SLAC – PUB – 3990
June 1986
T

BOUNDARY CONDITIONS: CHIRAL JACOBIANS AND WESS-ZUMINO TERMS*

DANIEL BOYANOVSKY

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

ABSTRACT

We show that the Jacobians associated with chiral rotations can be understood as arising from a change in the boundary conditions when fermionic theories are defined in a finite volume. In particular, a suitable choice of boundary conditions renders consistent an otherwise anomalous theory leading naturally to a Wess-Zumino term in the effective action.

Submitted to *Physics Letters B*

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.

It is by now fairly well known that because of the chiral anomaly, the fermionic path integral is not invariant under a chiral rotation even when the fermion masses are zero.

In a series of papers, Fujikawa[1] has shown that the fermionic measure is not invariant under a chiral rotation (change of variables) and there is a non-trivial Jacobian associated with this transformation.

This Jacobian plays a very important role and in many cases allows the exact computation of the low energy properties of the effective bosonic theories after "integrating out" the fermions.[2] Furthermore the evaluation of these fermion determinants yields a Wess-Zumino[3] term in the effective action of the bosonic fields.[4] Chiral Jacobians and Wess-Zumino terms also arise in anomaly free theories after decoupling heavy fermions and are fundamental for the consistency of the resulting low-energy effective theories.[5]

More recently it has been suggested that certain anomalous theories can be rendered consistent by adding a Wess-Zumino term to the effective action.[6,7] Usually the chiral Jacobians and ensuing Wess-Zumino terms are discussed in a path integral framework since they are related to the fermionic measure. It is therefore of interest to study these aspects at the level of canonical quantization.

In this letter we try to understand the effect of chiral rotations and the ensuing Jacobians in a Hamiltonian formulation of the theory. We first define the theory in a finite spatial volume and carefully analyze the nature of the boundary conditions and their effect on the ground state structure of the theory. Then the chiral Jacobians are related to the response of the ground state to the change in the boundary conditions under a chiral rotation. We are able to reproduce in a simple way the well-known Jacobians of the $1+1$ and $3+1$ dimensional abelian gauge theories.

Boundary conditions in $1+1$ dimensions:

The first quantized Dirac Hamiltonian in this case reads

$$H = -i\gamma^5 \, \frac{d}{dx} + \ldots \tag{1}$$

where the dots stand for mass terms and couplings to external (real) background fields. The theory is defined in the interval $-L \leq x \leq L$. The most general boundary conditions (b.c.) are the ones that render H self-adjoint; it is then straightforward to see[8] that they are classified as

$$\text{type 1}: \quad \psi(L) = e^{i(\alpha+\gamma^5\beta)} \, \psi(-L) \tag{2}$$

$$\text{type 2}: \quad \chi^+(L)\gamma^5\psi(L) = 0 \quad \chi^+(-L)\gamma^5\psi(-L) = 0 \tag{3}$$

where χ and ψ are any wavefunctions and α, β arbitrary real parameters. In this letter we only study massless theories and their chiral properties, for which only type 1 boundary conditions are suitable. In general, type 2 b.c. break chirality.

Consider the simplest case when there are no masses or background fields in Eq. (1). The eigenstates of H form right (R) ($\gamma^5 = +1$) and left (ℓ) ($\gamma^5 = -1$) branches with the b.c. given by Eq. (2).

$$\psi_R(L) = e^{i(\alpha+\beta)}\psi_R(-L) \qquad \psi_\ell(L) = e^{i(\alpha-\beta)}\psi_\ell(-L) \tag{4}$$

with eigenvalues (n is an integer)

$$E_{R,n} = \frac{\pi}{L}\left(n + \frac{\alpha+\beta}{2\pi}\right) \qquad E_{\ell,n} = -\frac{\pi}{L}\left(n + \frac{\alpha-\beta}{2\pi}\right) \, . \tag{5}$$

The twisted b.c. induce an asymmetry in the spectrum. Notice that we could have traded the twisted b.c. in Eq. (4a) for static background fields in the

Hamiltonian since the eigenstates of

$$H = -i\gamma_5 \frac{d}{dx} + \gamma^5 A_1 + A_0 \tag{6}$$

are of the form

$$\chi(x) = e^{i\left(kx - \int_{-L}^{x}[A_1(y)+\gamma^5 A_0(y)]dy\right)} \chi_0 \tag{7}$$

with χ_0 a constant spinor (eigenstate of γ_5) and choose

$$\alpha = \int\limits_{-L}^{L} A_1(y)dy\,; \quad \beta = \int\limits_{-L}^{L} A_0(y)dy$$

and $\chi(L) = \chi(-L)$.

