SLAC – PUB – 3982
May 1986
(T/E)

COMMENT ON THE ELECTRON ANOMALOUS MAGNETIC MOMENT BETWEEN CONDUCTING PLATES*

ANDREW C. TANG

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

Submitted to *Physical Review D*

*Work supported by the Department of Energy, contract DE–AC03–76SF00515.

1. Introduction

There have been a number of recent works that calculate the electron's magnetic moment between conducting plates separated by a distance a. These include Barton and Grotch,[1] Fischbach and Nakagawa,[2] Boulware, Brown and Lee,[3,4] and most recently, Bordag,[5] and Kreuzer and Svozil.[6] In this report, I will be considering the situation outlined by Fischbach and Nakagawa, and Kreuzer and Svozil: that of an essentially free electron moving in a weak magnetic field. It should be noted that this differs from Boulware, Brown and Lee, and Bordag's treatment of an electron trapped in the strong magnetic field of a Penning trap; and thus, the results of this paper are not directly applicable. The purpose of this comment is two-fold. First, the Abel-Plana and ϵ-averaging methods[7] of handling the difference of a divergent sum and integral are summarized and then applied to rederiving Kreuzer and Svozil's expression for the shift in the value of $a_e = \frac{1}{2}\,(g-2)$ between conducting plates from that in free space. I find the ϵ-averaging method particularly easy to follow, and hopefully, this method will make the final answer clearer. The second purpose is to clarify the present conflict in Fischbach and Nakagawa's and Kreuzer and Svozil's answer for Δa_e.

First, I need to derive an appropriate form for a_e. Let x denote the direction normal to the plates. Then doing the k_o and two of the $\vec{k}$ integrals in the usual Feynman rule expression for a_e,

$$a_e = \frac{2i\alpha m^2}{\pi^3} \int_0^1 dx \int d^4k \, \frac{x^2(1-x)}{[k^2 - m^2x^2]^3} \, , \tag{1.1}$$

gives the result

$$a_e = \frac{\alpha m^2}{\pi} \int_0^1 dx \int_0^\infty dk_x \, \frac{x^2(1-x)}{\left[k_x^2 + m^2x^2\right]^{3/2}} \quad . \tag{1.2}$$

An easier method of arriving at this answer is to do one of the $\vec{k}_\perp$ integrals in the light cone quantization expression for a_e,[8]

$$a_e = \frac{\alpha m^2}{\pi^2} \int_0^1 \frac{dx}{1-x} \int d^2\vec{k}_\perp \frac{1}{\left[m^2 - \frac{k_\perp^2 + m^2}{1-x} - \frac{k_\perp^2}{x}\right]^2} \quad . \tag{1.3}$$

Equation (1.2) was derived for an electron in free space. Naively, one might expect that the modification to this expression caused by placing the electron between conductors is to make the replacement

$$\int_0^\infty dk_x \;\rightarrow\; \frac{\pi}{a} \sum_{n=1}^{\infty} , \qquad k_x \;\rightarrow\; \frac{n\pi}{a} \quad . \tag{1.4}$$

Referring to the fully relativistic treatment of QED between conducting plates of Kreuzer and Svozil,[6] one finds that this is indeed the correction prescription for the geometry under consideration. Making this substitution and doing the x integral gives a final answer of

$$\Delta a_e = \frac{\alpha}{\pi} \left\{ h \sum_{\substack{n=1 \\ x=nh}}^{\infty} - \int_0^\infty dx \right\} \left[\log \frac{1+\sqrt{1+x^2}}{x} - 2\left(\sqrt{1+x^2} - x\right) \right] \tag{1.5}$$

where $h = \pi/ma$.

2. Mathematics of the Difference of a Divergent Sum and Integral

Most of what follows is from Barton's paper[7] on this topic. For a more detailed treatment, please refer to his work. The problem is to evaluate

$$D[f(n)] \equiv S - I \equiv \left[\sum_{n=0}^{\infty}{}' - \int_0^{\infty} dn \right] f(n) \tag{2.1}$$

where S and I are individually divergent. The primed sum means

$$\sum_{n=0}^{\infty}{}' f(n) \equiv \frac{1}{2}\, f(0) + \sum_{n=1}^{\infty} f(n) \quad . \tag{2.2}$$

To make sense of this expression, one needs to introduce a cutoff function $g(n|\lambda)$ and replace D by

$$D(\lambda) \equiv S(\lambda) - I(\lambda) \equiv \left[\sum_{n=0}^{\infty}{}' - \int_0^{\infty} dn \right] f(n|\lambda) \tag{2.3}$$

where

$$f(n|\lambda) \equiv f(n)\; g(n|\lambda) \quad . \tag{2.4}$$

The cutoff function g depends on the parameter λ and must satisfy a number of conditions including

i) $g(n|\lambda) \xrightarrow[\lambda\to\infty]{} 1$ and $\partial g^s/\partial n^s \xrightarrow[\lambda\to\infty]{} 0$ for $s = 1, 2, 3, \ldots$

ii) $g(n|\lambda)$ approaches zero fast enough so that $S(\lambda)$ and $I(\lambda)$ are convergent

iii) $D(\lambda)$ converges uniformly as $\lambda \to \infty$.

