring theories as fundamental theories of all interactions[1] has been accompanied by attempts[2-15] to understand the structure of string perturbation theory a la Polyakov.[2] Using this approach, much progress has been made recently on issues such as finiteness and vacuum stability to arbitrary order in the perturbative expansion.[5-15] With a few exceptions,[7-9,13-15] almost all of the recent work on string perturbation theory has been within the context of a flat space-time background. Ideally one would like to achieve the same kind of understanding for arbitrary backgrounds, such as Calabi-Yau compactifications.[16] However, the complexity of the interacting two-dimensional conformal field theories corresponding to Calabi-Yau backgrounds has blocked progress in computing arbitrary scattering amplitudes. On the other hand, orbifolds[17] and related vacua[18] describable in terms of free two-dimensional fields can give rise to phenomenologically interesting models, and yet being free theories they still may be treated exactly. Our theme in this paper will be to illustrate some of the main aspects of string perturbation theory on orbifold backgrounds, where the point group of the orbifold is a cyclic group — Z_N for some integer N.

String perturbation theory a la Polyakov entails calculation of correlation functions of vertex operators on Riemann surfaces of successively higher genera. Vertex operators for string theories compactified on orbifolds are of two types. There is a class of vertex operators that describe the emission of states in the untwisted sector of the theory. Those can be written explicitly in terms of the free fields (X^μ, ψ^μ, *etc.*) representing the fundamental degrees of freedom on the world-sheet. Correlation functions of such vertices are almost as straightforward to compute as those of the flat ten-dimensional theory. However, there is another class of vertex operators on orbifolds that describe the emission of states in twisted sectors. These involve conformal fields known as twist fields.[19-22] These fields do not possess a simple local representation in terms of the fundamental fields on the world-sheet; indeed the fundamental fields are multi-valued around

the twist fields. Consequently, correlation functions of these fields are usually calculated rather indirectly.

In this paper we calculate correlation functions of an arbitrary collection of Z_N twist fields on a Riemann surface R of arbitrary genus g by following the approach described in ref. [21] for the calculation of twist field correlation functions on the sphere. In this approach (henceforth referred to as the stress-tensor method) the idea that the stress tensor is the generator of deformations of the surface R — plus the local and global properties of twist fields — are used to derive integrable first order differential equations for the twist correlators. The same idea has been used to calculate spin field correlation functions at genus one[4] and at arbitrary genus.[11] As we shall see, the stress tensor method is also a powerful tool for calculation of twist field correlators at arbitrary genus, even though the global properties of the twist fields differ somewhat from those of the spin fields. Here we focus on the correlations of the fields σ which twist the bosonic coordinates X^i. Correlations of twist fields for fermionic coordinates ψ^i can be handled similarly; their correlations are simple generalizations of the spin field correlations of refs. [4,11].

An alternative approach to calculating Z_N twist field correlation functions on a Riemann surface R is to consider an appropriate N-fold covering surface $\tilde{R}$ for R (of higher genus than R), on which the fundamental fields X^i, ψ^i, *etc.*, are single-valued. This approach has been used[20] to explicitly calculate certain twist correlators on the sphere, and also[23] to calculate arbitrary-genus vacuum amplitudes (for some values of N), which could then be factorized[24] to yield lower-genus twist correlators. In the covering-surface approach one needs to relate the moduli characterizing the Riemann surface R with twist fields inserted at points z_i to the moduli of the N-fold cover $\tilde{R}$; this relation can be quite complicated for general twist configurations. Yet another way to calculate twist correlation functions would be to explicitly construct the twist field as an 'intertwining operator' which acts between the Fock spaces of different twisted sectors.[19,25] This operator approach also could be unwieldly for arbitrary collections of twists

on higher-genus surfaces.

We should remark that certain one-loop amplitudes for untwisted states in orbifold and related backgrounds have already been calculated explicitly. One-loop amplitudes of arbitrary numbers of untwisted external states for a Z_3 bosonic orbifold were calculated in ref. [26]. Also, the general one-loop calculation in ref. [14] of the Fayet-Iliopoulus D-term in an arbitrary background was accompanied by more explicit results for the $Spin(32)/Z_2$ heterotic string on a Z_3 orbifold. More specifically, the one-loop mass for a $U(1)$-charged scalar which is induced by the D-term was computed directly, and also by factorizing an amplitude for four untwisted space-time fermions onto the scalar pole. One-loop scattering amplitudes for three untwisted particles for Z_N orbifolds of the heterotic string were recently calculated in ref. [27]. Non-renormalization of Yukawa couplings at one-loop was exhibited when the orbifold preserves a four-dimensional supersymmetry, in agreement with general arguments.[8,9] For superstring models in which the internal degrees of freedom are free fermions with periodic or antiperiodic boundary conditions, the one-loop vacuum amplitude has been shown[28] to vanish when space-time supersymmetry is present, again in accord with general arguments.[8,9] *Non*-supersymmetric models in which the one-loop vacuum amplitude vanishes have been constructed by Moore.[29]

This paper is organized as follows: In section 2 we review some pertinent facts about twist fields and outline the stress tensor method for calculating their correlation functions. It will be convenient to split the coordinate field X into classical solutions X_{cl} and a quantum fluctuation X_{qu}, and to first calculate the quantum correlator Z_{qu} corresponding to the path integral for X_{qu} only. In section 3 we calculate Z_{qu} for an arbitrary configuration of Z_N twist fields on the torus. This exercise is a useful warm-up for the arbitrary-genus case because it avoids certain technical complications occurring for genus $g \geq 2$. The extension to arbitrary genus is carried out in section 4. Section 5 incorporates the contributions from the classical solutions X_{cl}; various properties of the full twist correlator are also analyzed there. Section 6 contains our conclusions. In Appendix A we collect a

few useful properties of theta functions on Riemann surfaces. In Appendix B we describe how to explicitly find a basis for the 'closed loops' on a Riemann surface with an arbitrary configuration of twist fields inserted. One needs to integrate certain 'cut abelian differentials' along these closed loops in the construction of the twist correlation function.

2. Twist Fields and the Stress Tensor Method

In this section we briefly review the properties of twist fields [19,20,21,22] and the stress tensor method [30,7,21] for determining their correlation functions, following the notation of ref. [21]. Twist fields are conformal fields which create states in the twisted sectors of the orbifold Hilbert space. In these sectors the coordinate fields X^i are not periodic on the world-sheet cylinder; instead they undergo some orthogonal rotation plus a translation, $X^i \to \theta^{ij}X^j + v^i$. The transformation (θ, v) is an element of the *space group* of the orbifold. In other words, the string in this sector only closes on the space-time torus modulo the rotation θ^{ij}. We consider here Z_N orbifolds, in which all the rotations are powers of a single rotation θ_0 of order N; θ_0 is an automorphism of some even-dimensional lattice Λ defining the space-time torus from which the orbifold is constructed.

We choose a complex basis for the coordinates X^i in order to diagonalize θ^{ij}, then focus on one of the complex coordinates, say $X = \frac{1}{\sqrt{2}}(X_1 + iX_2)$, $\overline{X} = \frac{1}{\sqrt{2}}(X_1 - iX_2)$. Since θ_0 has order N, X will acquire a phase $e^{2\pi ik/N}$ ($k \in \{0, 1, 2, \ldots, N-1\}$) under θ. This behavior on the world-sheet cylinder in the twisted sector fixes the local phase behavior (*monodromy*) of $X(z, \bar{z})$ near any twist field, independently of the location of other conformal fields, and independently of the world-sheet topology:

$$X(ze^{2\pi i}, \bar{z}e^{-2\pi i}) = e^{2\pi ik/N}X(z, \bar{z}) + v, \tag{2.1}$$

where we have mapped the twist field location to the origin $z = 0$. Thus Z_N twist fields can be represented as order-N branch points for the multi-valued

field X on the world-sheet Riemann surface, and there is a Z_N-valued sheet index associated with each point on the surface (See figure 1). Actually, each twist field is associated via (2.1) with not just a single space group element (θ, v) but an entire *conjugacy class* of the space group. For the Z_N case, each conjugacy class has a fixed rotation θ, and v runs over some coset of the lattice Λ. The different cosets of Λ for a given θ correspond to twisted states located at different fixed points of θ on the space-time torus. (See refs. [17,20,21] for details.)

Equation (2.1) holds for any twist field creating a state in a given twisted sector. Now we specialize to those twist fields, denoted by $\sigma(z,\bar{z})$, which create twisted sector ground states $|\sigma\rangle = \sigma(0)\,|0\rangle$. Ground states are annihilated by the positive-frequency mode operators $\alpha_{m-k/N}$ and $\bar{\alpha}_{m+k/N}$ which occur in the Laurent expansions of the fields $\partial_z X$ and $\partial_z \overline{X}$:

$$\begin{aligned} \alpha_{m-k/N}\,|\sigma\rangle &= 0, \qquad m > 0, \\ \bar{\alpha}_{m+k/N}\,|\sigma\rangle &= 0, \qquad m \geq 0. \end{aligned}$$

This determines the integer power of the singularity as the fields $\partial_z X$ and $\partial_z \overline{X}$ (and similarly $\partial_{\bar{z}} X$ and $\partial_{\bar{z}} \overline{X}$) approach σ:

$$\begin{aligned} \partial_z X\ \sigma(w,\bar{w}) &\sim (z-w)^{-(1-k/N)}\ \tau(w,\bar{w}) + \cdots\ , \\ \partial_z \overline{X}\ \sigma(w,\bar{w}) &\sim (z-w)^{-k/N}\ \tau'(w,\bar{w}) + \cdots\ , \\ \partial_{\bar{z}} X\ \sigma(w,\bar{w}) &\sim (\bar{z}-\bar{w})^{-k/N}\ \tilde{\tau}(w,\bar{w}) + \cdots\ , \\ \partial_{\bar{z}} \overline{X}\ \sigma(w,\bar{w}) &\sim (\bar{z}-\bar{w})^{-(1-k/N)}\ \tilde{\tau}'(w,\bar{w}) + \cdots\ , \end{aligned} \tag{2.2}$$

where the fields τ, τ', *etc.*, create excited states in the twisted sector.

Equations (2.2) contain all the *local* information necessary to construct correlation functions for ∂X, $\partial\overline{X}$, $\bar{\partial} X$ and $\bar{\partial}\overline{X}$ in the presence of twist fields. On the other hand, the information about the translation v in eq. (2.1) has been lost in passing to eqs. (2.2). This information reappears in *global* monodromy conditions on the correlation functions, which describe how X and $\overline{X}$ change

when they are carried around collections of twist fields, rather than just single twists. In particular, define a 'closed loop' γ on the Riemann surface R to be a loop which encloses a collection of fields with *net twist zero*, *i.e.* the product of the corresponding point group elements is the identity. Around such a loop X (also $\overline{X}$) is not rotated but merely translated,

$$\Delta_\gamma X = \oint_\gamma dz\ \partial_z X + \oint_\gamma d\bar{z}\ \partial_{\bar{z}} X = v_\gamma. \tag{2.3}$$

Here v_γ runs over some coset of the lattice Λ which is given by all possible products of space group elements (θ_i, v_i) for the twist fields which are encircled by γ.[21] So a closed loop returns to its starting point on the Riemann surface on the same sheet for X. The particular sheet is irrelevant, as two loops which differ only by the sheet index give rise to different elements v of the same coset of Λ in (2.3). Similarly, a closed loop which winds only around a single twist field is always trivial.

It is convenient to split X into a classical piece and a quantum fluctuation, $X = X_{cl} + X_{qu}$, such that the quantum piece does *not* change around each closed loop γ:

$$\Delta_\gamma X_{qu} = \oint_\gamma dz\ \partial_z X_{qu} + \oint_\gamma d\bar{z}\ \partial_{\bar{z}} X_{qu} = 0. \tag{2.4}$$

Because the functional integral for X is gaussian even in the presence of twists, the complete twist correlation function for a product of n complex pairs $(X, \overline{X})$ of orbifold coordinates takes the form[20,21]

$$Z = \prod_j \langle \sigma_j \rangle = \prod_{i=1}^{n} Z_{qu}^{(i)} \sum_{v,\bar{v}} e^{-S_{cl}(v,\bar{v})}, \tag{2.5}$$

where Z_{qu} is the quantum correlation function, *i.e.* Z_{qu} is evaluated using the

global monodromy conditions (2.4) for X. The classical action

$$S_{cl}(v,\bar{v}) = \frac{1}{2\pi}\int_R d^2z \sum_{i=1}^{n} (\partial_z X^i_{cl}\partial_{\bar{z}}\overline{X}^i_{cl} + \partial_{\bar{z}}X^i_{cl}\partial_z\overline{X}^i_{cl}) \tag{2.6}$$

is a sum of contributions from each complex pair. The sum over $(v,\bar{v})$ in (2.5) is a sum over all classical solutions $X^i_{cl}(z,\bar{z};v,\bar{v})$ obeying (2.3) for some vector v_γ in the appropriate coset of Λ, for all closed loops γ. Note that there may be a coupling between the different complex dimensions in this sum over cosets, so the full correlator need not decompose into a product of correlators for each complex pair of dimensions. On the other hand, the quantum contribution $Z_{qu} = \prod_i Z^{(i)}_{qu}$ does decompose and so may be treated one complex dimension at a time.

Now we review briefly how to calculate Z_{qu} via the stress tensor method. The stress tensor $T(z)$ enables one to derive simple differential equations for correlation functions of primary fields $\phi(z,\bar{z})$ in conformal field theory, due to the operator product expansion (OPE)

$$T(z)\phi(w,\bar{w}) \sim \frac{h_\phi}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + \cdots . \tag{2.7}$$

Hence if one can determine the expectation value for the stress tensor in the presence of primary fields,

$$\langle\langle T(z)\rangle\rangle \equiv \frac{\langle T(z)\prod_i \phi_i(z_i,\bar{z}_i)\rangle}{\langle\prod_i \phi_i(z_i,\bar{z}_i)\rangle} , \tag{2.8}$$

then one may evaluate the residue of the simple pole in (2.8) as $z \to z_i$ in order to obtain the logarithmic derivative of the primary field correlator with respect to z_i,

$$\frac{1}{\langle\prod_i \phi_i(z_i,\bar{z}_i)\rangle}\partial_{z_i}\left\langle\prod_i \phi_i(z_i,\bar{z}_i)\right\rangle . \tag{2.9}$$

Repeating this procedure for each z_i, and then integrating the first-order differential equations, one obtains the full z_i-dependence of the correlator $\langle\prod_i \phi_i(z_i,\bar{z}_i)\rangle$.