The ground state of the theory is defined by filling up all the negative energy states. As usual, the second quantized field is expanded in terms the single particle wavefunctions and creation and annihilation operators.

We define the ground state expectation values of the right (left) charges as the *total* number of *occupied* right (left) states, using a heat kernel regularization

$$Q_R = \lim_{S\to 0^+} \sum_{E_{R,n}\leq 0} e^{-|E_{R,n}|^2 S}\ ; \qquad Q_\ell = \lim_{S\to 0^+} \sum_{E_{\ell,n}\leq 0} e^{-|E_{\ell,n}|^2 S}. \tag{8}$$

Using (5) it is straightforward to find

$$Q_R = \lim_{S\to 0^+}\left[\frac{L}{S\pi} + \frac{\alpha+\beta}{2\pi} - 1\right]\ ; \quad Q_\ell = \lim_{S\to 0^+}\left[\frac{L}{S\pi} - \frac{(\alpha-\beta)}{2\pi} - 1\right]\ . \tag{9}$$

The above charges still have to be normal ordered. We choose to normal order with respect to $\alpha = \beta = 0$ (periodic b.c.). Actually Eq. (9) is true for $0 <$

$\alpha+\beta<1$ (same for $\alpha-\beta$). As α and β are changed adiabatically from zero to their final values we see that whenever $\alpha+\beta$, $(\alpha-\beta)$ pass through an integer, one right (left) level crosses $E=0$; this level is occupied (empty) in the adiabatic approximation. Alternatively, the charges could have been defined by $(-1/2)$ times the spectral asymmetry $\sum_{E_n} \mathrm{sign}(E_n)\, e^{-|E_n|S}$ giving the same result. The normal ordered *adiabatic* charges read

$$\overline{Q}_R = \frac{\alpha+\beta}{2\pi}\ , \overline{Q}_\ell = -\frac{(\alpha-\beta)}{2\pi}\ ;\ \overline{Q} = \overline{Q}_R + \overline{Q}_\ell = \frac{\beta}{\pi}\ ;\ \overline{Q}_5 = \overline{Q}_R - \overline{Q}_\ell = \frac{\alpha}{\pi}\ . \quad (10)$$

Consider the action (still in $1+1$ dimensions)

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ + \gamma^5 \partial\!\!\!/ \theta)\psi \qquad (11)$$

where $\theta(x)$ is a background field (taken as static in what follows). The Hamiltonian corresponding to Eq. (11) (defined in the interval $-L \le x \le L$) is

$$H = -i\gamma^5\partial_x + \frac{d\theta}{dx} \qquad (12)$$

and we assume periodic b.c. $\psi(L) = \psi(-L)$.

Performing the chiral rotation

$$\psi = \chi\, e^{-i\gamma^5\theta(x)} \qquad (13)$$

the Lagrangian (11) becomes a free field Lagrangian $\mathcal{L} = \overline{\chi}(i\partial\!\!\!/)\chi$, but now the boundary conditions for χ are

$$\chi(L) = e^{i\gamma^5\Delta\theta}\chi(-L) \qquad \Delta\theta = \theta(L) - \theta(-L)\ . \qquad (14)$$

Certainly this chiral rotation (change of variables) does not change the local physics, which can be seen as follows. The solutions of $H\psi = E\psi$ with H given by Eq. (12) are of the form

$$\psi_R(x) = \phi_R e^{ik_R x - i\theta(x)} \ , \quad \psi_\ell(x) = \phi_\ell e^{ik_\ell x + i\theta(x)} \qquad (\gamma_5 \phi_{R,\ell} = \pm\phi_{R,\ell}) \tag{15}$$

with $E_R = k_R$, $E_\ell = -k_\ell$. Periodic boundary conditions yield

$$k_R = \frac{\pi}{L}\left(n + \frac{\Delta\theta}{2\pi}\right) \qquad k_\ell = \frac{\pi}{L}\left(n - \frac{\Delta\theta}{2\pi}\right) .$$

