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DOES THE TRANSITION TO CHAOS DETERMINE THE DYNAMIC APERTURE?*

JOHN M. JOWETT†
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We review the important notion of the dynamic aperture of a storage ring with emphasis on its relation to general ideas of dynamical instability, notably the transition to chaos. Practical approaches to the problem are compared. We suggest a somewhat novel quantitative guide to the old problem of choosing machine tunes based on a heuristic blend of KAM theory and resonance selection rules.

INTRODUCTION

Nowadays, machine designers are much exercised by the *dynamic apertures* of their storage rings. Roughly speaking, the dynamic aperture, or non-linear acceptance, is the region around the central closed orbit in which single particle motion is stable. The term itself only recently came into use but neatly encapsulates the essence of what it is meant to describe.

Formerly, the aperture *tout court* of a synchrotron or storage ring was what is now variously described as the physical or mechanical acceptance or aperture; that is, the aperture determined by the vacuum chamber or other material obstruction presented to the beam.

To explain the distinction, we recall that, in terms of action-angle variables of linearised motion (R.D. Ruth in Ref. 1), $\mathbf{J} = (J_x, J_y, J_s)$, $\boldsymbol{\phi} = (\phi_x, \phi_y, \phi_s)$, the radial displacement of a particle at some azimuth $\theta = s/R$ may be written

$$x(\theta) = \sqrt{2J_x\beta_x(\theta)}\cos\left(\phi_x + \psi_x(\theta)\right) + \eta_x(\theta)\underbrace{\sqrt{2J_s}\cos\left(\phi_s + \psi_s(\theta)\right)}_{\delta}. \quad (1)$$

The vertical displacement y is similar and the Hamiltonian has the form

$$H(\boldsymbol{\phi}, \mathbf{J}) = \boldsymbol{\nu}\cdot\mathbf{J} + \{\text{nonlinear terms in } \boldsymbol{\phi} \text{ and } \mathbf{J}\}. \quad (2)$$

The actions, $\mathbf{J}$, are exact invariants only insofar as the oscillations are linear, although perturbation methods may be used to find approximate invariants in certain other cases. The average of $J_{x,y}$ over all the particles in the beam is the corresponding emittance, $\varepsilon_{x,y}$. Part of the displacement is attributed to the instantaneous momentum deviation from a central value, $\delta = (p - p_0)/p_0$,

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† Permanent address: CERN, CH-1211 Geneva 23.

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through the dispersion function, η_x. The amplitude of the betatron part of the oscillation is given by the β-function. If the beams are bunched, δ undergoes synchrotron oscillations and is also a dynamical variable, as indicated in (1). Often, however, synchrotron oscillations are so slow that they can be regarded as a parametric modulation of betatron motion. The β-, η- and ψ-functions are determined by the focusing structure, independently of any particle.

Generally, the phases $\boldsymbol{\phi}$ may take all values independently with equal probability so that the mean square beam size is

$$\langle x(\theta)^2\rangle = \beta_x(\theta)\langle J_x\rangle + \eta_x(\theta)^2\langle J_s\rangle = \beta_x(\theta)\varepsilon_x + \eta_x(\theta)^2\sigma_\delta^2. \tag{3}$$

Similarly for $\langle y^2\rangle$. To avoid loss of particles in the tails of the distribution, these dimensions have to be significantly less than the physical aperture. How much less is different for e^+e^- storage rings, where the emittance is determined by radiation effects, and hadron colliders, where it depends essentially on the injection system. The beam size must also stand in a similar relationship to the dynamic aperture and, particularly in the new generation of large colliders and synchrotron light sources, this can turn out to be the more stringent requirement. The dynamic aperture is a 6-dimensional subset of $(\boldsymbol{\phi}, \mathbf{J})$ space.

The number of betatron or synchrotron oscillations per turn (as $\mathbf{J} \to 0$) is called the *tune* of the machine and is also determined by the lattice, although small adjustments are easily made. The tune vector $\boldsymbol{\nu} = (\nu_x, \nu_y, \nu_s)$ plays a fundamental rôle in determining single-particle stability. Instability is associated with resonance conditions $\mathbf{k}\cdot\boldsymbol{\nu} = p$ where $\mathbf{k} = (k_1, k_2, k_3) \in \mathbf{Z}^3$ is an integer vector and $p \in \mathbf{Z}$ is an integer related to the harmonic of the revolution frequency at which nonlinear terms drive the resonance. A working point diagram like Figure 1(a) is a useful aid in the avoidance of resonances.