If the correlator is being evaluated on a genus g Riemann surface, it will also depend on the $3g-3$ moduli τ_i of the surface. This dependence can be determined[30,7] in a similar fashion by integrating the stress tensor expectation value $\langle\langle T(z)\rangle\rangle$ with respect to the Beltrami differentials. In fact one in general only needs to carry out this integration for $\langle\langle T(z)\rangle\rangle$ with *no* primary fields inserted, which yields the genus g partition function. The reason is that only a multiplicative prefactor $f(\tau_i,\bar{\tau}_i)$ for $\langle\prod_i \phi_i(z_i,\bar{z}_i)\rangle$ is left undetermined by the z_i integrations. Assuming that one knows the tree-level operator products of the ϕ_i, one can factorize the correlator onto the identity operator evaluated on the genus g surface, and thereby determine $f(\tau_i,\bar{\tau}_i)$ in terms of the genus g partition function.

The virtue of the stress tensor method for orbifold backgrounds is that the free fields $\partial_z X$, $\partial_{\bar{z}} X$, $\partial_z \overline{X}$ and $\partial_{\bar{z}} \overline{X}$ have precisely the same OPE's as in a flat background,

$$\begin{aligned} -\partial_z X \partial_w \overline{X} &\sim \frac{1}{(z-w)^2} + T(w) + \cdots , \\ \partial_{\bar{z}} X \partial_w \overline{X} &\sim \text{finite}. \end{aligned} \tag{2.10}$$

Therefore $\langle\langle T(z)\rangle\rangle$ is trivial to extract from the Green's function in the presence of twist fields,

$$g(z,w;z_i,\bar{z}_j) \equiv \frac{\langle -\partial_z X \partial_w \overline{X} \prod_{i=1}^{L} \sigma_i(z_i,\bar{z}_i)\rangle}{\langle \prod_{i=1}^{L} \sigma_i(z_i,\bar{z}_i)\rangle}. \tag{2.11}$$

One simply lets $w \to z$ and subtracts the double pole singularity:

$$\langle\langle T(z)\rangle\rangle = \lim_{w\to z}\left[g(z,w) - \tfrac{1}{(z-w)^2}\right] . \tag{2.12}$$

The auxiliary Green's function

$$h(\bar{z},w;z_i,\bar{z}_j) \equiv \frac{\langle -\partial_{\bar{z}} X \partial_w \overline{X} \prod_{i=1}^{L} \sigma_i(z_i,\bar{z}_i)\rangle}{\langle \prod_{i=1}^{L} \sigma_i(z_i,\bar{z}_i)\rangle} \tag{2.13}$$

is also required in order to fully determine $g(z,w)$ via the global monodromy constraints (2.4) on $X, \overline{X}$. In the next sections we will use complex function

theory on Riemann surfaces to explicitly construct these Green's functions and thereby extract the twist correlators.

3. Green's Functions and Quantum Twist Correlators at One Loop

In this section we use the stress tensor method to evaluate the quantum correlation function Z_{qu} of L twist fields on the torus. We will defer evaluation of the classical contribution, and a check of the asymptotic behavior of Z_{qu}, until after we have generalized the result for Z_{qu} to arbitrary genus in section 4.

3.1. Cut Abelian Differentials and Green's Functions

Here we determine the Green's functions $g(z,w)$ and $h(z,w)$ on the torus by exploiting the local and global monodromy requirements on $\partial_z X$ and $\partial_z \overline{X}$. First consider the local properties which $g(z,w)$ and $h(\bar{z},w)$ must satisfy. In addition to the usual singularity structure dictated by the OPE's (2.10),

$$\begin{aligned} g(z,w;z_i) &\sim \frac{1}{(z-w)^2} + \text{finite as } z \to w, \\ h(\bar{z},w;z_i) &\sim \text{ finite as } z \to w, \end{aligned} \tag{3.1}$$

they must exhibit the same behavior as $\partial_z X$, $\partial_{\bar{z}} X$ and $\partial_w \overline{X}$ as they approach the locations of the L twist fields $\sigma_i(z_i,\bar{z}_i)$ (OPE's (2.2)):

$$\begin{aligned} g(z,w;z_i) &\sim (z-z_i)^{-(1-\frac{k_i}{N})} \quad \text{as } z \to z_i, \\ &\sim (w-z_i)^{-\frac{k_i}{N}} \quad \text{as } w \to z_i, \\ h(\bar{z},w;z_i) &\sim (\bar{z}-\bar{z}_i)^{-\frac{k_i}{N}} \quad \text{as } \bar{z} \to \bar{z}_i, \\ &\sim (w-z_i)^{-\frac{k_i}{N}} \quad \text{as } w \to z_i. \end{aligned} \tag{3.2}$$

These *local* properties are identical to those obeyed by the Green's functions on the sphere.

The first step in the construction of the Green's functions is to build certain holomorphic and antiholomorphic one-forms which are doubly-periodic on the torus yet have the appropriate behavior near the twist fields, as dictated by (2.2). We denote such one-forms by $\langle \partial_z X \rangle$, $\langle \partial_{\bar{z}} X \rangle$, $\langle \partial_z \overline{X} \rangle$ and $\langle \partial_{\bar{z}} \overline{X} \rangle$, and refer to them as *cut (abelian) differentials.* When properly normalized (in the next section), the cut differentials will also serve as the classical solutions $\partial_z X_{cl}$, *etc.*, in the presence of twist fields. Since $h(\bar{z}, w)$ remains finite as $\bar{z} \to w$, it consists of the product of an antiholomorphic and a holomorphic cut differential. Similarly $g(z, w)$ is written as the sum of two terms, the first developing the required double pole as z approaches w and the second consisting of a product of holomorphic cut differentials in z and w.

Now let us construct the cut differentials. To generate the appropriate local monodromy, define*

$$\begin{aligned} \gamma_{N-k}(z) &= \prod_{i=1}^{L} \vartheta_1(z - z_i)^{-(1-\frac{k_i}{N})} , \\ \gamma_k(z) &= \prod_{i=1}^{L} \vartheta_1(z - z_i)^{-\frac{k_i}{N}} . \end{aligned} \tag{3.3}$$

While $\gamma_{N-k}(z)$ and $\gamma_k(w)$ exhibit the local monodromy of $\partial_z X$ and $\partial_w \overline{X}$, the transformation properties of theta functions (equations (A.3)) show that they are not periodic on the torus. One can make them periodic as $z \to z + 1$ by multiplying them by a product of $L - M$ and M theta functions, respectively. Here we have defined $M = \sum_{i=1}^{L} \frac{k_i}{N}$; M is an integer, because the correlation function must have zero net twist on the torus in order not to vanish.[17,20,21] Since the fraction $\frac{k_i}{N}$ is defined to be between 0 and 1, we have $M \geq 1$ and $M \leq L - 1$. Thus $\langle \partial_z X \rangle$ is constructed by multiplying $\gamma_{N-k}(z)$ by $\prod_{j=1}^{L-M} \vartheta_1(z - Z_j)$. The only constraint on the Z_j is that $\sum_{j=1}^{L-M} Z_j = \sum_{i=1}^{L} (1 - \frac{k_i}{N}) z_i$, so that $\langle \partial_z X \rangle$ is periodic as $z \to z + \tau$.

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* See Appendix A for our ϑ-function conventions.

A set of linearly independent differentials for $\langle \partial_z X \rangle$ is

$$\omega_{N-k}^{\alpha_i}(z) = \gamma_{N-k}(z)\vartheta_1(z - z_{\alpha_i} - Y_{N-k}) \prod_{j\neq i}^{L-M} \vartheta_1(z - z_{\alpha_j}) , \qquad i = 1, \ldots, L-M, \tag{3.4}$$

where $\{z_{\alpha_1}, \ldots, z_{\alpha_{L-M}}\}$ is an arbitrarily chosen set of $L-M$ twist insertion points and Y_{N-k} is determined by periodicity as $z \to z + \tau$ to be

$$Y_{N-k} = \sum_{i=1}^{L} \left(1 - \frac{k_i}{N}\right) z_i - \sum_{j=1}^{L-M} z_{\alpha_j} . \tag{3.5}$$

The $\omega_{N-k}^{\alpha_i}(z)$ are linearly independent since by our construction $\omega_{N-k}^{\alpha_i}(z_{\alpha_j}) = 0$ when $i \neq j$. To see that they span the space of cut differentials with appropriate local monodromy consider the doubly-periodic meromorphic function $\lambda(z)$ which is constructed from an arbitrary cut differential $\omega_{N-k}(z)$ and a reference differential, say $\omega_{N-k}^{\alpha_1}(z)$:

$$\lambda(z) = \frac{\omega_{N-k}(z)}{\omega_{N-k}^{\alpha_1}(z)} - \sum_{i=1}^{L-M} \frac{C_i\, \omega_{N-k}^{\alpha_i}(z)}{\omega_{N-k}^{\alpha_1}(z)}. \tag{3.6}$$

Given $\omega_{N-k}(z)$ — and assuming $z_i \neq z_j$ for all $i \neq j$ — one can adjust the constants C_i, $i > 1$, so that the residues of $\lambda(z)$ at the $L-M$ poles $z_{\alpha_2}, \ldots, z_{\alpha_{L-M}}$ and $z_{\alpha_i} + Y_{N-k}$ vanish. For these values of C_i, (3.6) has no poles and hence is a constant, which we can set to zero by adjusting C_1.

Similarly, we find a complete set of M linearly independent cut differentials for $\langle \partial_z \overline{X} \rangle$,

$$\omega_k^{\beta_i}(z) = \gamma_k(z)\vartheta_1(z - z_{\beta_i} - Y_k) \prod_{j\neq i}^{M} \vartheta_1(z - z_{\beta_j}), \qquad i = 1, \ldots, M, \tag{3.7}$$

where

$$Y_k = \sum_{i=1}^{L} \frac{k_i}{N} z_i - \sum_{j=1}^{M} z_{\beta_j} \tag{3.8}$$

and $\{z_{\beta_1}, \ldots, z_{\beta_M}\}$ are M twist insertion points, not necessarily related to $\{z_{\alpha_i}\}$.

With this basis of cut differentials we now write down the Green's functions:

$$g(z,w) = g_s(z,w) + \sum_{i=1}^{L-M}\sum_{j=1}^{M} A_{ij}\ \omega_{N-k}^{\alpha_i}(z)\omega_k^{\beta_j}(w)$$

and

$$h(\bar{z},w) = \sum_{i=1}^{M}\sum_{j=1}^{M} B_{ij}\ \bar{\omega}_k^{\beta_i}(\bar{z})\omega_k^{\beta_j}(w) \tag{3.9}$$

where

$$g_s(z,w) = \gamma_{N-k}(z)\gamma_k(w)\left[\frac{\vartheta_1'(0)}{\vartheta_1(z-w)}\right]^2 P(z,w).$$

The constants A_{ij} and B_{ij} will be determined using global monodromy after choosing $P(z,w)$. $P(z,w)$ is uniquely determined — up to shifts in the non-singular portion of $g(z,w)$, *i.e.* shifts in A_{ij} — by the requirement that $g_s(z,w)$ is doubly-periodic on the torus in the variables z and w, and has the double-pole singularity (3.1) as $z \to w$.

One way to construct $P(z,w)$ explicitly is via the parametrization

$$P(z,w) = \sum_A C_A F_A(z,w) \prod_{i\in A}^{M} \vartheta_1(w-z_i) \prod_{j\in A^*}^{L-M} \vartheta_1(z-z_j), \tag{3.10}$$

where the sum on A is over all $\binom{L}{M}$ subsets of $\{1,2,\ldots,L\}$ containing exactly M elements, and A^* is the complement of A in $\{1,2,\ldots,L\}$. The $\binom{L}{M}$ constants C_A are chosen to satisfy the M equations $\sum_{A|i\in A} C_A = \frac{k_i}{N}$, $i = 1,\ldots,M$, which also imply that $\sum_A C_A = 1$. $F_A(z,w)$ is given by

$$F_A(z,w) = \frac{\vartheta_1(z-w+U_A)}{\vartheta_1(U_A)}\frac{\vartheta_1(z-w+Y_A-U_A)}{\vartheta_1(Y_A-U_A)} \tag{3.11}$$

where

$$Y_A = \sum_{i=1}^{L}\frac{k_i}{N}z_i - \sum_{j\in A}^{M} z_j = -\left[\sum_{i=1}^{L}\left(1-\frac{k_i}{N}\right)z_i - \sum_{j\in A^*}^{L-M} z_j\right].$$

It is easily verified that with $P(z,w)$ as written in (3.10) $g_s(z,w)$ is doubly-periodic and has a double pole with coefficient $\sum_A C_A = 1$ as $z \to w$, since

$F_A(w,w) = 1$ for all $\binom{L}{M}$ sets A. The single pole is absent when the constants C_A satisfy the above M equations, provided also that $\partial_z F_A(z,w)|_{z=w} = 0$ for all A. But $\partial_z F_A(z,w)|_{z=w}$ is a meromorphic function of U_A with a simple pole at $U_A = 0$ and one at $U_A = Y_A$. Hence it has two zeroes, at say $U_A = U_A^0$ and $U_A = Y_A - U_A^0$; these choices of U_A both give the same function $F_A(z,w)$ in (3.11), which when inserted into (3.10) (for each set A) yields a $P(z,w)$ satisfying all the required constraints. Note that $\binom{L}{M} > M$ for $M \neq 1, L-1$, so there are in general more constants C_A than equations for them. The resulting ambiguity in $\{C_A\}$ corresponds to a shift in the nonsingular portion of g_s, which may be absorbed into a shift in A_{ij} in (3.9).