These are the same eigenvalues obtained from the Hamiltonian of Eq. (12) with $d\theta/dx = 0$ but with the wave functions obeying the twisted boundary conditions Eq. (14). In fact, the chiral rotation cancelled the background field in Eq. (12) but at the expense of twisting the b.c. The physics has not changed when the proper change in the b.c. is taken into account. In both cases the vacuum charge is given by (see Eqs. (10)).

$$\overline{Q} = \frac{\Delta\theta}{\pi} . \tag{16}$$

In terms of path-integrals, we can write

$$\int [\mathcal{D}\psi] \ e^{i\int \overline{\psi}(i\partial\!\!\!/ + \gamma^5 \partial\!\!\!/\theta)\psi d^2x} = J[\theta] \int [\mathcal{D}\chi] e^{i\int \overline{\chi}(i\partial\!\!\!/)\chi d^2x} . \tag{17}$$

Where the Jacobian accounts for the change in the measure, or

$$J[\theta] = \frac{\det[i\partial\!\!\!/ + \gamma^5 \partial\!\!\!/\theta]}{\det[i\partial\!\!\!/]} . \tag{18}$$

In the above discussion we assumed that the eigenfunctions of H obeyed periodic b.c.; these are the b.c. assumed on the left-hand side of Eq. (17). Since the fermionic path integral on the r.h.s. of (17) corresponds to a *free* theory, periodic b.c. are assumed in the denominator of (18). Then performing the chiral rotations on the fields and taking into account the change in the b.c. yields

$$J[\theta] = \frac{\det[i\partial\!\!\!/]_{T.b.c.}}{\det[i\partial\!\!\!/]_{P.b.c.}} \tag{19}$$

where T.b.C (P.b.C) stands for twisted (periodic) b.c. clearly,

$$\overline{Q} = \int dx \, \frac{i\delta \ell n \det[i\partial\!\!\!/ + \gamma^5 \partial\!\!\!/ \theta]}{\delta(\partial_x \theta)} = \int dx \, \frac{i\delta \ell n \, J[\theta]}{\delta(\partial_x \theta)} = \frac{1}{\pi} \int dx \, \frac{d\theta}{dx} \, . \tag{20}$$

In Eq. (20) the charge is normal ordered with respect to the vacuum with periodic b.c as imposed by Eq. (19). Assuming locality and Lorentz covariance $J[\theta]$ can be read off as

$$J[\theta] = e^{(i/2\pi)\int (\partial_\mu \theta)^2 d^2x} \, . \tag{21}$$

The physics of the Jacobian now becomes clear; it is compensating for the change in the boundary conditions of the fields. The observables (currents, charges) are invariant under a chiral change of variables,when the change in b.c. is properly accounted for, and that is exactly what the Jacobian does. It is straightforward to generalize Eq. (21) when background gauge fields are present. Now consider the Hamiltonian of the chiral Schwinger model in the interval $-L \leq x \leq L \quad (A_\mu(\pm L) = 0)$

$$H = (-i\gamma_5)\partial_x + [A_0(x) + A_1(x)] \, \left(\frac{1+\gamma_5}{2}\right) \, . \tag{22}$$

Since only right-handed fermions are coupled to A_μ this theory is anomalous.[9]

The eigenstates of (22) are the right-handed spinors

$$\psi_R(x) = \chi_R e^{i\left(kx - \int_{-L}^{x} [A_0(y)+A_1(y)]dy\right)} \qquad \gamma_5 \chi_R = +\chi_R \tag{23}$$

and we assume the right-handed b.c. (the left-handed are assumed to have periodic b.c. and decouple)[10]

$$\psi_R(L) = e^{i(\alpha+\beta)}\psi_R(-L) \quad \alpha = -\int_{-L}^{L} \pi(y)dy \quad \beta = \int_{-L}^{L} dy\, \partial_y \phi(y) = \Delta\phi \ . \tag{24}$$

where we have introduced the local interpolating fields π, ϕ. With the solutions (23) and the b.c. (24) we find

$$\overline{Q}_R = \frac{1}{2\pi}\,(\delta + \alpha + \beta) \qquad \delta = \int_{-L}^{L} (A_0 + A_1)\, dy \ . \tag{25}$$

