SLAC–PUB–4271
March 1987
T/E

THE BETHE-MAXIMON RESULT*

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ABSTRACT

We develop and illustrate a very simple eikonal technique in the framework of one-particle quantum mechanics for calculating radiation during scattering and apply it to derive the celebrated result of Bethe and Maximon for bremsstrahlung from a point coulomb target with a large charge Ze. This approach is a modification of that developed in a previous paper for calculating radiation during the collision of electron-positron beam pulses (beamstrahlung). The differences in the physics of scattering from a localized target and from an extended target are clarified. A second reason for the present note is the simplicity of our method and its possible extension to other processes.

Submitted to *Physical Review D*

⋆ Work supported by the Department of Energy, contract DE–AC03–76SF00515.

1. Introduction and Motivation

In 1954, Bethe and Maximon[1] calculated the differential cross sections for bremsstrahlung by high energy electrons and for electromagnetic lepton pair production in a coulomb field of arbitrary charge Ze without the use of the Born approximation. Their paper was a major tour de force of analysis utilizing appropriate approximations to exact continuum relativistic coulomb wave functions. The first treatment to lowest order in $Z\alpha$ was of course given by Bethe and Heitler.[2]

Since this pioneering work, several useful approximations for treating high energy scattering processes have been developed. These lead to considerable simplification in the treatment of radiation during the scattering of electrically charged high energy particles. These include in particular the eikonal method in potential scattering,[3–6] operator methods for calculating radiation from an electron orbiting in a homogeneous purely magnetic field,[7] and light-cone quantization techniques applied to QED in the infinite momentum limit.[8]

In this paper we develop and illustrate a very simple eikonal technique in the framework of one particle quantum mechanics for calculating radiation during scattering. This method was developed in connection with a recent study[9] of radiation during the collision of two beam pulses ('beamstrahlung '), but it was found to be inappropriate (*i.e.*, wrong) for that problem because of the extended nature of the target pulses- in the rest frame of one of the pulses, its length grows as $E/m = \gamma$. However for finite potential sources, this method leads directly and simply to the same expression for the amplitude as found by Bjorken, Kogut, and Soper,[8] and confirms the results of Bethe and Maximon for bremsstrahlung during scattering in a pure coulomb field of arbitrary charge Ze.

2. T-Matrix

Consider the T-Matrix for the scattering of a particle with free Green's function G in an external vector potential wV, where w is a scale parameter that will eventually be set equal to one,

$$T = wV\,(1 - GwV)^{-1}\,. \tag{2.1}$$

Introducing the electromagnetic potential A for the radiation field in the standard gauge invariant manner leads to the substitution

$$wV \rightarrow wV - A \cdot j \tag{2.2}$$

and the T-matrix becomes

$$T[A] = (wV - A \cdot j)\,(1 - G(wV - A \cdot j))^{-1}\,. \tag{2.3}$$

The matrix elements relevant for elastic scattering

$$T[0, wV] = R^T\,wV = wV R\,, \tag{2.4}$$

and for the emission of one photon

$$T[1, wV] = -\,R^T\,A \cdot j\,R\,, \tag{2.5}$$

take a simple form. The resolvents for the coulomb problem have been introduced as

$$R = (1 - GwV)^{-1} \qquad \text{and} \qquad R^T = (1 - wVG)^{-1} \; . \tag{2.6}$$

The obvious generalization of (2.5) to describe the emission of n photons is

$$T[n, wV] = (-1)^n R^T (A \cdot j \; g)^{n-1} A \cdot j \, R \; , \tag{2.7}$$

where the Green's function g in the external field wV is

$$g = G \, R^T = R G \; . \tag{2.8}$$

Note for later use the relations

$$\frac{dR}{dw} = R G V R \qquad \text{and} \qquad \frac{dR^T}{dw} = R^T V G R^T \; . \tag{2.9}$$

Now the derivative of $T[1, wV]$ with respect to the scale of the potential w is easily expressed as

$$\frac{dT[1, wV]}{dw} = - R^T [V \, g \, A \cdot j + A \cdot j \, g \, V] R \; , \tag{2.10}$$