It turns out that the explicit form (3.10) for $P(z,w)$ is not required to calculate the twist correlator $Z = \langle \prod_i \sigma_i(z_i, \bar{z}_i) \rangle$. The following identity is easily derived from constraints imposed on $P(z,w)$ by the singularity structure (3.1) of $g(z,w)$, and is sufficient to eliminate all explicit dependence of Z_{qu} on $P(z,w)$:

$$\gamma_{N-k}(w)\gamma_k(w)\left(\frac{\partial}{\partial z}\frac{\partial}{\partial w}P(z,w)\mid_{z=w} + \frac{\partial^2}{\partial z^2}P(z,w)\mid_{z=w}\right)$$
$$= \sum_{i=1}^{L}\left(1-\frac{k_i}{N}\right)\frac{\partial}{\partial w}\left[\frac{\vartheta_1'(w-z_i)}{\vartheta_1(w-z_i)}\right] + \left[\sum_{i=1}^{L}\frac{\vartheta_1'(w-z_i)}{\vartheta_1(w-z_i)}\right]\left[\sum_{i=1}^{L}\left(1-\frac{k_i}{N}\right)\frac{\vartheta_1'(w-z_i)}{\vartheta_1(w-z_i)}\right] \tag{3.12}$$

On the other hand, the expression (3.10) is required in order to explicitly determine via global monodromy all the constants A_{ij} and B_{ij} occurring in the Green's functions $g(z,w)$ and $h(\bar{z},w)$. Explicit forms for the Green's functions are needed to calculate the correlation functions involving fields other than the ground state twists $\sigma(z,\bar{z})$.[21]

The global monodromy conditions for $g(z,w)$ and $h(\bar{z},w)$ follow from eq. (2.4):

$$0 = \oint_\gamma dz\; g(z,w) + \oint_\gamma d\bar{z}\; h(\bar{z},w) \tag{3.13}$$

for all closed loops γ. The number of independent equations represented by

(3.13) is just the number generators of the homology group for the closed loops, keeping in mind the equivalence of loops under shifts of the sheet index, *etc.* We show in Appendix B that this number always equals the total number of linearly independent cut abelian differentials, namely $L+2g-2$ for arbitrary genus g, and also show how to construct the homology generators, which we refer to as a *basis* for the closed loops. It suffices to impose (3.13) on the $L+2g-2$ elements of this basis, as we can then invert the global monodromy conditions (3.13) (and also similar conditions on the classical solutions).

For the torus ($g=1$) we define the elements of the L by L 'cut period matrix'* to be

$$\begin{aligned} W_a^i &= \oint_{\gamma_a} dz\ \omega_{N-k}^{\alpha_i}(z), \qquad i=1,\ldots,L-M, \\ W_a^{L-M+i} &= \oint_{\gamma_a} d\bar{z}\ \bar{\omega}_k^{\beta_i}(\bar{z}), \qquad i=1,\ldots,M, \end{aligned} \tag{3.14}$$

where the γ_a, $a=1,\ldots,L$, form a basis for the closed loops. The antiholomorphic differentials $\bar{\omega}_k^{\beta_i}(\bar{z})$ are the conjugates of (3.7) and appear because of the antiholomorphic dependence in $h(\bar{z},w)$. As long as the twist insertion points z_i do not coincide, the inverse of W_a^i exists:

$$\sum_{a=1}^{L} \left(W^{-1}\right)_i^a W_a^j = \delta_i^j\ , \qquad \sum_{i=1}^{L} W_a^i \left(W^{-1}\right)_i^b = \delta_a^b\ . \tag{3.15}$$

With $g_s(z,w)$ as previously defined, the Green's functions satisfying both local

---

* Since the cut period matrix defined here has both holomorphic and antiholomorphic elements, it does not correspond precisely to (elements of) the usual holomorphic period matrix on the covering surface.

and global monodromy are given as

$$
\begin{aligned}
g(z,w) &= g_s(z,w) - \sum_{i=1}^{L-M} \omega_{N-k}^{\alpha_i}(z) \sum_{a=1}^{L} (W^{-1})_i^a \oint_{\gamma_a} dy\; g_s(y,w), \\
h(\bar{z},w) &= -\sum_{i=1}^{M} \bar{\omega}_k^{\beta_i}(\bar{z}) \sum_{a=1}^{L} (W^{-1})_{L-M+i}^a \oint_{\gamma_a} dy\; g_s(y,w).
\end{aligned}
\tag{3.16}
$$

Although the Green's functions are unique, the particular *representation* (3.16) is basis-dependent since it requires a choice of $L-M$ points $\{z_{\alpha_i}\}$ and M points $\{z_{\beta_i}\}$, as well as a basis of linearly independent loops $\{\gamma_a\}$.

3.2. Twist Correlators

The stress tensor expectation value $\langle\langle T(z)\rangle\rangle$ in the presence of twists is obtained from the Green's function $g(z,w)$ in the limit $w \to z$, using eq. (2.12). With $g(z,w)$ given by eq. (3.16), and the identity (3.12), we find

$$
\begin{aligned}
\langle\langle T(z)\rangle\rangle = \frac{1}{2} &\left[\sum_{i=1}^{L} \frac{k_i}{N} \frac{\vartheta_1'(z-z_i)}{\vartheta_1(z-z_i)}\right] \left[\sum_{i=1}^{L} \left(1-\frac{k_i}{N}\right) \frac{\vartheta_1'(z-z_i)}{\vartheta_1(z-z_i)}\right] \\
&- \frac{1}{2}\gamma_{N-k}(z)\gamma_k(z) \frac{\partial}{\partial z}\frac{\partial}{\partial w} P(z,w)\,|_{w=z} - \frac{1}{3}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} \\
&- \sum_{i=1}^{L-M} \omega_{N-k}^{\alpha_i}(z) \sum_{a=1}^{L} (W^{-1})_i^a \oint_{\gamma_a} dy\; g_s(y,z).
\end{aligned}
\tag{3.17}
$$

Similarly, the OPE's between the stress tensor and the twist fields yield differential equations for the quantum portion Z_{qu} of the twist correlator $\langle \prod_{i=1}^{L} \sigma_i \rangle$:

$$
\partial_{z_i} \ln Z_{qu} = \lim_{z \to z_i} \left[(z-z_i)\langle T(z)\rangle - \frac{h_i}{(z-z_i)} \right] , \tag{3.18}
$$

where $h_i = \frac{1}{2}\frac{k_i}{N}(1-\frac{k_i}{N})$ is the conformal dimension of the twist field σ_i. As a check, observe that the $1/(z-z_i)$ poles in (3.18) cancel. A similar set of differential equations $\partial_{\bar{z}_i} \ln Z_{qu} = \cdots$ are obtained by considering the OPE's of the antiholomorphic stress tensor $\overline{T}(\bar{z})$ with the twist fields.

Because the i^{th} cut abelian differential $\omega^{\alpha_i}_{N-k}(z)$ is more singular than the others as $z \to z_{\alpha_i}$, the last term in the stress tensor (3.17) has a particularly simple structure in this limit; hence the differential equations for the $L-M$ variables z_{α_i} are also simple. So let us first consider the differential equations for these variables. To deal with the rest of the insertion points, we can then change the set $\{z_{\alpha_i}\}$ and thus the basis of abelian differentials so that the point that we are differentiating with respect to always occurs in that set. Hence the problem of integrating the differential equations for all the twist insertion points reduces to showing how to integrate the equations for a particular set $\{z_{\alpha_i}\}$, and then showing that changing the set preserves the integrability. The quantum correlator $Z_{qu}(z_i, \bar{z}_i)$ should not depend on which set $\{z_{\alpha_i}\}$ was chosen, and this will be demonstrated.

From (3.17) and (3.18), the differential equation for the point z_{α_i} is

$$\begin{aligned}\partial_{z_{\alpha_i}} \ln Z_{qu} =& \frac{1}{2}\sum_{j\neq\alpha_i}^{L}\left[\frac{k_{\alpha_i}}{N}\left(1-\frac{k_j}{N}\right)+\left(1-\frac{k_{\alpha_i}}{N}\right)\frac{k_j}{N}\right]\frac{\vartheta_1'(z_{\alpha_i}-z_{\alpha_j})}{\vartheta_1(z_{\alpha_i}-z_{\alpha_j})} \\ &- \frac{1}{2\vartheta_1'(0)}\prod_{j\neq\alpha_i}^{L}\frac{1}{\vartheta_1(z_{\alpha_i}-z_j)}\frac{\partial}{\partial z}\frac{\partial}{\partial w}P(z,w)\mid_{z_{\alpha_i}} \\ &- \sum_{a=1}^{L}(W^{-1})^a_i\oint_{\gamma_a} dz\;\omega^{\alpha_i}_{N-k}(z)\Lambda_i(z)\;.\end{aligned} \tag{3.19}$$

The function $\Lambda_i(z)$ which occurs in the last term is given by

$$\Lambda_i(z) = \lim_{w\to z_{\alpha_i}}\left[\frac{\omega^{\alpha_i}_{N-k}(w)}{\gamma_{N-k}(w)}\right]\frac{\gamma_{N-k}(z)}{\omega^{\alpha_i}_{N-k}(z)}\frac{\vartheta_1'(0)P(z,z_{\alpha_i})}{\vartheta_1^2(z-z_{\alpha_i})\prod_{j\neq\alpha_i}^{L}\vartheta_1(z_{\alpha_i}-z_j)}\;. \tag{3.20}$$

By Taylor expanding $P(z,z_{\alpha_i})$ about z_{α_i}, and using the constraints on it mentioned earlier, the singular behavior of $\Lambda_i(z)$ near z_{α_i} is easily shown to be

$$\Lambda_i(z) \sim \frac{1-\frac{k_{\alpha_i}}{N}}{z-z_{\alpha_i}} + \cdots .$$

But this is also the leading singularity of $[\omega^{\alpha_i}_{N-k}(z)]^{-1}\partial_{z_{\alpha_i}}\omega^{\alpha_i}_{N-k}(z)$ near z_{α_i}, as

long as Y_{N-k} is non-vanishing. The difference between the two functions is meromorphic with poles at the $L-M$ zeroes of $\omega^{\alpha_i}_{N-k}(z)$. It is therefore equal to $[\omega^{\alpha_i}_{N-k}(z)]^{-1}\sum_{j=1}^{L-M} T^i_j\,\omega^{\alpha_j}_{N-k}(z)$, where the $L-M$ coefficients T^i_j are determined by equating the residues of the $L-M$ poles and the leftover constant function. This establishes the identity

$$\omega^{\alpha_i}_{N-k}(z)\Lambda_i(z) = \partial_{z_{\alpha_i}}\omega^{\alpha_i}_{N-k}(z) + \sum_{j=1}^{L-M} T^i_j\,\omega^{\alpha_j}_{N-k}(z). \tag{3.21}$$

When the expansion (3.21) is inserted into the differential equation (3.19), only the T^i_i term in the sum over abelian differentials contributes, because of (3.15). T^i_i is determined by letting $z\to z_{\alpha_i}$ in $\Lambda_i(z) - [\omega^{\alpha_i}_{N-k}(z)]^{-1}\partial_{z_{\alpha_i}}\omega^{\alpha_i}_{N-k}(z)$:

$$T^i_i = \frac{\partial_z^2 P(z,z_{\alpha_i})\,|_{z=z_{\alpha_i}}}{2\vartheta_1'(0)\prod_{j\neq\alpha_i}^{L}\vartheta_1(z_{\alpha_i}-z_j)} - \left(1-\frac{k_{\alpha_i}}{N}\right)\sum_{j\neq i}^{L-M}\frac{\vartheta_1'(z_{\alpha_i}-z_{\alpha_j})}{\vartheta_1(z_{\alpha_i}-z_{\alpha_j})}\,. \tag{3.22}$$

By applying the identity (3.12) to the $P(z,z_{\alpha_i})$ terms in (3.22) and (3.19), we find that the $P(z,z_{\alpha_i})$-dependence of the correlator disappears.

After some rearranging, the differential equations in z_{α_i} for the quantum correlator assume a simple form:

$$\begin{aligned}\partial_{z_{\alpha_i}}\ln Z_{qu} = &-\sum_{a=1}^{L}\left(W^{-1}\right)^a_i\frac{\partial}{\partial z_{\alpha_i}}W^i_a \\ &+\partial_{z_{\alpha_i}}\ln\left[\prod_{l\neq i}^{L-M}\vartheta_1(z_{\alpha_i}-z_{\alpha_l})^{(1-\frac{k_{\alpha_i}}{N})}\prod_{j\neq\alpha_i}^{L}\vartheta_1(z_{\alpha_i}-z_j)^{-(1-\frac{k_{\alpha_i}}{N})(1-\frac{k_j}{N})}\right].\end{aligned} \tag{3.23}$$

The first term can be rewritten as $-\partial_{z_{\alpha_i}}\ln|W| + \sum_{j\neq i}^{L-M}\sum_{a=1}^{L}\left(W^{-1}\right)^a_j\partial_{z_{\alpha_i}}W^j_a$, where $|W|$ is the determinant of the cut period matrix W. Evaluation of the z_{α_i} derivatives on W^j_a $(j\neq i)$, is similar to the analysis of $\Lambda_i(z)$. There exist

coefficients $U_l^{(i)j}$ such that

$$\partial_{z_{\alpha_i}} \omega_{N-k}^{\alpha_j}(z) = \sum_{l=1}^{L-M} U_l^{(i)j} \omega_{N-k}^{\alpha_l}(z) \ . \tag{3.24}$$

This is proved by dividing (3.24) by $\omega_{N-k}^{\alpha_j}(z)$ and using the $L-M$ constants $U_l^{(i)j}$ to match the residues at the $L-M$ poles on each side. The difference between the two sides is then a constant which can be absorbed in a redefinition of $U_j^{(i)j}$. The only coefficients which contribute to (3.23) are the $U_j^{(i)j}$, because of the contraction with $\left(W^{-1}\right)_j^a$, and they are easily determined by letting $z \to z_{\alpha_j}$ in (3.24). We find that

$$\begin{aligned} \sum_{j\neq i}^{L-M} \sum_{a=1}^{L} \left(W^{-1}\right)_j^a \frac{\partial}{\partial z_{\alpha_i}} W_a^j &= \sum_{j\neq i}^{L-M} U_j^{(i)j} \\ &= \frac{\partial}{\partial z_{\alpha_i}} \ln \left[\vartheta_1(Y_{N-k})^{L-M-1} \prod_{j\neq i}^{L-M} \vartheta_1(z_{\alpha_j} - z_{\alpha_i})^{\frac{k_j}{N}} \right] \ . \end{aligned} \tag{3.25}$$

Putting all this together, we can now write down the quantum correlator which solves the differential equations in all the variables $\{z_{\alpha_i}\}$:

$$\begin{aligned} Z_{qu} =& f(\tau; k_l, z_l \notin \{z_{\alpha_i}\}, \bar{z}_l) |W|^{-1} \vartheta_1(Y_{N-k})^{(L-M-1)} \\ & \times \prod_{\substack{i,j=1 \\ i<j}}^{L-M} \vartheta_1(z_{\alpha_i} - z_{\alpha_j}) \prod_{\substack{i,j=1 \\ i<j}}^{L} \vartheta_1(z_i - z_j)^{-(1-\frac{k_i}{N})(1-\frac{k_j}{N})} \ , \end{aligned} \tag{3.26}$$

where $f(\tau; k_l, z_l \notin \{z_{\alpha_i}\}, \bar{z}_l)$ may depend on the remaining M points, the modulus τ, the twist integers k_l and all the antiholomorphic twist insertions $\bar{z}_l$.