When the background gauge fields vary in time adiabatically, $\dot{\overline{Q}}_R = \frac{\dot{\delta}(t)}{2\pi}$. Since the type 1 b.c. (Eq. (2)) implies $J_x(L) = J_x(-L)$, the total change in the charge can be written as

$$\begin{aligned} \Delta Q_R &= \int_0^T dt \int_{-L}^{L} \partial_\mu J^{\mu R} \\ \partial_\mu J^{\mu R} &= \frac{1}{2\pi}\,(\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu) + \ldots \end{aligned} \tag{26}$$

where we used Lorentz covariance. The dots in (26) stand for terms that cannot be obtained from the above arguments. They reflect the ambiguity in the gauge variant piece in (26)[8] and can be obtained from known results.[10] They are not relevant for what follows (alternatively we can work in a transverse gauge). From

Eq. (25) and (26) we see that if we *assume* that the b.c. (24) are dynamical and $\pi(y,t) = \partial_0\phi(y,t)$ then Eqs. (25) and (26) imply

$$
\begin{aligned}
J^{\mu R} &= \frac{1}{2\pi}\ \left[(g^{\mu\nu}+\epsilon^{\mu\nu})\,A_\nu - (g^{\mu\nu}-\epsilon^{\mu\nu})\,\partial_\nu\phi\right] \\
\partial_\mu J^{\mu R} &= \frac{1}{2\pi}\ \left[(\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu) - \partial_\mu\partial^\mu\phi\right]\ .
\end{aligned}
\tag{27}
$$

. In order to cancel the anomaly ϕ must satisfy the equation of motion obtained from

$$
\mathcal{L}_{W-Z}[\theta, A] = \frac{1}{4\pi}\ (\partial_\mu\phi)^2 + \frac{\phi}{2\pi}\ (\partial_\mu A^\mu + \epsilon^{\mu\nu}\partial_\mu A_\nu)\ . \tag{28}
$$

. However, we now see that $\mathcal{L}_{W-Z}$ is the Wess-Zumino action proposed in Refs. (6-7). The theory whose action is $\mathcal{L}[\psi, A] - \mathcal{L}_{W-Z}[\theta, A]$ is then free of anomalies. At this stage, assuming dynamical b.c. to cancel anomalies is perhaps as unphysical as *postulating* the above form for $\mathcal{L}_{W-Z}$. However, we are investigating the possibility that $\mathcal{L}_{W-Z}$ can be obtained after decoupling of heavy fermions in an anomaly free theory.[11]

The same idea allows one to compute the Jacobian in 3 + 1 dimensions. Consider a constant background (Abelian) magnetic field $A_\mu = (0, 0, Bx, 0)$ ($\overrightarrow{B} = B\hat{e}_z$). We choose the chiral basis and solve for the eigenstates of

$$
H\psi = (-i\overrightarrow{\alpha}\cdot\overrightarrow{\nabla} - \overrightarrow{\alpha}\cdot\overrightarrow{A})\psi = E\psi \qquad \psi(x,y,t) = e^{ik_y y}e^{ik_z z}\phi(x)\ . \tag{29}
$$

The solutions are the familiar Landau levels with center at $x_0 = k_y/B$, and the wave functions are the solutions of the harmonic oscillator. The eigenvalues are

$(B > 0)$[12]

$$E_n(k_z) = \pm\sqrt{(2n+1)|B| - \sigma_z B + k_z^2} \qquad n \neq 0 \; . \tag{30}$$

For $n = 0$ there are two Weyl branches (right and left) with

$$E_R = +k_z \qquad E_\ell = -k_z \; . \tag{31}$$

Since the wave functions are localized in the x-direction, only those Landau levels that are a distance $1/\sqrt{|B|}$ from the x-boundaries will feel the boundary conditions, hence any effect of the b.c. along the x-directions will vanish as $L_x \to \infty$.