The matrix element of (2.10) between appropriate initial and final (electron) states has the graphical interpretation illustrated in Fig. 1: (a) the exact wave function for the incident electron in the field wV radiates a quantum $(A \cdot j)$, propagates again in the external field wV and is scattered by the potential V into its final state (again an exact scattering state in the field wV); (b) the second term of (2.6) reverses the order of scattering and radiation in the usual way as demanded by gauge invariance. Once we develop a suitable approximation for (2.10) , the radiation matrix element of interest is given by the integral

$$T_e[1,V] = \int_0^1 dw \frac{dT[1,wV]}{dw} \quad . \tag{2.11}$$

3. Eikonal Approximation

In this section we use the physical picture and approximations of eikonal propagation to derive a tractable expression for the scattering amplitude valid in the high energy regime. An eikonal-type approximation for the spatial part of the wave function including the higher order corrections was discussed in reference 9 . In this approach, the resolvent R acting to the right on the incident plane wave produces a wave with outgoing boundary conditions. For a such a wave with momentum $\vec{p}_i$ along the $z-$axis, the result is

$$R \exp(i\vec{p}_i \cdot \vec{r})\, u(p_i) = \exp(i\Phi_i(x))\, u(p_i) \ , \tag{3.1}$$

where

$$\Phi_i = p_i z - \chi_0(z,b) - \frac{1}{p}\left[\chi_1(z,b) + i\chi_2(z,b)\right] - O(1/p^2) \ , \tag{3.2}$$

with the eikonal phase given by

$$\chi_0(z,b) = w \int_{-\infty}^{z} dz' \, V(z',b) \ . \tag{3.3}$$

The leading $(1/p)$ corrections are given in ref 9 but will not be needed here as we shall demonstrate.

Similarly, the resolvent kernel acting to the left on the final state plane wave produces

$$u^{\dagger}(p_f)\exp(i\vec{p}_f \cdot \vec{r})\, R_T = u^{\dagger}(p_f)\, \exp(i\Phi_f(x)) \tag{3.4}$$

where

$$\Phi_f = \vec{p}_f \cdot \vec{r} + \tau_0(z,b) + \frac{1}{p}\left[\tau_1(z,b) + i\tau_2(z,b)\right] - O(1/p^2) \ , \tag{3.5}$$

with

$$\tau_0(z,b) = w \int_z^\infty dz' V(z',b) \ . \tag{3.6}$$

The line integrals in these phases are along the incident z–direction for the small scattering angles relevant to our problem.

The higher order terms, χ_1 and τ_1, that it proved necessary to retain in the bremsstrahlung analysis for extended targets can be dropped here as negligible, and truly of order $(1/p)$. The same is true for χ_2 and τ_2, and they are also negligible. In ref 9 the beam pulse was Lorentz *expanded* in its rest frame by a factor γ. In this situation, the seemingly small higher order term in $(1/p)$ was in fact promoted to leading order by the effects of the target length. Here, we are concerned with *localized* scattering potentials and it is easy to see that the higher terms are indeed small and vanish for sufficiently high energies. In particular, for a coulomb potential, straightforward calculation shows that

$$\frac{\chi_1}{p\chi_0} \sim \frac{Z\alpha}{\ln\gamma} \to 0 \tag{3.7}$$

and can be neglected for the situation of interest where $Z\alpha \leq 1$. In contrast, for finite range potentials the ratio is even smaller, $\frac{\chi_1}{p\chi_0} \sim \frac{1}{\gamma} \to 0$.

The eikonal Green's function in the external field wV can be directly computed in a simple form. From (3.1) and (3.2) we see that the effect of the interaction is summarized in an eikonal phase so that all we need compute is the free Green's function for straight line propagation along the direction of the very large momentum $\vec{p}\,'$ where $p'^2 \sim m^2$. Approximating

$$(p'+l)^2 - m^2 = 2p'\cdot l + l^2 + p'^2 - m^2 \sim 2p'\cdot l$$

for the propagation from $r_1 = (\vec{b}, z_1)$ to $r_2 = (\vec{b}, z_2)$, we obtain

$$g(r_2, r_1; p') = g_0(r_2, r_1; p')\,(\not{p}' + m)\ , \tag{3.8}$$

where

$$\begin{aligned} g_0(r_2, r_1; p') &= -i \exp\left[-iw\chi_{21}\right] \int \frac{d^4 l}{(2\pi)^4} \frac{\exp[-i(p'+l)\cdot(r_2 - r_1)]}{+2p'\cdot l} \\ &= -\frac{1}{2} \exp\left[-iw\chi_{21}\right] \int_0^\infty dt \exp(-itm^2)\delta^4(x - y - p't)\ , \end{aligned} \tag{3.9}$$

with the shorthand notation

$$\chi_{21} = \int_{z_1}^{z_2} dz' V(z', b)\ .$$

The optimal choice for the parameter p' depends on details of the process and kinematics. For calculating the matrix element (2.10) for an electron with p_i to radiate a photon k and scatter to a final electron with p_f, the natural choice is $p' \equiv p_i - k$ for the amplitude corresponding to Fig 1a. In this case g describes the eikonal correction to free propagation after the incident electron emits the photon. Correspondingly for Fig 1b, $p' \equiv p_f + k$.