To find the full z_l-dependence, consider the change of basis induced by exchanging one of the $L-M$ z_{α_i}'s for one of the remaining M points. To be concrete let $\alpha_1 = 1$ and let the rest of the α_i's be any set of $L-M-1$ other insertion points. Denote the basis of cut abelian differentials with this choice

of $\{z_{\alpha_i}\}$ as $\omega^{\alpha_i}_{N-k(1)}$, where the extra subscript indicates that $z_1 \in \{z_{\alpha_i}\}$. Now consider changing the basis by letting $\alpha_1 = a$ instead, where z_a is not one of the other z_α's. (Such a z_a always is present since $M \geq 1$.) Denote this new basis of cut abelian differentials by $\omega^{\alpha_i}_{N-k(a)}$. Since both bases are complete, there exists the relation

$$\omega^{\alpha_i}_{N-k(a)}(z) = \sum_{j=1}^{L-M} C^i_j \, \omega^{\alpha_j}_{N-k(1)}(z). \tag{3.27}$$

To determine the coefficients C^i_j , let $z \to z_{\alpha_i}$ for each i. The resulting transformation of the determinant $|W_1|$, where the subscript denotes the first basis of abelian differentials, is

$$|W_1| \to |W_a| = |W_1| \left(\frac{\vartheta_1(Y_{N-k}\,|_{\alpha_1=a})}{\vartheta_1(Y_{N-k}\,|_{\alpha_1=1})} \right)^{L-M-1} \prod_{j=2}^{L-M} \frac{\vartheta_1(z_{\alpha_j} - z_a)}{\vartheta_1(z_{\alpha_j} - z_1)} \, . \tag{3.28}$$

This change in the determinant is cancelled by the change in the additional α_i-dependent terms in the correlator (3.26), so that (3.26) is invariant under the change of basis. Hence (3.26) already contains explicitly the dependence on *all* of the z_i.

The dependence on the antiholomorphic coordinates $\bar{z}_i$ is found in the same way, using Green's functions $\bar{g}(\bar{z}, \bar{w})$ and $\bar{h}(z, \bar{w})$. Inspection of the behavior (2.2) of $\bar{\partial} X$ and $\bar{\partial}\overline{X}$ near twist fields shows that the antiholomorphic dependence is given by simply conjugating all coordinates $z, z_{\beta_i}, \ldots \leftrightarrow \bar{z}, \bar{z}_{\beta_i}, \ldots$ and interchanging k_i with $N - k_i$ everywhere. Now, the determinant $|W|$ in the quantum correlator (3.26) does not factorize holomorphically. However, $|W|$ is invariant (up to a sign) under complex conjugation accompanied by interchange of differentials of the type γ_{N-k} with γ_k, so that it is compatible with the antiholomorphic

differential equations. The full quantum correlator is

$$
\begin{aligned}
Z_{qu} = & f(\tau;k_l)|W|^{-1}\vartheta_1(Y_{N-k})^{L-M-1}\overline{\vartheta_1(Y_k)}^{M-1} \\
& \prod_{\substack{i,j=1 \\ i<j}}^{L-M} \vartheta_1(z_{\alpha_i}-z_{\alpha_j}) \prod_{\substack{i,j=1 \\ i<j}}^{M} \overline{\vartheta_1(z_{\beta_i}-z_{\beta_j})} \\
& \prod_{\substack{i,j=1 \\ i<j}}^{L} \vartheta_1(z_i-z_j)^{-(1-\frac{k_i}{N})(1-\frac{k_j}{N})}\overline{\vartheta_1(z_i-z_j)}^{-\frac{k_i}{N}\frac{k_j}{N}} \ .
\end{aligned}
\tag{3.29}
$$

The normalization factor $f(\tau;k_l)$ can be determined by factorizing the twist correlator on the torus partition function.

4. Extension to Arbitrary Genus

We again apply the stress tensor method to calculate the quantum twist correlator Z_{qu}, now at arbitrary genus. The basic outline of the calculation is identical to the genus one case: We write down the cut abelian differentials and from them construct the Green's functions $g(z,w)$ and $h(\bar{z},w)$ using local and global monodromy, and from the Green's functions we subsequently extract the stress tensor and differential equations for the correlators. There are complications associated with function theory on a Riemann Surface R of genus $g > 1$, and we have listed some relevant mathematical results in Appendix A for convenience. We also refer the reader to the works of Mumford[31] and Fay,[32] and to the reviews [33,34,35].

The local monodromy of the cut abelian differentials can again be generated by the functions $\gamma_{N-k}(z)$ and $\gamma_k(z)$ defined in (3.3) if $\vartheta_1(z-z_i)$ is replaced by any Riemann theta function[31] of odd characteristic $\vartheta\left[\begin{smallmatrix}\vec{a}_0 \\ \vec{b}_0\end{smallmatrix}\right](\vec{z}-\vec{z}_i)$ where $\vec{z} = \int_{P_0}^{z} dz\ \vec{\omega}(z)$ is a point on the Jacobian Variety $Jac(R)$ of the Riemann Surface R. From now on we will suppress the characteristic label $\left[\begin{smallmatrix}\vec{a}_0 \\ \vec{b}_0\end{smallmatrix}\right]$. Now, $\vartheta(\vec{z}-\vec{w})$ has, in addition to a zero at $z=w$, 'spurious' zeroes at $z=R_i$, $i=1,\ldots,g-1$, which are independent of the location of w (see Appendix A). Therefore $\gamma_{N-k}(z)$

and $\gamma_k(z)$ have spurious poles, of order M and $L-M$ respectively, at each point in the set $\{R_i\}$. These poles must not appear in the cut differentials. We wish to show that the one-forms

$$\omega_{N-k}^{\alpha_i}(z) = h^2(z)\gamma_{N-k}(z)\vartheta(\vec{z}-\vec{z}_{\alpha_i}-\vec{Y}_{N-k})\prod_{l\neq i}^{L-M}\vartheta(\vec{z}-\vec{z}_{\alpha_l}) \qquad (4.1)$$

are linearly independent cut differentials for $\langle\partial_z X\rangle$. Here $h^2(z) = \vec{\omega}(z)\cdot\partial_{\vec{z}}\vartheta(0)$ enables $\omega_{N-k}^{\alpha_i}(z)$ to transform as a one-form rather than a function. $h^2(z)$ has double zeroes at each of the points R_i (see Appendix A) which cancel off spurious poles from $\gamma_{N-k}(z)$ in (4.1). The double zeroes also mean that the 'square root' $h(z)$ is a well-defined half-order differential, which we will need shortly. The vector $\vec{Y}_{N-k}$ is a point on $Jac(R)$ defined to make $\omega_{N-k}^{\alpha_i}(z)$ periodic about B cycles:

$$\vec{Y}_{N-k} = \sum_{i=1}^{L}\left(1-\frac{k_i}{N}\right)\vec{z}_i - \sum_{j=1}^{L-M}\vec{z}_{\alpha_j}\ . \qquad (4.2)$$

For generic insertion points z_i, the vector $\vec{z}_{\alpha_i}+\vec{Y}_{N-k}\in Jac(R)$ is *not* the image $\vec{y}$ of a point y on the Riemann surface R, because the set of vectors $\vec{y}$ is only a one complex-dimensional subspace of the g complex-dimensional $Jac(R)$. Due to this fact, the Riemann vanishing theorem (Appendix A) shows that the set $\{Q_1,\ldots,Q_g\}$ of zeroes of $\vartheta(\vec{z}-\vec{z}_{\alpha_i}-\vec{Y}_{N-k})$ is in general disjoint from $\{R_i\}$.* Hence the $L-M$ spurious poles at R_i from $\gamma_{N-k}(z)$ are not quite cancelled by the $L-M-1$ zeroes from $\prod_j\vartheta(\vec{z}-\vec{z}_{\alpha_j})$; this is why we have used the differential $h^2(z)$, with a double zero at each R_i, in (4.1). Note that $\omega_{N-k}^{\alpha_i}(z)\sim z-R_j$ as z approaches each R_j. Hence each of these cut abelian differentials vanishes at the $g-1$ points $\{R_i\}$ and at the $L-M-1$ points $\{z_{\alpha_j}\}$, $j\neq i$.

---

* This assumes that $\vartheta(\vec{z}-\vec{z}_{\alpha_i}-\vec{Y}_{N-k})$ does not vanish identically — which is the case for generic insertion points z_l.

We have constructed $L-M$ linearly independent differentials for $\langle \partial_z X \rangle$. But they do not span the space of all such cut differentials. This can be seen by applying the Riemann–Roch theorem,[33] eq. (A.5), to the meromorphic function

$$\lambda(z) = \omega_{N-k}(z)/\omega^{\alpha_1}_{N-k}(z),$$

where $\omega_{N-k}(z)$ is an arbitrary cut differential for $\langle \partial_z X \rangle$. Now $\lambda(z)$ generically has simple poles at the points $\{z_{\alpha_2},\ldots,z_{\alpha_{L-M}},R_1,\ldots,R_{(g-1)},Q_1,\ldots,Q_g\}$ coming from the zeroes of $\omega^{\alpha_1}_{N-k}(z)$; the Q_j are the zeroes of $\vartheta(\vec{z}-\vec{z}_{\alpha_i}-\vec{Y}_{N-k})$. The divisor $\aleph$ corresponding to simple zeroes at these points has degree $d[\aleph] = L-M+2g-2$. Since the degree of the divisor for an abelian differential is $2g-2$ there are no nonsingular differentials which are multiples of $\aleph$ and so $i[\aleph] = 0$. The dimension of the space of cut differentials $\omega_{N-k}(z)$ equals the dimension of the space of functions $\lambda(z)$; by eq. (A.5) this number is $r[\aleph^{-1}] = L-M+g-1$. Therefore we need to add $g-1$ independent cut differentials to the set (4.1) in order to have a basis. Similar arguments yield a dimension of $M+g-1$ for the space of cut differentials for $\langle \partial_z \overline{X} \rangle$, giving a total of $L+2g-2$ cut abelian differentials.

Notice that the $L-M$ differentials (4.1) vanish at all but one of the $L-M+g-1$ special points $\{z_{\alpha_i},R_j\}$, the exception being z_{α_l} for some l. To fill out a basis, it is natural to build $g-1$ cut differentials which vanish at all the z_{α_i} and at all but one of the R_j. This can be achieved by considering a theta function $\tilde{\vartheta}(\vec{z}) \equiv \vartheta\left[\begin{smallmatrix}\vec{a}_1\\ \vec{b}_1\end{smallmatrix}\right](\vec{z})$ with a different odd characteristic from that of $\vartheta(\vec{z})$. (There are $2^{g-1}(2^g-1) > 1$ odd characteristics for $g > 1$.) The theta function $\tilde{\vartheta}(\vec{z}-\vec{R}_j)$ has a zero at R_j and spurious zeroes at $P_1,\ldots,P_{g-1}$. Define the abelian differential $\tilde{h}^2(z) = \vec{\omega}(z)\cdot\partial_{\vec{z}}\tilde{\vartheta}(0)$; the half-order differential $\tilde{h}(z)$ has single zeroes at the P_i. Therefore, the cut abelian differentials

$$\omega^{R_j}_{N-k}(z) = h(z)\tilde{h}(z)\gamma_{N-k}(z)\frac{\tilde{\vartheta}(\vec{z}-\vec{R}_j-\vec{Y}_{N-k})}{\tilde{\vartheta}(\vec{z}-\vec{R}_j)}\prod_{l=1}^{L-M}\vartheta(\vec{z}-\vec{z}_{\alpha_l}) \qquad (4.3)$$

have no singularities except the desired ones (3.2) as $z \to z_i$, vanish at the z_{α_i} and at all but one of the R_i, and will fill out our basis of differentials for $\langle \partial_z X \rangle$.

The construction of $M+g-1$ independent differentials for $\langle \partial_z \overline{X} \rangle$ is entirely analogous. For convenience, we will label the complete set as

$$\begin{aligned}
\omega_i(z) &= \omega_{N-k}^{\alpha_i}(z), && i = 1,\ldots,L-M, \\
\omega_{i+L-M}(z) &= \omega_{N-k}^{R_i}(z), && i = 1,\ldots,g-1; \\
\omega_{i+L-M+g-1}(z) &= \omega_k^{\beta_i}(z), && i = 1,\ldots,M, \\
\omega_{i+L+g-1}(z) &= \omega_k^{R_i}(z), && i = 1,\ldots,g-1,
\end{aligned} \tag{4.4}$$

where the $\{z_{\beta_i}\}$ are a set of M insertion points not necessarily related to the $\{z_{\alpha_i}\}$, as in the one-loop case.