Some resonances do not lead directly to instability but instead cause beating of amplitudes. Nonetheless it is important to avoid these too since the beating may lead to particle loss at the physical aperture. Thus, the two concepts of aperture are not completely distinct.

If the machine were perfect, the betatron oscillations could be made linear (with pure quadrupole focusing). The dynamic aperture would be infinite in both betatron degrees of freedom but the momentum acceptance would be almost zero. Since $\sigma_\delta \neq 0$, the natural chromaticity (*i.e.* betatron tune dependence on momentum) has to be compensated by the introduction of sextupoles. Careful arrangements of sextupole families are used in large machines to cancel the linear and quadratic parts of the chromaticity. A few other harmful effects, *e.g.* systematic excitation of low order resonances, can be removed at the same time. However the interactions of one sextupole with others (or itself on later turns) generate driving terms for resonances of arbitrarily high order and there remain the field errors (especially in superconducting magnets).

The oscillations are then necessarily non-linear and the betatron tunes, where they can be defined at all, are functions $\boldsymbol{\nu}(\mathbf{J})$ with many singularities and folds related to resonances and chaotic layers. It is impossible to avoid this, since the resonance planes form a dense web in $\boldsymbol{\nu}$-space and, generically, there is no way of eliminating the tune-dependence on amplitude. It is now a commonplace that the interactions of many resonances will inevitably lead to chaotic motion in some regions of the phase space.

THE PRACTICAL APPROACH: TRACKING

In practice, most estimates of dynamic aperture are made by computer tracking of particles through models of the lattice. Simulated random errors in magnetic fields or magnet positions may or may not be included.

Limited computer time forces a number of compromises on us: fast algorithms (*e.g.* thin lens approximation) have to be preferred to more accurate but slower methods; only a 2- or 3-dimensional section of initial condition phase space can be sampled; the number of turns of the machine must be limited, typically to a few hundred; the sampling of possible errors has to be extremely limited; although tracking results are very strongly affected by the tune values, only a few of these can be tested, *etc.*

In the face of such overwhelming difficulties, the practical stability criterion in fairly general use might be baldly paraphrased as: *If, for a given lattice design, with a few typical sets of random errors, and a particular set of tune values, a particle with initial betatron emittances ε_x, ε_y, an initial momentum deviation δ and initial phase zero in all three degrees of freedom survives for (say) 400 turns, then particles with the same amplitude and arbitrary phases will be stable for the same lattice with most other set of errors and most reasonable tunes.* A "reasonable" tune satisfies a number of well-known criteria concerning the avoidance of the resonances one can expect to be driven in a given lattice. For hadron colliders particularly, one may add the further caveat that the tracked orbit must remain essentially indistinguishable from a linear motion.

The dynamic aperture is then taken to be the set of all initial conditions generated in this way from the largest connected set of stable conditions (usually this includes the origin) in the sampling section.

Clearly this stability criterion is far removed from anything which can be found in the mathematical theory of dynamical systems. From a strictly logical point of view, it also sounds like wild optimism since we know that even millions of turns of apparently regular, stable motion may be just the initial segment of a chaotic orbit which will eventually find its way to large amplitude—such orbits can sometimes be found more quickly by finely scanning the sampling section.

Yet this approach seems to work quite well!

It is difficult to believe that the survival of the *last* invariant torus is an adequate criterion for stability of a particle beam. With or without the help of external perturbations, particles may easily jump or bypass such a flimsy barrier. One cannot avoid including some narrow chaotic regions in the dynamic aperture. On the other hand, regular orbits, particularly those associated with resonant beating, may have to be rejected.