Inserting (3.1) , (3.2) , and (3.3) into (2.10) we obtain the eikonal approximation to the matrix element:

$$\left\langle p_f; k \left| \frac{dT[1, wV]}{dw} \right| p_i \right\rangle = \int d^4 r_2 d^4 r_1 \frac{e}{\sqrt{2k}} \times \\ \exp[ip_f \cdot r_2 - i\tau_0(z_2)]\ U\ \exp[-ip_i \cdot r_1 - i\chi_0(z_1)]\ , \tag{3.10}$$

where

$$U \equiv u^\dagger(p_f)\Big(V(r_2)g(r_2,r_1;p_i-k)\epsilon\cdot j(r_1)\,\exp(ik\cdot r_1)$$
$$+\,\epsilon\cdot j(r_2)\exp(ik\cdot r_2)\,g(r_2,r_1;p_f+k)V(r_1)\Big)u(p_i)\ . \tag{3.11}$$

Defining $q = p_f + k - p_i$, and $\chi(b) = \tau_0(z_2) + \chi(z_2, z_1) + \chi_0(z_1)$, the matrix element becomes

$$\left\langle p_f;k\left|\frac{dT[1,wV]}{dw}\right|p_i\right\rangle \propto \int d^4r\,\exp[iq\cdot r - iw\chi(b)]\,V(r)\frac{e}{\sqrt{2k}}U_0\ , \tag{3.12}$$

where

$$U_0 = \int_0^\infty dt\,\exp(-itm^2)\,u^\dagger(p_f)M_0\,u(p_i)\ , \tag{3.13}$$

with

$$M_0 = \exp[-i(k-p_i)\cdot(p_i-k)t](\not{p}_i - \not{k} + m)\epsilon\cdot j(r-(p_i-k)t)+$$
$$\exp[+i(k+p_f)\cdot(k+p_f)t]\epsilon\cdot j(r+(p_f+k)t)(\not{p}_f + \not{k} + m)\ . \tag{3.14}$$

We also find explicitly the standard eikonal phase,

$$\chi(b) = \int_{-\infty}^{\infty} dz' V(z',\vec{b})\ . \tag{3.15}$$

The key to further simplifications is to realize that the arguments in the current operators tend to asymptotically free regions outside the range of the potential. For example, $(p_i - k)t \sim \gamma m t$ along the z−axis, and since the range in the matrix element is $0 < t \sim \frac{1}{m^2}$, we have

$$j(r-(p_i-k)t) \;\rightarrow\; j(z\rightarrow -\infty) \equiv j_{initial} \equiv \vec{\alpha} \tag{3.16}$$

where $\vec{\alpha}$ is the Dirac free current operator. Similarly

$$j(r+(p_f+k)t) \;\rightarrow\; j(z\rightarrow +\infty) \equiv j_{final} \equiv \vec{\alpha} \;\; . \tag{3.17}$$

This reduction of the current j to asymptotic free currents was not applicable in our beamstrahlung study that involved Lorentz extended pulses. Here however, it permits (3.12) to be simplified by directly performing the t integral:

$$\left\langle p_f;k \left| \frac{dT[1,wV]}{dw} \right| p_i \right\rangle \propto \int d^4r \; \exp[iq\cdot r - iw\chi(b)]\, V(r) \frac{e}{\sqrt{2k}} U_1 \;, \tag{3.18}$$

where

$$U_1 \;=\; u^\dagger(p_f) \left[\frac{(\not p_i - \not k + m)}{2p_i\cdot k} \epsilon\cdot\vec{\alpha} - \;\epsilon\cdot\vec{\alpha} \frac{(\not p_f + \not k + m)}{2p_f\cdot k} \right] u(p_i) \;. \tag{3.19}$$

This can be written as

$$\equiv\; i\frac{e}{\sqrt{2k}}\, \delta(energy) \int d^3r\, V(r) \epsilon\cdot J_{free}\; \exp[i\vec{q}\cdot\vec{r} - iw\chi(b)] \;. \tag{3.20}$$