The Green's functions $g(z,w)$ and $h(\bar{z},w)$ at arbitrary genus have the same local properties as at tree-level and at one loop. Again they are determined by local plus global monodromy requirements. The term in $g(z,w)$ which contains the double pole is:

$$g_s(z,w) = \gamma_{N-k}(z)\gamma_k(w)\left[\frac{h(z)h(w)}{\vartheta(\vec{z}-\vec{w})}\right]^2 P(z,w). \tag{4.5}$$

The half-order differentials enable $g_s(z,w)$ (and subsequently $g(z,w)$) to transform correctly in both z and w. The previous constraints on the function $P(z,w)$ generalize straightforwardly to the arbitrary genus case, and the parametrization (3.10) can be lifted to arbitrary genus by replacing the Jacobi theta functions with their higher genus Riemann theta function counterparts. $P(z,w)$ must in addition cancel the spurious poles of $\gamma_{N-k}(z)\gamma_k(w)$ and this is explicit in the construction (3.10). As before, the identity (3.12), when generalized to higher genus, is sufficient to construct the quantum twist correlators.

The number of linearly independent loops at genus g is $L+2g-2$. (See Appendix B.) As we have equal numbers of linearly independent loops and differentials we can again use global monodromy to fix the coefficients A_{ij} and B_{ij} occurring in the Green's functions. We extend the definition of the 'cut period

matrix' to the arbitrary genus case:

$$W_a^i = \oint_{\gamma_a} dz\ \omega_i(z), \qquad i = 1, \ldots, L-M+g-1,$$
$$W_a^{i+L-M+g} = \oint_{\gamma_a} d\bar{z}\ \bar{\omega}_{i+L-M+g}(\bar{z}), \qquad i = 1, \ldots, M+g-1. \tag{4.6}$$

In terms of ω_i, g_s and W, the Green's functions take exactly the same form as at one-loop:

$$\begin{aligned} g(z,w) =& g_s(z,w) - \sum_{i=1}^{L-M+g-1} \omega_i(z) \sum_{a=1}^{L+2g-2} (W^{-1})_i^a \oint_{\gamma_a} dy\ g_s(y,w), \\ h(\bar{z},w) =& - \sum_{i=L-M+g}^{L+2g-2} \bar{\omega}_i(\bar{z}) \sum_{a=1}^{L+2g-2} (W^{-1})_i^a \oint_{\gamma_a} dy\ g_s(y,w)\ . \end{aligned} \tag{4.7}$$

From these Green's functions the stress tensor is easily evaluated. A set of differential equations can be derived and integrated in exactly the same fashion as in the one-loop case. Here we shall suppress the details and quote only the result for the quantum correlator:

$$\begin{aligned} Z_{qu} =& f(\Omega; k_l) |W|^{-1} \vartheta(\vec{Y}_{N-k})^{L-M-1}\ \tilde{\vartheta}(\vec{Y}_{N-k})^{g-1}\ \overline{\vartheta(\vec{Y}_k)}^{M-1}\ \overline{\tilde{\vartheta}(\vec{Y}_k)}^{g-1} \\ & \prod_{\substack{i,j=1 \\ i<j}}^{L-M} \vartheta(\vec{z}_{\alpha_i} - \vec{z}_{\alpha_j}) \prod_{\substack{i,j=1 \\ i<j}}^{M} \overline{\vartheta(\vec{z}_{\beta_i} - \vec{z}_{\beta_j})} \\ & \prod_{\substack{i,j=1 \\ i<j}}^{L} \vartheta(\vec{z}_i - \vec{z}_j)^{-(1-\frac{k_i}{N})(1-\frac{k_j}{N})} \overline{\vartheta(\vec{z}_i - \vec{z}_j)}^{-\frac{k_i}{N}\frac{k_j}{N}} \\ & \prod_{i=1}^{g-1} \Bigg\{ \prod_{j=1}^{L-M} \vec{\omega}(R_i) \cdot \partial_{\vec{z}} \vartheta(\vec{R}_i - \vec{z}_{\alpha_j}) \prod_{k=1}^{M} \overline{\vec{\omega}(R_i) \cdot \partial_{\vec{z}} \vartheta(\vec{R}_i - \vec{z}_{\beta_k})} \\ & \prod_{l=1}^{L} \Big[\vec{\omega}(R_i) \cdot \partial_{\vec{z}} \vartheta(\vec{R}_i - \vec{z}_l) \Big]^{-(1-\frac{k_l}{N})} \overline{\Big[\vec{\omega}(R_i) \cdot \partial_{\vec{z}} \vartheta(\vec{R}_i - \vec{z}_l) \Big]}^{-\frac{k_l}{N}} \Bigg\} \\ & \prod_{l=1}^{L} |h^2(z_l)|^{\frac{k_l}{N}(1-\frac{k_l}{N})}\ . \end{aligned} \tag{4.8}$$

The power of $h^2(z_{\alpha_i})$ in Z_{qu} is precisely the conformal dimension of the twist field $\sigma(z_{\alpha_i})$; therefore Z_{qu} transforms correctly under changes of coordinates for the twist locations z_i. Note also that the powers of $\vec{\omega}(R_i)$ cancel, in the sense that the correlator transforms with respect to each of the R_i as a form of weight zero. Finally, note that (4.8) reduces to (3.29) for $g = 1$.

It might appear that (4.8) has branch cuts in z_i terminating at the $(g-1)$ spurious zeroes R_j, due to the fact that $\vartheta(\vec{z} - \vec{z}_l) \sim (z_l - R_j)$ as $z_l \to R_j$. We now show that Z_{qu} is in fact non-singular as $z_l \to R_j$, for the case that $l \in \{\alpha_i\}$. (The analysis for the case $l \notin \{\alpha_i\}$ is similar and leads to the same conclusion.) In this case, the determinant $|W| \sim \epsilon^{-(1-\frac{k_j}{N})(L-M+g-1)}\epsilon^{(L-M+g-2)+1}$, where $\epsilon \equiv z_l - R_j$. The factor $\gamma_{N-k}(z)$ in the $L-M+g-1$ cut differentials for $\langle \partial_z X \rangle$ contributes the fractional powers of ϵ, the extra $\vartheta(\vec{z} - \vec{z}_l)$ in all but one of the cut differentials contributes an additional $L-M+g-2$ powers of ϵ, and the degeneration of the basis $(\omega^{z_l}_{N-k}(z) \to \omega^{R_j}_{N-k}(z))$ contributes the final power of ϵ in the determinant factor. Also, as $z_l \to R_j$ the term $\vec{\omega}(R_i) \cdot \partial_{\vec{z}}\vartheta(\vec{R}_i - \vec{z}_l)$ develops a single zero for $i \neq j$ and a double zero when $i = j$. Combining these factors of ϵ with those from $h^2(z_l)$ and the $\vartheta(\vec{z}_i - \vec{z}_l)$'s in Z_{qu}, one finds that all factors cancel, as desired.

In the analysis that led to the quantum correlator (4.8) we assumed that the quantum fluctuations $(X, \overline{X})$ were strictly periodic around all closed loops γ, including the $2g$ generators of the canonical homology basis for the genus g surface. In fact we need to consider also the cases where X and $\overline{X}$ acquire Z_N phases, $\{1, e^{2\pi i/N}, \ldots, e^{2\pi i(N-1)/N}\}$, around the latter cycles. The set of Z_N phases acquired by X around the $2g$ cycles is referred to as a *twist structure*, by analogy to the more familiar *spin structure* which specifies the Z_2 phases acquired by a world-sheet fermion around the $2g$ cycles. As for the spin-structure case, modular invariance dictates that we sum up correlation functions calculated in different twist structures, with well-defined relative coefficients. To find the correlators in other twist structures we could repeat the entire procedure leading to (4.8), starting with Green's functions $g(z,w)$ and $h(\bar{z},w)$ which are not periodic

in z and w but acquire instead the appropriate Z_N phases.

Alternatively, a more efficient way of calculating a twist correlator in any nontrivial twist structure is to notice that translating the argument of a twist field along a nontrivial homology cycle leaves a branch cut running around that cycle, and hence changes the twist structure (see figure 2). It is easy to check that translation of the twist location z_i around a cycle — *i.e.* $\vec{z}_i \to \vec{z}_i + \Omega\vec{m} + \vec{n}$ on the Jacobian $Jac(R)$, for some integer vectors $\vec{m}, \vec{n}$ — does not in general leave the correlator (4.8) invariant. Instead it transforms into the same correlator evaluated in a different twist structure. So to calculate correlators of any collection of twist fields $\{\frac{k_i}{N}\}$ we can add to the set a twist-antitwist pair $\frac{1}{N}, -\frac{1}{N}$, calculate the correlator of the new collection in the trivial twist structure using (4.8), and then generate all other twist structures by translating the location of the $\frac{1}{N}$ twist along the $2g$ canonical cycles an appropriate number of times. To recover the original correlator for the different twist structures we then factor the $\frac{1}{N}, -\frac{1}{N}$ pair of twists onto the identity. The relative phases of contributions from different twist structures can be determined by demanding that the full sum is periodic in all twist locations. This construction has been carried out explicitly for the analogous case of spin fields in ref. [4].

5. Classical Contribution to Twist Correlators

In order to evaluate the full twist correlator (2.5) one also needs to calculate the action S_{cl} (eq. (2.6)) for all classical solutions $X_{cl}(z,\bar{z}), \overline{X}_{cl}(z,\bar{z})$ which obey the proper global monodromy conditions. The classical action is a sum of contributions from each complex pair of coordinates $(X, \overline{X})$, so we can focus on the contribution from one pair:

$$S_{cl} = \frac{1}{2\pi} \int_R d^2z (\partial_z X_{cl} \partial_{\bar{z}} \overline{X}_{cl} + \partial_{\bar{z}} X_{cl} \partial_z \overline{X}_{cl}). \tag{5.1}$$

Now we construct the classical solutions which have the correct local and

global monodromy. Each classical solution $X_{cl}(z,\bar{z};v,\bar{v})$ satisfies

$$\Delta_{\gamma_a} X_{cl} = v_a, \qquad a = 1,\cdots,L+2g-2, \tag{5.2}$$

where $\Delta_\gamma X$ is given by (2.3) and v_a is a particular element of the appropriate coset of the lattice Λ for each of the $L+2g-2$ independent closed loop γ_a. Note that v denotes the collection of coset elements for a given classical solution, $\{v_1,\ldots,v_{L+2g-2}\}$ (see section 2). There is a similar equation for $\overline{X}_{cl}$ and we denote its shift by $\bar{v}_a$. The derivatives of the classical solutions $\partial_z X_{cl},\ldots$ are linear combinations of the cut differentials (4.4) which satisfy these global monodromy constaints. Solving (5.2) is an exercise in linear algebra and we find:

$$\begin{aligned} \partial_z X_{cl}(v,\bar{v}) =& v_a (W^{-1})^a_{i'}\,\omega_{i'}(z), \\ \partial_{\bar{z}} X_{cl}(v,\bar{v}) =& v_a (W^{-1})^a_{i''}\,\bar{\omega}_{i''}(\bar{z}). \end{aligned} \tag{5.3}$$

where a contraction of i' denotes a sum over the first $L-M+g-1$ indices of $\omega_i(z)$, a contraction of i'' denotes a sum over the remaining $M+g-1$ indices, and a contraction over the loop parameter a is over all $L+2g-2$ basis loops. Expressions for $\partial_{\bar{z}}\overline{X}_{cl}$ and $\partial_z\overline{X}_{cl}$ are obtained by complex conjugation of (5.3). The classical action for each pair of coordinates $(X,\overline{X})$ is given by

$$\begin{aligned} S_{cl}(v,\bar{v}) = \frac{1}{2\pi} v_a \bar{v}_b \big\{ &(W^{-1})^a_{i'} (\overline{W}^{-1})^b_{j''} (\omega_{i'},\omega_{j'}) \\ &+ (W^{-1})^a_{i''} (\overline{W}^{-1})^b_{j''} (\omega_{j''},\omega_{i''}) \big\} \quad , \end{aligned} \tag{5.4}$$

where $(\omega_i,\omega_j) = i\int_R \omega_i \wedge \bar{\omega}_j$ is the hermitian inner product on the differential forms ω_i, which are given in local analytic coordinates as $\omega_i = \omega_i(z)dz$.

The next step is to evaluate (ω_i,ω_j) in terms of matrix elements of W and $\overline{W}$. One technique is to follow the proof of the Riemann bilinear relations which allow one to evaluate the same inner product between uncut meromorphic abelian differentials. First we perform a *canonical dissection* (see *e.g.* [31]) of the surface R along a set of $2g$ disjoint cycles meeting at a point, in order to form a simply

connected region Π. Next we arrange the branch cuts so that they connect the points $\{z_1, \cdots, z_{L-1}\}$ to z_L, cut out the points $\{z_i\}$ and the branch cuts which connect them, and finally cut out a thin neck to join this boundary with the boundary of Π. This procedure generates another simply connected region, denoted as Π' and depicted in figure 3 for a four twist configuration. On Π', ω_i is holomorphic and there exists a holomorphic function $f_i(z)$ such that $\omega_i = df_i$. Hence we can apply Green's Theorem to the inner product of differentials on Π' :

$$(\omega_i, \omega_j) = i \int_{\partial \Pi'} f_i \, \overline{\omega}_j .$$

We label by C_k the part of the boundary $\partial \Pi'$ which starts close to z_L, winds clockwise around z_k, and returns to z_L. Note that any two cut differentials appearing together in an inner product in (5.4) have the same local monodromy. Using this fact, and writing $f_i(z) = f_i(z_k) + \int_{z_k}^{z} \omega_i$ on C_k, it is straightforward to derive the following bilinear relation:

$$\begin{aligned} \frac{1}{i}(\omega_{i'}, \omega_{j'}) = & \sum_{k=1}^{g} \left[\int_{A_k} \omega_{i'} \int_{B_k} \overline{\omega}_{j'} - \int_{B_k} \omega_{i'} \int_{A_k} \overline{\omega}_{j'} \right] \\ & + \sum_{\substack{l,k=1 \\ l<k}}^{L-1} \int_{C_l} \omega_{i'} \int_{C_k} \overline{\omega}_{j'} + \sum_{l=1}^{L-1} \frac{1}{1-\alpha^{-k_l}} \int_{C_l} \omega_{i'} \int_{C_l} \overline{\omega}_{j'} , \end{aligned} \tag{5.5}$$

where $\alpha \equiv e^{2\pi i/N}$. Conjugation of α yields the bilinear relation for $(\omega_{i''}, \omega_{j''})$.