Along the y-direction a twist in b.c. will shift the values of k_y, thereby shifting the position of the Landau levels. By the previous reasoning, the b.c. in the y-direction will not affect the physics. But from Eq. (30) and (31), we see that a twist in the b.c. in the z-direction produces a shift in k_z and changes the energies eigenvalues. However, for the $n \neq 0$ branches, there is no asymmetry in the spectrum, (because of the $\pm$ in (30)). Only the Weyl branches (Eq. 31) have an asymmetry and give rise to a change in the vacuum charge. If H in Eq. (29) has a term $H^1 = \alpha_z \gamma^5 \partial_z \theta(z)$, this term can be rotated away by a chiral rotation at the expense of twisting the boundary conditions in the interval $-L_z \leq z \leq L_z$.

$$\psi(x, y, L_z) = e^{i\gamma^5 \Delta\theta} \psi(x, y, -L_z) \qquad \Delta\theta = \theta(L_z) - \theta(-L_z) \; . \tag{32}$$

Since the Weyl branches have the same dispersion relation as the 1+1 dimensional examples, and since the $n \neq 0$ branches do not contribute we find

$$Q = \frac{B(4L_x L_y)}{2\pi} \, \frac{\Delta\theta}{\pi} \tag{33}$$

where $B(4L_x L_y)/2\pi$ is the total number of states in the $n = 0$ (Weyl branches)[12] and $\Delta\theta/\pi$ is the result in one dimension (Eq. (16)). Lorentz covariance and locality then imply

$$J_\mu[A,\theta] - J_\mu[A,0] = -\frac{1}{4\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\partial_\nu\theta\,F_{\rho\sigma} \tag{34}$$

which is the result found in Ref.(13). Now from the expression $J^\mu(x) = -\frac{i\delta \ell n\, Z}{\delta A_\mu}$, and following the steps leading to Eqs. (17)-(20) the Jacobian is defined such that

$$\frac{i\delta \ell n\, J[\theta]}{\delta\, A_\mu} = J^\mu[A,\theta] - J^\mu[A,0]$$

The right hand side can be written in a way analogous to Eq. (19) but with the gauge fields included (the twist is now in the z-direction). Hence

$$J[\theta] = e^{-(i/8\pi^2)\int d^4x\;\epsilon^{\mu\nu\rho\sigma}\theta(x)\,F_{\mu\nu}\,F_{\rho\sigma}}\;. \tag{35}$$

This argument can be also carried out for a gauge vortex configuration[14] in the $x-y$ plane. The 2-dimensional zero modes yield the Weyl branches Eq. (31). More details of the results presented in this letter, as well as the treatment of massive theories, will be given in a forthcoming article.

ACKNOWLEDGEMENTS

It is a pleasure to thank R. Blankenbecler, T. Banks, M. Peskin, J. Kiskis, A. Dhar, E. Fradkin and M. Weinstein for helpful conversations and remarks.

REFERENCES

1. K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979); Phys. Rev. D21, 2848 (1983).
2. See for example: Selected Topics in Gauge Theories, W. Dittrich and M. Reuter (Lecture Notes in Physics, Springer 1985). C. M. Naón, Phys. Rev. D31, 2035 (1985) and references therein.
3. J. Wess and B. Zumino, Phys. Lett. 37B, 95 (1971).
4. See for example, P. DiVecchia in Recent Developments in Quantum Field Theory, edited by J. Ambjorn et al., North-Holland (1985).
5. E. D'Hoker and E. Farhi, Nucl. Phys. B248, 59 and 77 (1984).
6. L. D. Faddeev and S. L. Slatashvili, Phys. Lett. 167B, 255, L. Faddeev, Nuffield Workshop, Proc. (1985), to be published.
7. I. Halliday, E. Rabinovici, A. Schwimmer and M. Chanowitz, Nucl. Phys. B268, 413 (1986); M. Chanowitz, Phys. Lett. B171, 280 (1986).
8. See for example, H. Yamagishi, Phys. Rev. D27, 2383 (1983); B. Grossman, Phys. Rev. Lett. 50, 464 (1983); M. Stone, Phys. Rev. B31, 6112 (1985); S. M. Roy and Virendra Singh, Phys. Lett. 143B, 179 (1984); M. Weinstein, unpublished.
9. See for example, R. Jackiw in Relativity, Groups and Topology II, edited by B. deWitt and R. Stora (north-Holland 1984).
10. Alternatively twisted b.c. can be assumed for the left-handed fields and periodic for right handed and take into account the currents of the left sector.
11. T. Banks and D. Boyanovsky, unpublished.

12. H. B. Nielsen and Masao Ninomiya, Phys. Lett. 130B, 389 (1983).

13. C. G. Callan and J. A. Harvey, Nucl. Phys. B250, 427 (1985).

14. J. Kiskis, Phys. Rev. D15, 2329 (1977); D. Boyanovsky, R. Blankenbecler and R. Yahalom, to appear in Nucl. Phys. B[FS].