Certain other conditions on g and f may exist for the specific method employed. This will become clearer in what follows. An example of a cutoff function is $g(n|\lambda) = \exp(-n/\lambda)$. If $\lim\limits_{\lambda\to\infty} D(\lambda)$ exists as defined above and has a finite value independent of the specific form of g, one equates this with $D[f(n)]$.

Two methods of calculating $D(\lambda)$ are considered: the Abel-Plana formula and the ϵ-averaging method. The Abel-Plana formula is applicable to $f(z|\lambda)$ that are analytic in $z = x + iy$ and free of singularities either in the positive 1/2 plane $x \geq x_0$ for some $x_0 < N$ (N is a non-negative integer), or to the right of the wedge formed by the lines $y = \pm\ (x - x_0)\ \tan\phi,\ 0 < \phi \leq \frac{\pi}{2}$. $f(z|\lambda)$ must also vanish fast enough so that contour integrals draw no contribution from their arcs at infinity. For this class of f,

$$\left[\sum_{n=N}^{\infty}{}' - \int_{N}^{\infty} dn\right] f(n|\lambda) = f(N|\lambda)\left(\frac{1}{2} - \frac{\phi}{\pi}\right) + \int_{0}^{\infty} d\rho \left\{ \frac{e^{i\phi} f\left(N + \rho e^{i\phi}|\lambda\right)}{\exp\left(-2\pi i \rho e^{i\phi}\right) - 1} + \frac{e^{-i\phi} f\left(N + \rho e^{-i\phi}|\lambda\right)}{\exp\left(2\pi i \rho e^{-i\phi}\right) - 1} \right\} . \tag{2.5}$$

Once again, the details of deriving this are in Ref. 7. If the right hand side of this expression converges, the limit $\lambda \to \infty$ is taken by simply replacing $f(z|\lambda)$ by $f(z)$. For the case $N = 0,\ \phi = \pi/2$,

$$D[f(n)] = \lim_{\lambda \to \infty} D(\lambda) = i \int_{0}^{\infty} \frac{d\rho}{e^{2\pi\rho} - 1} \left[f(i\rho) - f(-i\rho)\right] . \tag{2.6}$$

Of course, for a specific f, one still needs to verify that a satisfactory cutoff does indeed exist so that $S(\lambda)$, $I(\lambda)$ converge and $D(\lambda)$ converges uniformly.

For cases in which the sum and integral can be done for some finite upper limit λ, the ϵ-averaging technique is usually easier to implement than the Abel-Plana formula. The procedure is as follows. First, separate λ into its integer and non-integer parts.

$$\lambda \equiv [\lambda] + \epsilon = \nu + \epsilon \tag{2.7}$$

and take

$$g(n|\lambda) = \theta(\lambda - n) . \tag{2.8}$$

At this point, the limit $\lambda \to \infty$ of

$$D(\lambda) = \left[\sum_{n=0}^{\infty}{}' - \int_0^{\infty} dn \right] \theta(\lambda - n) f(n)$$

$$= \left[\sum_{n=0}^{\nu}{}' - \int_0^{\lambda} dn \right] f(n) \equiv S(\nu) - I(\lambda) \tag{2.9}$$

is not well defined since $g(n|\lambda)$ clearly does not satisfy the three conditions outlined earlier (see following example for $f(n) = n$). In fact, one notes that due to the step-like nature of $S(\nu)$, $D(\lambda)$ oscillates violently as λ increases from one integer to the next. What needs to be done is a smoothing out of $S(\nu)$ by taking an average of $S(\nu)$ over values of ϵ. That is, formally replace ν by $\lambda - \epsilon$ and define

$$\bar{S}(\lambda) \equiv \int_0^1 d\epsilon \; S(\lambda - \epsilon) \; . \tag{2.10}$$

One then identifies

$$D[f(n)] = \lim_{\lambda \to \infty} \left[\bar{S}(\lambda) - I(\lambda) \right] \; . \tag{2.11}$$

The compatibility of this answer with that derived using the Abel-Plana formula is described in Ref. 7.