Finally, we wish to relate the line integrals over the paths C_k to a basis of linearly independent 'closed loops' γ_a. (The canonical A and B cycles can always be made elements of a basis.) When k_L is relatively prime with respect to N, this relationship is very simple. As discussed in Appendix B, we can in this case combine the $L-2$ loops of Bershadskii and Radul[22] with the $2g$ canonical

homology cycles for genus g in order to form a basis. Then

$$\int_{C_a} \omega_i = -\oint_{\gamma_a} dz\, \omega_i(z), \qquad a = 1, \cdots, L-1.$$

The first $L-2$ of the contour integrals over γ_a are elements of the cut period matrix (4.6). The last one is a linear combination of the other $L-1$ cut period matrix elements. For simple twist configurations which are not of the Bershadskii-Radul type, one can rearrange the boundary of Π' so that it is closely related to a simple basis of closed loops, yet still surrounds all the branch points and cuts. This procedure may be more economical than deriving a general relationship between the paths C_k and the basis of loops γ_a found in Appendix B. For the most general configuration, one can always follow the strategy described in Appendix B of replacing a single twist k_1/N with two twists $1/N$ and $(k_1-1)/N$, solving the problem for that set of $L+1$ twists, and then taking the limit as the two twists coalesce. The final answer always takes the form

$$\begin{aligned} \frac{1}{i}(\omega_{i'}, \omega_{j'}) &= W_a^{i'} \overline{W}_b^{j'}\, M^{ab}, \\ \frac{1}{i}(\omega_{i''}, \omega_{j''}) &= \overline{W}_a^{i''} W_b^{j''}\, \overline{M}^{ab}, \end{aligned} \tag{5.6}$$

where the matrix elements are simple rational expressions in α and $\bar{\alpha}$. Note that because the inner product is hermitian, $\overline{M}^{ab} = -M^{ba}$.

It is important to verify that the full twist correlator (2.5) has the correct local behavior as two points z_i and z_j coalesce. We shall consider here the holomorphic behavior of Z_{qu}; the antiholomorphic behavior is similar. The conformal dimension of a twist field σ_i is $\frac{1}{2}\eta_i(1-\eta_i)$, where $\eta_i \equiv 1 - \frac{k_i}{N}$. As two twist fields come together, they should factor onto another twist field:

$$\sigma_i(z_i, \bar{z}_i)\sigma_j(z_j, \bar{z}_j) \sim (z_i - z_j)^{\kappa_{ij}} (\bar{z}_i - \bar{z}_j)^{\kappa_{ij}} \sigma_{i+j}(z_j, \bar{z}_j) + \cdots \text{ as } z_i \to z_j. \tag{5.7}$$

The twist σ_{i+j} rotates X by $e^{2\pi i(k_i+k_j)/N}$; therefore the conformal dimension of σ_{i+j} is $\frac{1}{2}(\eta_i+\eta_j)(1-\eta_i-\eta_j)$ if $\eta_i+\eta_j \le 1$ and $\frac{1}{2}(\eta_i+\eta_j-1)(2-\eta_i-\eta_j)$

if $\eta_i + \eta_j \geq 1$. If $\eta_i + \eta_j = 1$, the 'twist field' σ_{i+j} is actually the identity operator I of the untwisted sector. By equating the conformal dimensions of both sides of the OPE (5.7), we see that $\kappa_{ij} = -\eta_i\eta_j$ when $\eta_i + \eta_j \leq 1$ and $\kappa_{ij} = -(1-\eta_i)(1-\eta_j)$ when $\eta_i + \eta_j \geq 1$. We want to verify that the asymptotic behavior of (3.29) is consistent with these values of κ_{ij}.

The subtler factors come from the cut period matrix W. Certain matrix elements diverge as $z_i \to z_j$ because closed loops γ become 'pinched' between z_i and z_j, where the cut abelian differentials are singular. To be more concrete, choose local world-sheet coordinates so that $z_i = 0$ and define $\delta \equiv z_j$. Then the local behavior of the cut differentials is $\omega^l_{N-k}(z) \sim (z)^{-\eta_i}(z-\delta)^{-\eta_j}$. (Some of the cut differentials may be less singular than this.) For this analysis we choose a basis for the closed loops where precisely one of the loops γ_p passes between the two coinciding points. Such a basis must exist, simply because the basis of $L+2g-2$ loops for the L twist configuration at genus g must evolve into a basis of $(L-1)+2g-2$ loops for the $L-1$ twist configuration, losing exactly one loop in the process. The contour integral of the most singular abelian differentials around γ_p diverges as $\delta^{-(\eta_i+\eta_j-1)}$ as $\delta \to 0$ for the case that $\eta_i + \eta_j > 1$, and is convergent for $\eta_i + \eta_j < 1$. If z_i and z_j both belong to the set $\{z_{\alpha_i}\}$, then there is an additional asymptotic factor of δ in the determinant $|W|$, coming from degeneration of the basis of cut differentials: $\omega^{\alpha_i}_{N-k}(z) \to \omega^{\alpha_j}_{N-k}(z)$. However, this factor is cancelled by a δ from the 'extra' $\vartheta(\vec{z}_i - \vec{z}_j)$ in (4.8). Adding these contributions to κ_{ij} to the $-\eta_i\eta_j$ coming from $\vartheta(\vec{z}_i - \vec{z}_j)^{-\eta_i\eta_j}$ in (4.8), we obtain the correct κ_{ij} for both $\eta_i + \eta_j < 1$ and $\eta_i + \eta_j > 1$.

In spite of these factors of δ, we will now show that the classical action (5.4) remains finite as $z_i \to z_j$. There are two potential sources of divergences. The first is when the basis of cut differentials degenerates. Consider the $(W^{-1})^a_{i'}W^{i'}_c$ term in (5.4). When $\eta_i + \eta_j < 1$, the only way that this term can diverge is if both z_i and z_j belong to $\{z_{\alpha_i}\}$, in which case the matrix elements $(W^{-1})^a_{i'}$ and $(W^{-1})^a_{j'}$ diverge like δ^{-1}. To see this, write $(W^{-1})^a_k = (-1)^{k+a}|W|^k_a/|W|$, where $|W|^k_a$ is the determinant of the cofactor matrix obtained from W by deleting the

a^{th} row and the k^{th} column. The determinant of W vanishes like δ, as does the determinant $|W|^k_a$ when k does not equal i or j. But when k is either i or j , $|W|^k_a$ does not vanish because the sub basis does not degenerate. To see that they are absent from the classical action, one can use the the identity

$$(W^{-1})^a_{i'} W^{i'}_b = \delta^a_b - (W^{-1})^a_{i''} W^{i''}_b . \tag{5.8}$$

Then the holomorphic vanishing of the determinants due to degeneration of the basis always cancels in the ratio for $(W^{-1})^a_{i''}$. Similarly, consider the $(\overline{W}^{-1})^b_{j''}\overline{W}^{j''}_d$ term. When z_i and z_j both belong to $\{z_{\beta_i}\}$ and $\eta_i + \eta_j > 1$, two of the $(\overline{W}^{-1})^b_{j''}$ terms diverge like δ^{-1}. These divergences are again removed by using the complex conjugate of (5.8). For convience, let S^{ab}_{cl} be the $v_a \bar{v}_b$ component of the full classical action. After applying (5.8), and using the definitions (5.6), S^{ab}_{cl} can be written as

$$S^{ab}_{cl} = \frac{1}{2\pi}\left[(\overline{W}^{-1})^b_{j'} \overline{W}^{j'}_d M^{ad} + (W^{-1})^a_{i''} W^{i''}_c \overline{M}^{bc} \right] \tag{5.9}$$

The second potential source of divergences is from the matrix elements $W^{i'}_p$ and $\overline{W}^{j''}_p$, when $\eta_i + \eta_j > 1$, where p denotes the pinched loop. However the offending terms are not present in (5.9). (The matrix elements $W^{i''}_a$ and $\overline{W}^{j'}_a$ are integrals over antiholomorphic cut differentials.)

In the above factorization onto twist fields σ_{i+j}, with $\eta_i + \eta_j \neq 1$, the quantum contribution to to the correlation function gave the correct leading behavior — the value of κ_{ij}. But this is no longer true for factorization onto untwisted fields, *i.e.* $\eta_i + \eta_j = 1$. In this case the quantum correlator develops an additional logarithmic singularity as $z_i \to z_j$, due to the $\ln \delta$ behavior of the integral over the pinched loop γ_p entering into W. This logarithmic singularity is cancelled by the classical contribution to the correlator after performing a Poisson resummation on the lattice cosets for v_p and $\bar{v}_p$. This converts the sum over windings around the pinched loop into sums over the momenta of states in the untwisted intermediate channel (See figure 4).

The classical contribution to the correlator will cancel the logarithmic divergence in the quantum part when resummed if the S_{cl}^{pp} term vanishes like $1/\ln\delta$. Consider the case where the basis of cut differentials is non-singular as z_i and z_j coalesce, *i.e.* neither $\{z_{\alpha_i}\}$ or $\{z_{\beta_i}\}$ contain both z_i and z_j. Then the there are divergent matrix elements of W are $W_p^{i'}$ and $\overline{W}_p^{j''}$ which diverge as $\ln\delta$. Both $(W^{-1})_k^p$ and $(\overline{W}^{-1})_k^p$ vanish as $1/\ln\delta$ for all $k = 1, \cdots, L+2g-2$, so the leading order terms in S_{cl}^{pp} vanish as desired. Finally, if the basis of cut differentials diverges as $z_i \to z_j$, the same result is easily obtained.

The function $f(\Omega; k_l)$ is determined on a case by case basis by factorization. Consider for instance a twist/antitwist configuration. As the two points coalesce, the full correlator — a product of the partition function and both the quantum and classical parts of the correlator, should factor onto the identity. More complicated twist configurations can be subsequently normalized with respect to lower point functions by this method.

6. Conclusion

In this paper we have concentrated on the most non-trivial aspect of string perturbation theory for Z_N orbifold vacua, namely the calculation of correlators of bosonic twist fields on Riemann surfaces of arbitrary genus. Using the stress tensor method, as well as some results from complex function theory on Riemann surfaces, we have given expressions for these correlators in terms of Riemann theta functions. These expressions, together with the expressions for the Green's functions in the presence of twists, can be used to calculate any correlation function in the bosonic Z_N orbifold conformal field theory, at any order in the loop expansion. Scattering amplitudes are obtained by integrating correlation functions with respect to the locations of vertex operators on the Riemann surface and with respect to the moduli of the surface. The measure for the former integration is simply $\int_R \prod_i d^2 z_i$. Determining the latter measure explicitly

requires knowing the Beltrami differentials, but is no more difficult than for the case of the flat space-time background.

Of course one is more interested in scattering amplitudes for orbifold backgrounds of the heterotic string. In this case one also needs to calculate correlators of the fields which twist the world-sheet fermions ψ^i, *etc.* As remarked above, these correlators are simple generalizations of the spin field correlators calculated in refs. [4,11]. One must sum over spin structures as well as twist structures; the former sum is the crucial one for showing that explicit orbifold results (non-renormalization of terms in the superpotential,[27] D-terms,[13,14] *etc.*) are consistent with expectations from a low-energy supersymmetric four-dimensional effective field theory.

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APPENDIX A

Basic Properties of Theta Functions on Compact Riemann Surfaces

A Riemann surface R of genus g has a $2g$-dimensional *canonical homology basis* consisting of g cycles (loops) a_i and g cycles b_i with intersection numbers $I(a_i, a_j) = I(b_i, b_j) = 0$, $I(a_i, b_j) = \delta_{ij}$ (see figure 5). The components ω_i of the g-dimensional vector $\vec{\omega}$ are the *canonical holomorphic differentials* of R, which are normalized by the relations $\oint_{a_i} dz\ \omega_j = \delta_{ij}$. The holomorphic period matrix Ω is defined by $\Omega_{ij} = \oint_{b_i} dz\ \omega_j$. Given a base point P_0, each point z on R is associated with a unique point

$$\vec{z} = \int_{P_0}^{z} dz\ \vec{\omega}(z) \tag{A.1}$$

on the *Jacobian variety* $Jac(R)$, which is the g-complex-dimensional torus defined

by the lattice $L_\Omega = \{\vec{m} + \Omega\vec{n} \mid \vec{m}, \vec{n} \in \mathbf{Z}^g\}$.

The Riemann theta functions with characteristic $\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right]$ are defined on $Jac(R)$ by the series

$$\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z},\Omega) = \sum_{\vec{n}\in\mathbf{Z}^g} \exp\left[\pi i(\vec{n}+\vec{a})\cdot\Omega\cdot(\vec{n}+\vec{a}) + 2\pi i(\vec{n}+\vec{a})\cdot(\vec{z}+\vec{b})\right] ,$$

and are defined as quasi-periodic functions on R using the map (A.1). Their transformation properties around the cycles a_i and b_i on R follow from

$$\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z}+\vec{m}) = \exp(2\pi i\vec{a}\cdot\vec{m})\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z}),$$
$$\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z}+\Omega\vec{m}) = \exp(-2\pi i\vec{b}\cdot\vec{m})\exp(-\pi i\vec{m}\cdot\Omega\cdot\vec{m} - 2\pi i\vec{m}\cdot\vec{z})\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z}),$$

where the period matrix Ω is now implicit in $\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z})$. If the characteristic $\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right]$ is odd — meaning that $\vec{a},\vec{b} \in (\frac{1}{2}\mathbf{Z}/\mathbf{Z})^g$ with $4\vec{a}\cdot\vec{b}$ odd — then $\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{0}) = 0$, and $\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z}-\vec{w}) \sim z - w$ as $z \to w$ on the Riemann surface.