Integrating with respect to w and setting $w=1$ we finally obtain

$$T(p_f,k;p_i) \;=\; i\frac{e}{\sqrt{2k}}\, \delta(energy)\, \epsilon\cdot J_{free} \int d^3r\, V(r)\; \exp[i\vec{q}\cdot\vec{r}]\, \frac{1-\exp[-i\chi(b)]}{\chi(b)} \;\; . \tag{3.21}$$

In the high energy limit the longitudinal momentum transfer vanishes as $1/\gamma$. For a localized potential $V(r)$ we can replace $\vec{q}\rightarrow\vec{q}_\perp$ and further reduce the above to

$$T(p_f, k; p_i) = i\frac{e}{\sqrt{2k}}\,\delta(energy)\,\epsilon\cdot J_{free}\int d^2b\,\exp(i\vec{q}_\perp\cdot\vec{b})\,\left(1-\exp[-i\chi(b)]\right)\ , \tag{3.22}$$

a result derived earlier by Bjorken, Kogut, and Soper[8] .

It is now trivial to display the result of Bethe and Maximon: For a coulomb potential,

$$V \equiv \frac{Z\alpha}{\sqrt{z^2+b^2}}\ , \tag{3.23}$$

the necessary integrals are

$$\begin{aligned}\chi(b) &= \lim_{L\to\infty}\int_{-L}^{L}\frac{Z\alpha dz}{\sqrt{z^2+b^2}} \\ &= Z\alpha\ln\left(\frac{4L^2}{b^2}\right)\end{aligned} \tag{3.24}$$

and

$$\int d^2b\,\exp(i\vec{q}_\perp\cdot\vec{b})\left(\frac{1}{b^2}\right)^{iZ\alpha} = \frac{1}{q^{2(1-iZ\alpha)}}\,\frac{\Gamma(1-iZ\alpha)}{2^{2iZ\alpha}\Gamma(iZ\alpha)}\ . \tag{3.25}$$

The rapidly oscillating phase factors disappear when we take the modulus of the matrix element; viz

$$\left|\int d^2b\,\exp(i\vec{q}_\perp\cdot\vec{b})\left(\frac{1}{b^2}\right)^{iZ\alpha}\right|^2 = \frac{Z^2\alpha^2}{|q_\perp|^4}\ . \tag{3.26}$$

Thus bremsstrahlung cross-section approaches the Born approximation result at very high energies. From the above our main result evidently follows, $d\sigma = d\sigma_{Bethe-Maximon}$.

This same limit does not apply for pair-production for which corrections in $(Z\alpha)^2$ remain finite in the high energy limit.

For pair production it is necessary to include the difference in paths between the electron and positron; otherwise their charges cancel ($\sim$ locally) and there is no scattering in the coulomb field. Thus a simple one-particle eikonal treatment is inadequate and one has to reproduce, in some form or other, the result in ref. 8 for the internal wave function of the photon. Not surprisingly one finds corrections[1] to the Born approximation in this case.

4. Summary

We have studied the problem of bremsstrahlung from a localized target in a form that allows a direct comparision with our treatment of scattering from an extended target. Conditions for the validity of the eikonal approximation in these cases and the physical differences in the two processes are discussed.

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FIGURE CAPTIONS

Figure 1a. A graphical interpretation of the formula for the derivative of the T-matrix with respect to the potential strength is given. The incident electron propagates in the field wV radiates a quantum $(A \cdot j)$, propagates again in the external field wV and is scattered by the potential V into its final state (again an exact scattering state in the field wV).

Figure 1b. The same as above except that the electron scatters from the potential V before it radiates the photon.

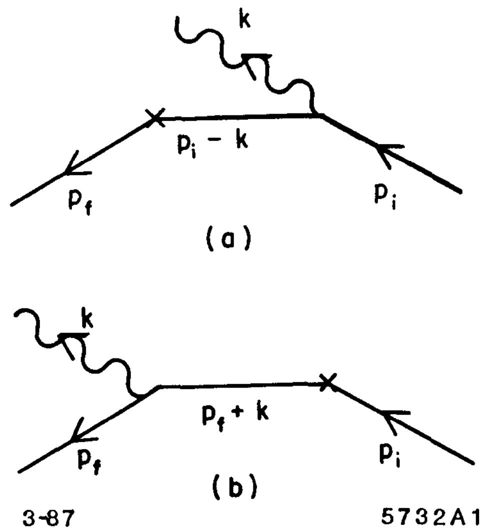

FIG. 1