As a simple example, consider $f(n) = n$. From the Abel-Plana formula or the ϵ-averaging method,

$$D[n] = -\frac{1}{12} \; ; \tag{2.12}$$

whereas the naive method of calculating sum minus integral gives $D[n] = +\infty$:

$$D[n] \equiv \lim_{\nu \to \infty} \left[\sum_{n=0}^{\nu}{}' - \int_0^{\nu} dn \right] n = \lim_{\nu \to \infty} \frac{\nu}{2} \; . \tag{2.13}$$

3. Calculation of $g-2$ between Conductors

I will apply the results of the previous section to the problem of calculating Δa_e. For any physical system, electromagnetic modes with frequency greater than the plasma frequency of the conductors, Λ, do not experience the effect of the walls and are essentially free; thus, the upper limit in (1.5) should be replaced by Λ (see Refs. 2, 6). However, I will show that the dependence on Λ is weak and can be neglected for physical conductors. Making this replacement gives

$$\Delta a_e = \frac{\alpha}{\pi}\left\{h \sum_{\substack{n=1 \\ x=nh}}^{\nu} - \int_0^{\lambda} dx\right\} \left[\log \frac{1+\sqrt{1+x^2}}{x} - 2\left(\sqrt{1+x^2}-x\right)\right] \tag{3.1}$$

where $\lambda = \Lambda/mh = \Lambda a/\pi$ and $\nu = [\lambda] =$ integer part of λ.

Note that at this point, the sum and integral in Eq. (3.1) are individually finite for $\nu,\ \lambda \to \infty$ since the quantity within [...] goes like $\frac{-1}{4x^3}$ as $x \to \infty$. Unfortunately, I do not know of a closed expression for some of the sums involved. On the other hand, I note that for a typical metal, $\Lambda \sim 1$ eV, for which $\nu h \approx \Lambda/m = 1.96 \times 10^{-6} \ll 1$. This allows me to expand [...] in x, then keeping only the terms of $O(x^0)$, one obtains

$$\Delta a_e = \frac{\alpha h}{\pi}\left\{\sum_{n=1}^{\nu} - \int_0^{\lambda} dn\right\}\Big[\log 2 - 2 - \log h - \log n + O(nh)\Big] \,. \tag{3.2}$$

Now the sum and integral diverge, and I will make use of ϵ-averaging to define this expression. One defines

$$\Delta^{\lambda}\Big[f(n)\Big] \equiv \left[\overline{\sum_{n=1}^{\nu}} - \int_0^{\lambda} dn\right] f(n) \equiv \bar{S}(\lambda) - I(\lambda) \tag{3.3}$$

where the bar indicates ϵ-averaged. The only difference between Δ^{λ} and D

defined in Sec. 2 is the term $\frac{1}{2}\, f(0)$ in the sum. For $f(n) = 1$,

$$I(\lambda) = \lambda, \quad S(\nu) = \nu, \quad \bar{S}(\lambda) = \lambda - \frac{1}{2} \ ,$$

$$\Delta^{\lambda}[1] = \frac{1}{2} \ . \tag{3.4}$$

For $f(n) = \log n$,

$$\begin{aligned} I(\lambda) &= \lambda \log \lambda - \lambda \ , \\ S(\nu) &= \sum_{n=1}^{\nu} \log n = \log \nu! \end{aligned} \tag{3.5}$$

At this point, a further approximation needs to be made. For $a \gg 10^{-5}$ cm and $\Lambda = 1$ eV, $\Lambda a \approx \nu \gg 1$. Using the asymptotic form[9] for $\log \nu!$

$$S(\nu) = \nu \log \nu - \nu + \frac{1}{2} \log 2\pi\nu + \frac{1}{12\nu} - \frac{1}{360\nu^3} + O\left(\frac{1}{\nu^5}\right) \ , \tag{3.6}$$

and then substituting $\nu = \lambda - \epsilon$ and dropping terms of $O\left(\frac{1}{\lambda}\right)$ gives

$$S(\lambda - \epsilon) = \left(\lambda + \frac{1}{2} - \epsilon\right) \log \lambda - \lambda + \frac{1}{2} \log 2\pi + O\left(\frac{1}{\lambda}\right) \ . \tag{3.7}$$

ϵ-averaging this expression produces

$$\bar{S}(\lambda) = \lambda \log \lambda - \lambda + \frac{1}{2} \log 2\pi + O\left(\frac{1}{\lambda}\right) \ , \tag{3.8}$$

resulting in

$$\Delta^{\lambda}\Big[\log n\Big] = \frac{1}{2} \log 2\pi + O\left(\frac{1}{\lambda}\right) \ . \tag{3.9}$$

Using (3.4) and (3.9) in (3.2) gives us

$$\Delta a_e = \frac{\alpha h}{\pi}\left\{\left[1 - \frac{1}{2} \log \frac{4\pi}{h}\right] + O\left(\frac{1}{\Lambda a}\right) + O\left(\frac{\Lambda}{m}\right)\right\} \ . \tag{3.10}$$

Dropping the small $\frac{1}{\Lambda a}$, $\frac{\Lambda}{m}$ corrections gives a final answer of

$$\Delta a_e = \frac{\alpha}{2ma}\left(2 - \log 4ma\right) \ . \tag{3.11}$$

Note that this result is effectively independent of the type of conductor since all terms involving the conductor (*i.e.*, involving Λ) are 4-5 orders of magnitude smaller for a real conductor. Using the Abel-Plana formula (2.6) and looking up Barton's[7] answer for $D[\ln(n+\eta)]$ and $D[1]$ also results in (3.11).