Theta functions for $g > 1$ have a peculiar property which makes the construction of cut abelian differentials tricky: In addition to the zero at $\vec{z} = \vec{w}$, $\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z}-\vec{w})$ has additional 'spurious' zeroes at $g-1$ points $R_1,\ldots,R_{g-1}$. This is a consequence of the Riemann Vanishing Theorem[31]: Depending on the characteristic $\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right]$ and on the vector $\vec{Y} \in Jac(R)$, the function $f(z) \equiv \vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z}-\vec{Y})$ either vanishes identically for all $z \in R$ or else $f(z)$ has g zeroes $Q_1,\ldots,Q_g$ on R. In addition, there exists a vector $\vec{\Delta}(\vec{a},\vec{b})$ on $Jac(R)$ such that the points $\{Q_i\}$ satisfy

$$\sum_{i=1}^{g} \vec{Q}_i = \vec{\Delta}(\vec{a},\vec{b}) + \vec{Y} \qquad (\text{mod } L_\Omega). \tag{A.2}$$

Suppose now that $\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right]$ is an odd characteristic and $\vec{Y}$ is the image of a point on the Riemann surface, say $\vec{Y} = \vec{w} = \int_{P_0}^{w} dz\,\vec{\omega}(z)$ for some $w \in R$. Since $\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](0) = 0$, we can let $Q_g = w$ and denote the remaining, spurious zeroes by R_i, $i = 1,\ldots,g-1$.

The R_i are independent of w and satisfy $\sum_{i=1}^{g-1} \vec{R}_i = \vec{\Delta}(\vec{a},\vec{b})$ (mod L_Ω). On the other hand, if $\vec{Y}$ is an arbitrary vector in $Jac(R)$, all the zeroes Q_i depend on $\vec{Y}$ and are in general disjoint from the set $\{R_i\}$ of zeroes of $\vartheta\left[\begin{smallmatrix}\vec{a}\\ \vec{b}\end{smallmatrix}\right](\vec{z}-\vec{w})$.

A useful abelian differential to define is

$$h^2(z) \equiv \vec{\omega}(z) \cdot \partial_{\vec{z}}\vartheta(0),$$

where $\vartheta(\vec{z})$ is a theta function of odd characteristic. Differentiating the relation $\vartheta(\vec{z}-\vec{R}_i) \equiv 0$ once or twice with respect to z and setting $z = R_i$, one learns that $h^2(z)$ has double zeroes at each of the $g-1$ points R_i. Because these are all of the zeroes of $h^2(z)$, the 'square root' $h(z)$ is a well-defined half-order differential.[32]

The Riemann–Roch theorem can be stated as follows. Given points P_i on a compact Riemann surface R of genus g, one formally defines a *divisor* $\aleph = P_1^{l_1} \ldots P_n^{l_n}$, with degree $d[\aleph] = \sum_i l_i$. (See refs. [34,33,35].) The dimension of the vector space of meromorphic functions which are multiples of $\aleph^{-1}$, *i.e.* which have poles of order at most l_i at the points P_i, is denoted by $r[\aleph^{-1}]$. Similarly, $i[\aleph]$ denotes the dimension of the vector space of abelian differentials which are multiples of $\aleph$, *i.e.* have zeroes of order at least l_i at P_i. Then

$$r[\aleph^{-1}] = d[\aleph] + i[\aleph] - g + 1. \qquad \text{(A.3)}$$

Finally we mention the simplifications which occur for the case of the torus ($g = 1$): The theta functions have no spurious zeroes, and the vector indices on $\vec{z}, \vec{w}, \vec{z}_i$ can be dropped, as R is now isomorphic to $Jac(R)$. The period matrix Ω is replaced by the single modulus τ. Also, there is only one theta function of odd characteristic, conventionally denoted as $\vartheta_1(z) \equiv \vartheta\left[\begin{smallmatrix}1/2\\ 1/2\end{smallmatrix}\right](z)$.

APPENDIX B

Homology Basis for Closed Loops

In this appendix, we will discuss how to find a basis of linearly independent 'closed loops' on a Riemann Surface R of genus g with L twist field insertions. The main result, that the number of independent loops is $L + 2g - 2$, holds for any configuration of L twists having net twist zero. This number matches the total number of linearly independent cut differentials at genus g, a result which is needed in order to be able to determine the coefficients A_{ij} and B_{ij} appearing in the Green's functions (3.9) and to normalize the classical solutions via eq. (5.2). Of the $L + 2g - 2$ independent loops for genus g, $2g$ can be taken to be the canonical homology basis for the surface without branch cuts. By adding multiples of these $2g$ loops, any closed loop can be made into a homologically trivial loop (on the uncut surface). Thus the problem is reduced to finding $L-2$ linearly independent loops for the L twist configuration on the sphere (See figure 5). Recall that loops which are related by a shift of sheets are not considered independent because they lead to the same monodromy constraints.

Bershadskii and Radul[22] found a basis for the case in which one of the twist fields, say $\sigma_L(z_L, \bar{z}_L)$, has an integer $\frac{k_L}{N}$ which is relatively prime with respect to N. Their construction is described in this paragraph and illustrated in figure 6. One runs a branch cut from the 'base point' z_L to each of the other twist locations $z_1, \ldots, z_{L-1}$. To generate a loop γ_i, start at some point P_0 on the complex plane near the cut from z_L to z_i and rotate clockwise about z_L a number of times until the point returns to P_0 on a sheet which differs from the original sheet by k_i (mod N). This can always be done because k_L is relatively prime to N. Now rotate counterclockwise once around z_i to complete the loop γ_i. Only $L-2$ of the $L-1$ loops so constructed are linearly independent, because the sum of all of them can be pulled off to infinity. On the other hand, the independent loops do provide a basis because we can sequentially subtract these loops from an arbitrary closed loop, until the remaining loop winds only about z_L some multiple of N times,

which means that it is trivial.

Now consider the example of a Z_6 orbifold with two 1/2 twists and three 1/3 twists inserted onto the sphere. No twists are relatively prime with respect to N, so we cannot build up the kind of basis described above. We would like to explicitly construct $5 - 2 = 3$ linearly independent closed loops, in a way that will generalize to arbitrary collections of twists. Our approach is to start with a *six* twist configuration, replacing one of the 1/3 twists by two 1/6 twists in close proximity. Since 1 is trivially relatively prime to $N = 6$, we can use one of the 1/6 twists as the base point in a Bershadskii-Radul basis of loops for the new configuration. This basis is described by the following table:

loop \ twist	1/6	1/6	1/3	1/3	1/2	1/2
γ_1	1	-1	0	0	0	0
γ_2	2	0	-1	0	0	0
γ_3	2	0	0	-1	0	0
γ_4	3	0	0	0	-1	0
γ_5	3	0	0	0	0	-1

Table 1.

The values in the table denote the number of clockwise encirclements of each point and all of the branch cuts originate at the first 1/6 twist.

Only four of the five loops are linearly independent; we can generate γ_5 by subtracting the first four from the trivial loop $(-1,-1,-1,-1,-1,-1)$. (Of course for non-zero genus this loop may not be homotopically trivial, but it will be homologically trivial.) We would like to form from the four independent loops three independent linear combinations which circle both 1/6 twists an equal number of times (mod 6) on each sheet, so that they are not 'pinched' when the

two 1/6 twists coalesce. Then these unpinched loops can be used as a basis for the original five-point configuration.

First let us change the basis of loops $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ for the six twist configuration so that it includes the loop $\gamma_0 = \gamma_4 - \gamma_2 = (1,0,1,0,-1,0)$. This can be done for example by replacing γ_2 and γ_4 by γ_0 and γ_4', where

$$\begin{pmatrix} \gamma_0 \\ \gamma_4' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} \gamma_2 \\ \gamma_4 \end{pmatrix}. \tag{B.1}$$

Equation (B.1) is a legitimate change of basis because the 2×2 integer matrix has determinant one and hence is invertible over the integers, so that the original basis vectors can be recovered from the new basis. (A general change of n-dimensional basis is a $SL(n, \mathbf{Z})$ transformation.) Next we subtract multiples of γ_0 from γ_1, γ_3 and γ_4' to get three loops winding equal times about the two 1/6 twists on each sheet. These three loops together with γ_0 also form a basis for the six twist configuration, because the transformation matrix is in $SL(4, \mathbf{Z})$. And the three loops by themselves give the desired basis for the five twist configuration. By further manipulations, this basis can be simplified into the basis displayed in figures 7 (a)–(c). Note that we may need to use 'sheet-shifted copies' of the loops in the above transformations in order that the final three loops do not pass between the two 1/6 twists – otherwise a loop may wind around one of the twists clockwise on one sheet but counterclockwise on another sheet (for example) so as to remain pinched between the two twists.

Now we turn to the most general L twist configuration. If all the twist integers k_i contain a common factor n_c, then the problem of constructing the independent loops is equivalent to the problem for the integers k_i/n_c. (Adjacent sheets can be taken to correspond to a phase difference of $e^{2\pi i n_c/N}$ rather than $e^{2\pi i/N}$.) So we can assume that the k_i have no common factor. The general strategy is the same as in the previous example. We consider first a $L+1$ twist configuration where the k_1/N twist is replaced by a $1/N$ twist and a $(k_1-1)/N$ twist. For this configuration, we use the Bershadskii and Radul basis of L loops

γ_i (of which $L-1$ are linearly independent), taking the $1/N$ twist as base point. Table 2 lists how many times each loop encircles each of the $L+1$ twists.

loop \ twist	$1/N$	$(k_1-1)/N$	k_2/N	k_3/N	$\cdots$	k_L/N
γ_1	k_1	-1	0	0	$\cdots$	0
γ_2	k_2	0	-1	0	$\cdots$	0
γ_3	k_3	0	0	-1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
γ_L	k_L	0	0	0	$\cdots$	-1

Table 2.

Define the 'pinch number' for a loop to be the number of times it encircles the $1/N$ twist minus the number of times it encircles the $(k_1-1)/N$ twist, *i.e.* the difference between the entries in columns 1 and 2 in table 2. We now change the basis $\{\gamma_1,\ldots,\gamma_{L-1}\}$ to one in which $L-2$ of the basis loops have pinch number zero; these loops serve as the desired basis for the original L twist configuration. The change of basis proceeds in two steps. The first step is to construct a $(L-1)$-dimensional basis including a loop γ_0 with a pinch number m which is relatively prime to N. The second (easier) step is to add multiples of γ_0 to the remaining $L-2$ basis loops, so as to convert them to the desired $L-2$ independent loops with pinch number zero (mod N). This can be done because m and N have no common factor, and the replacement of the $L-2$ loops is a legitimate change of basis.

All that remains is to construct a basis containing γ_0, which we will do iteratively. Denote the greatest common divisor of a set of n integers $\{k_i\}$ by $d=(k_1,\ldots,k_n)$. If $d=(k_1,k_2)$, then one can always find two integers p_1 and p_2

such that $p_1 k_1 + p_2 k_2 = d$. Therefore the transformation

$$\begin{pmatrix} \gamma_1' \\ \gamma_2' \end{pmatrix} = \begin{pmatrix} p_1 & p_2 \\ -\frac{k_2}{d} & \frac{k_1}{d} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{B.2}$$

is a change of basis — the matrix has determinant one. The new basis includes a loop γ_1' with pinch number $d = (k_1, k_2)$. Similarly we can construct a linear combination of the loops γ_1' and γ_3 with pinch number $(d, k_3) = (k_1, k_2, k_3)$ and incorporate this loop into the basis. By iteration we find a basis including a loop γ_0 which has pinch number $m = (k_1, \ldots, k_{L-1})$. Now recall that $\sum_{i=1}^{L} k_i = MN$ with $M \in \mathbf{Z}$. Suppose that m and N have a common factor. Then $k_L = MN - \sum_{i=1}^{L-1} k_i$ and m would have a common factor, so all L of the k_i would have a common factor $n_c = (m, k_L) > 1$, and we described above how to treat this case. Therefore we can take m and N to be relatively prime, and we are done. Hence every L twist configuration on a genus g surface has a homology basis for the cut abelian differentials of rank $L + 2g - 2$.