It is not easy to distinguish regular from chaotic motion in systems with three degrees of freedom. Two-dimensional phase projections usually show only a cloud of points, but much more information can be obtained from watching a "movie" of the motion. Otherwise, computational means of making the distinction include Lyapounov exponents, reversal tests (A. Wrulich in Ref. 1) and

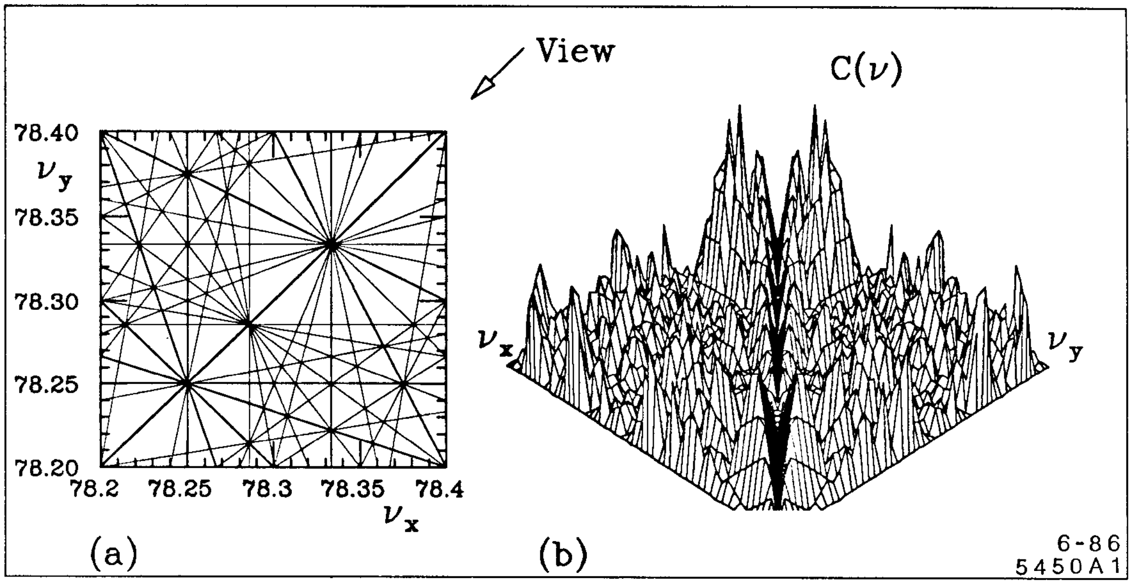

Fig. 1 Tune diagram and $C(\boldsymbol{\nu})$ for $\mathcal{K} = \{\mathbf{k} \in \mathbf{Z}^3 : \|\mathbf{k}\| \leq 7,\ k_3 = 0\}$

fractal dimensions of orbit power spectra (J.M. Jowett in Ref. 2).

A GUIDE TO CHOOSING $\boldsymbol{\nu}$

KAM theory suggests the definition of a function

$$C(\boldsymbol{\nu}) = \min_{\substack{\mathbf{k}\in\mathcal{K} \\ p\in\mathcal{P}}} \frac{|\mathbf{k}\cdot\boldsymbol{\nu} - p|}{\|\mathbf{k}\|^4}, \qquad \|\mathbf{k}\| \stackrel{\text{def}}{=} |k_1| + |k_2| + |k_3| \tag{4}$$

which, in a well-defined sense, measures the distance of $\boldsymbol{\nu}$ from the resonance most likely to influence it *a priori*; large values of $C(\boldsymbol{\nu})$ indicate a better chance of stability. The sets of integer vectors $\mathcal{K} \subset \mathbf{Z}^3$ and harmonics $\mathcal{P} \subset \mathbf{Z}$ are chosen with the help of selection rules reflecting the best available judgment of which resonances are important, ***e.g.***, if the lattice has superperiodicity N, then p must be a multiple of N for the systematic resonances. Placing an upper bound on the order $\|\mathbf{k}\|$ limits $\mathbf{k}$ to an octahedron. Figure 1(b) shows $C(\boldsymbol{\nu})$ in a small portion of the ν_x-ν_y plane. In one dimension, maximising $C(\boldsymbol{\nu})$ leads, in principle, to a $\boldsymbol{\nu}$ value related to the "golden mean" but its 3-dimensional analogue is not yet fully understood. If detailed information on resonance widths were available, it could be incorporated into (4). The width of the peaks of $C(\boldsymbol{\nu})$ is also important. In other words, to accommodate the distribution of amplitudes, and therefore tunes, we should look for regions, rather than points, in $\boldsymbol{\nu}$ space where $C(\boldsymbol{\nu})$ is large.

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