4. Conclusions

Much care must be taken when calculating sum minus integral. When this is done correctly,

$$\Delta a_e = \frac{\alpha}{2ma}\left(2 - \log 4ma\right) \tag{4.1}$$

which agrees with the answer of Kreuzer and Svozil.[6] Recall that the derivation of this answer is valid for $\Lambda/m \ll 1$ and $\Lambda a \approx \nu \gg 1$. Kreuzer and Svozil show that Eq. (5.1) is also valid for the region $\Lambda/m \gg 1$. Note that for a perfect conductor ($\Lambda \to \infty$), the $\frac{1}{12}\left(\frac{\pi}{\Lambda a}\right) - \frac{1}{360}\left(\frac{\pi}{\Lambda a}\right)^3 \ldots$ correction to (5.1) goes to zero, giving a finite limit for Δa_e. Also note that, as expected, $\Delta a_e \to 0$ as $a \to \infty$.

I now need to say a few words about Fischbach and Nakagawa's answer[2] of

$$\Delta a_e = -\frac{\alpha}{2ma} \log 2a\Lambda \ . \tag{4.2}$$

First of all, this answer has the undesirable behavior of being infinite as $\Lambda \to \infty$. Formally, this answer can be derived by the following procedure. Define

$$\tilde{\Delta}^{\lambda}\Big[f(n)\Big] \equiv \Big[\sum_{n=1}^{\lambda} - \int_0^{\lambda} dn\Big] f(n) \tag{4.3}$$

with $\lambda =$ an integer. $\tilde{\Delta}^{\lambda}[f(n)]$ corresponds to using the non-differentiable cutoff function $g(n|\lambda) = \theta(\lambda - n)$ and not doing an ϵ-average. One easily finds

$$\tilde{\Delta}^{\lambda}[1] = 0, \qquad \tilde{\Delta}^{\lambda}\Big[\log n\Big] = \frac{1}{2}\,\log 2\pi\lambda\ . \tag{4.4}$$

Using $\tilde{\Delta}^{\lambda}$ in place of Δ^{λ} in (3.2) gives the result (4.2).

Finally, to get a feel for the size of the answer (4.1), let us set $a = 1$ cm. For this value,

$$\Delta a_e = -3.09 \times 10^{-12}\ . \tag{4.5}$$

For general interest's sake, I quote Van Dyck, Schwinberg and Dehmelt's[10] recent experimental result of

$$\frac{g_e}{2} = 1.001\ 159\ 652\ 209 \pm 0.000\ 000\ 000\ 031 \tag{4.6}$$

for the electron magnetic moment. I should note, however, that this paper considers the case of a free electron between conducting plates and therefore is not applicable to the Dehmelt experiment in which the electron is in a bound orbit.

ACKNOWLEDGEMENTS

The author would like to acknowledge useful discussions with Professor Stanley Brodsky of the Stanford Linear Accelerator Center and communications with Professors Maxmillian Kreuzer and Karl Svozil of the Technische Universität of Vienna.

REFERENCES

1. G. Barton and H. Grotch, J. Phys. **A10**, 1201 (1977); M. Babiker and G. Barton, Proc. R. Soc. London **A326**, 277 (1972).

2. E. Fischbach and N. Nakagawa, Phys. Rev. **D30**, 2356 (1984); Phys. Lett. **149B**, 504 (1984).

3. D. G. Boulware, L. S. Brown and T. Lee, Phys. Rev. **D32,** 729 (1985).

4. D. G . Boulware and L. S. Brown, Phys. Rev. Lett. **55,** 133 (1985).

5. M. Bordag, Phys. Lett. **171B,** 133 (1986).

6. M. Kreuzer and K. Svozil, Phys. Rev. **D34,** 1429 (1986).

7. G. Barton, J. Phys. **A14,** 1009 (1981).

8. S. J. Brodsky and S. D. Drell, Phys. Rev. **D22,** 2236 (1980).

9. CRC Standard Mathematical Tables, 24th ed., p. 408 (1976).

10. R. S. Van Dyck, in *Proc. of the Ninth Int. Conf. on Atomic Physics,* World Scientific, Singapore, eds. R. S. Van Dyck and E. N. Fortson (1985).