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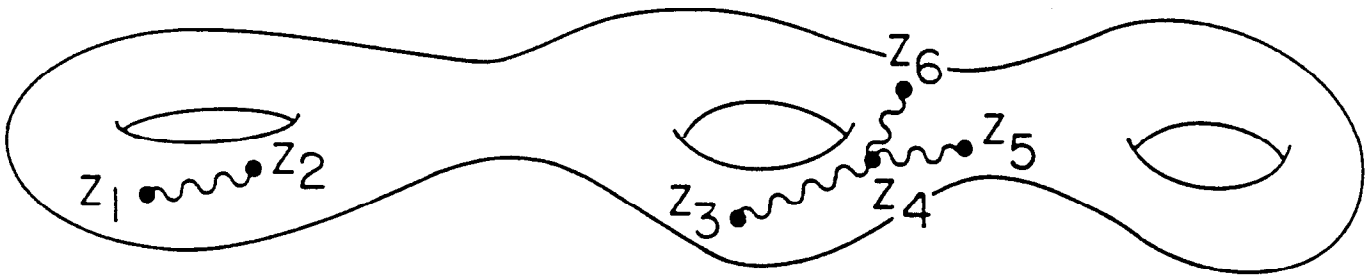

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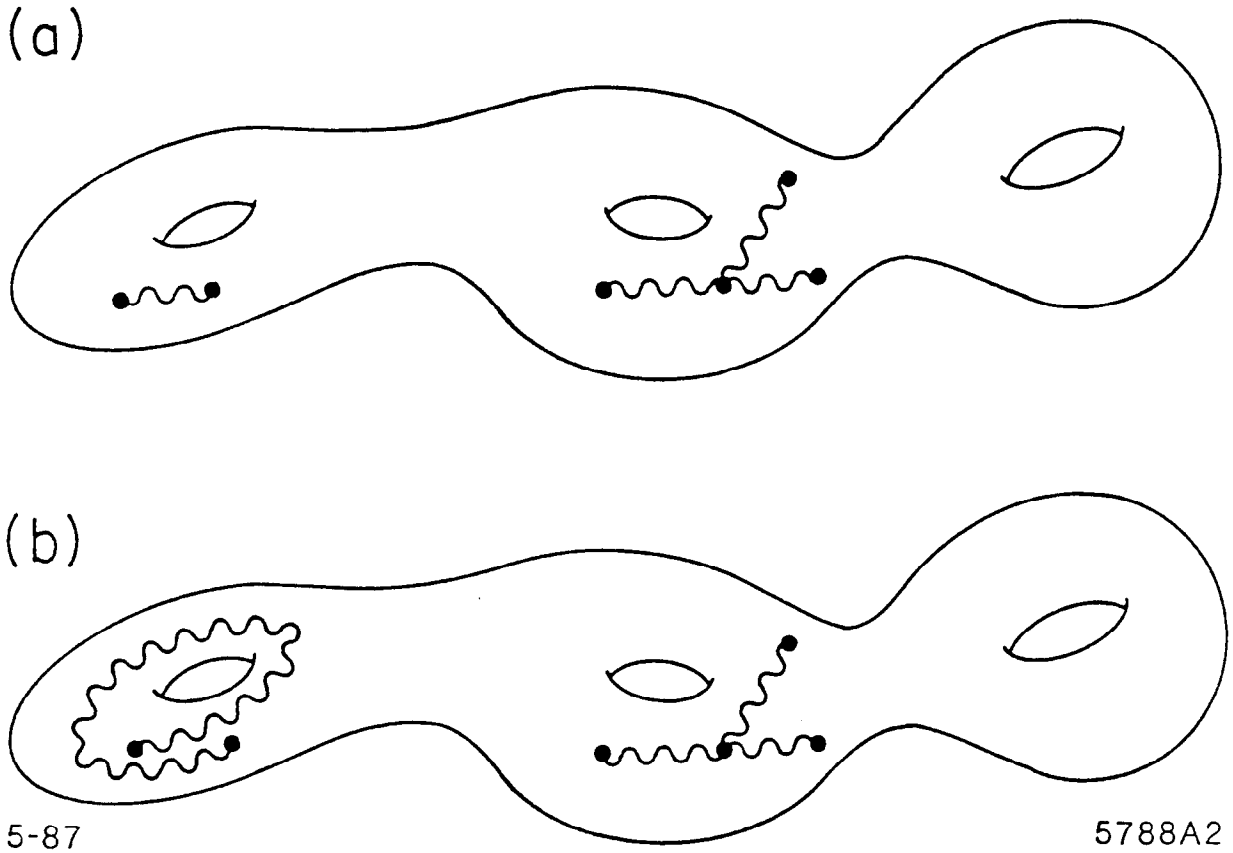
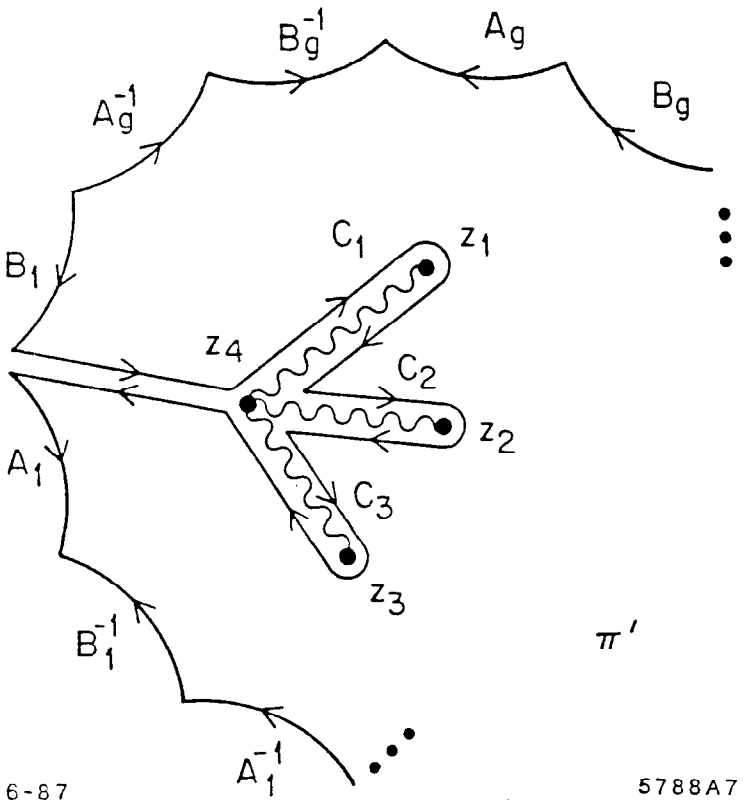


Fig. 2



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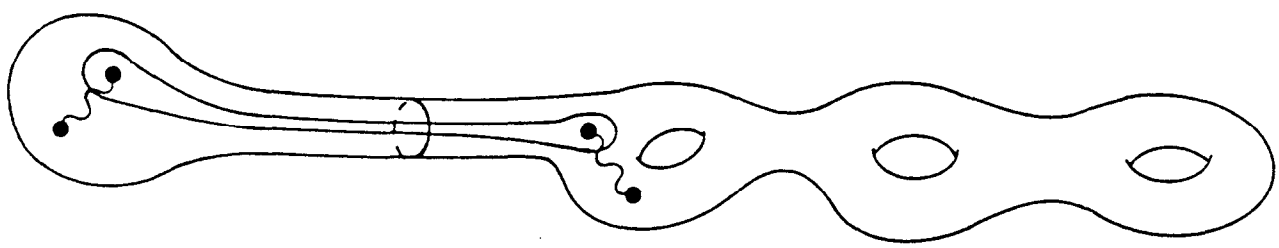

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Fig. 4

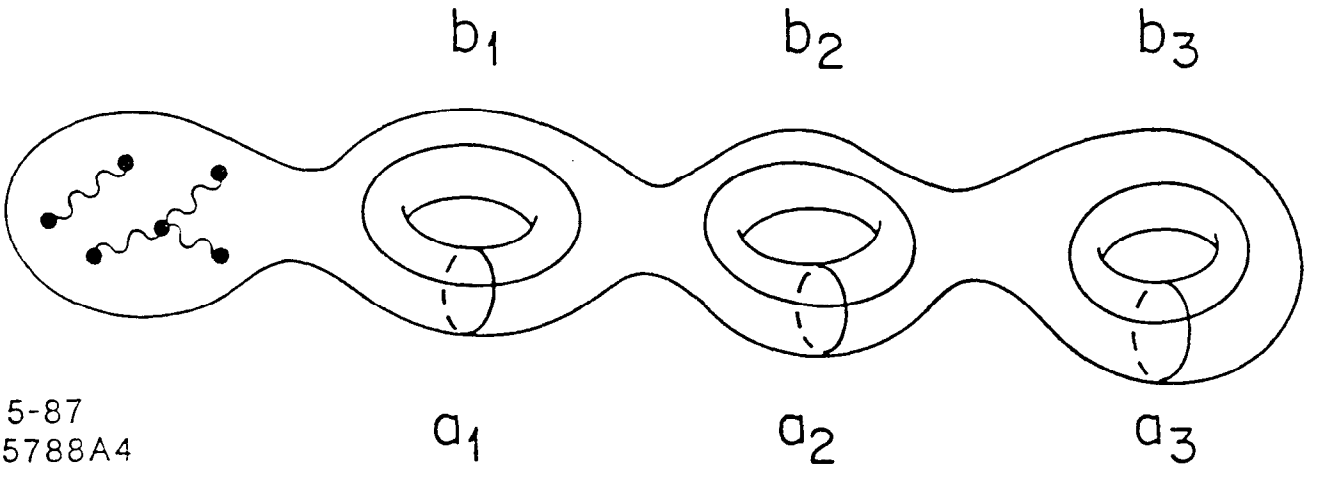

Fig. 5

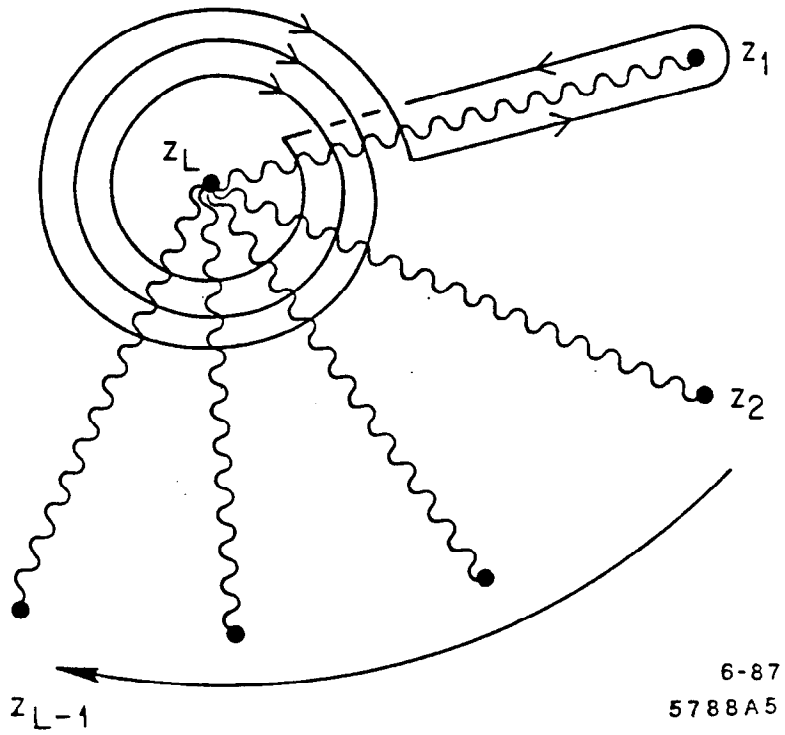

Fig. 6

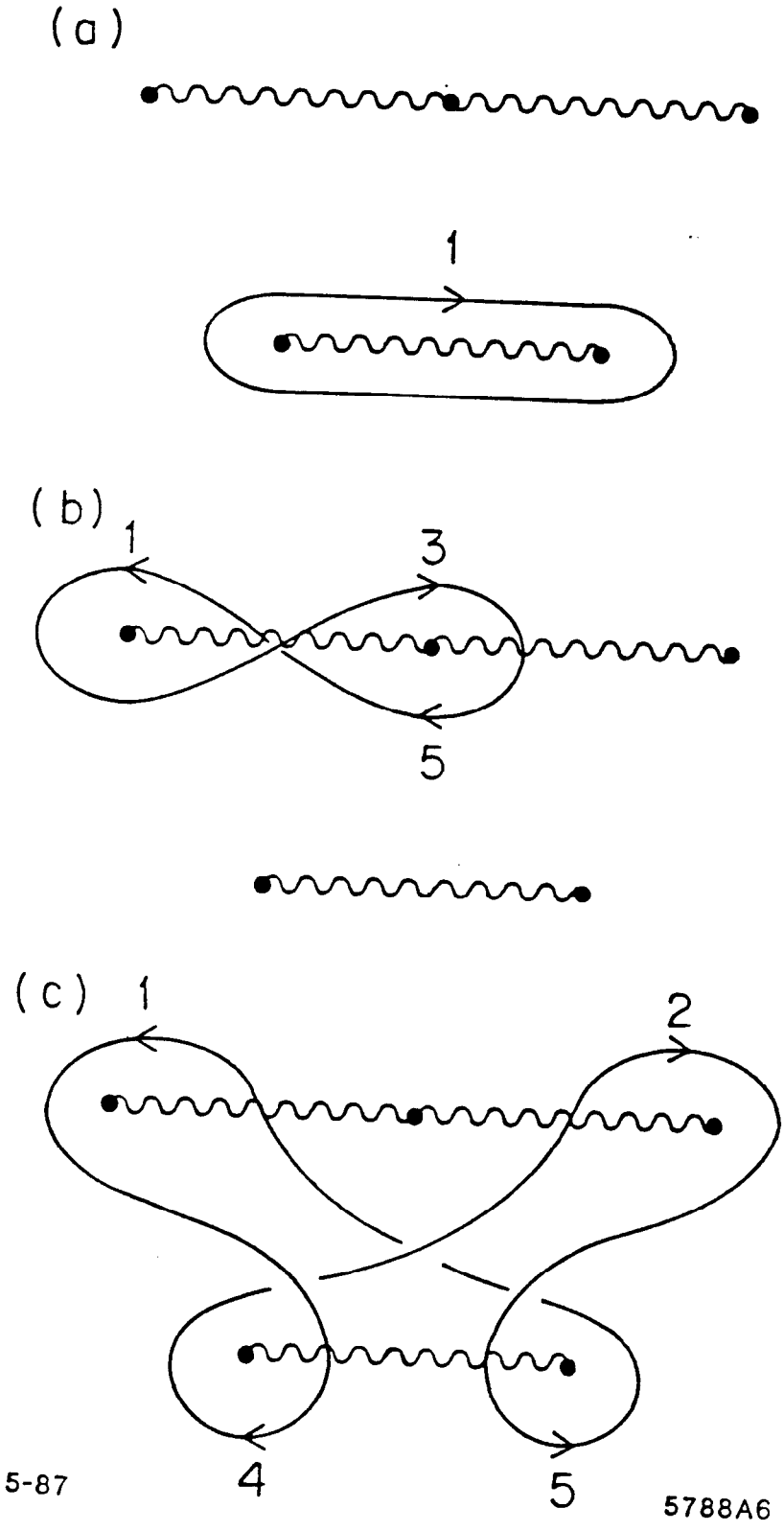

Fig. 7