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Normal Coordinates, θ-Expansion and Strings on Curved Superspace*

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ABSTRACT

A method for normal coordinate expansion of superspace σ-model actions is developed and used to obtain a covariant θ-expansion of the heterotic Green-Schwarz superstring in curved superspace. It is shown that in the absence of background fermions the expansion terminates at order θ^2 in the light-cone gauge. An expression for the gauge-fixed action including background metric, torsion and Yang-Mills fields is given.

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Two-dimensional nonlinear σ-models seem to provide a suitable framework for studying string propagation in background fields. Recent investigations [1-9] of these models have given valuable insight into the structure of string theories. A remarkable fact that has emerged is that quantum consistency of these models requires the background fields to satisfy equations of motion identical to those derived from the field theory limit of strings [2, 7-9]. Although these studies have used both the Neveu-Schwarz-Ramond (NSR) [10] and the Green-Schwarz (GS) [11] formulations of the superstring, virtually all detailed quantum calculations have been carried out within the NSR formulation. The GS formulation has the advantage of being manifestly space-time supersymmetric, but progress with it has been hampered by the complicated form of the σ-model superfield action and by the apparent impossibility [12] of covariant quantization of the model. In the light-cone gauge, however, the flat superspace action (the free GS superstring) drastically simplifies and quantization of the theory is then straightforward. One might expect a similar simplification in the curved superspace action in this gauge. Should this be the case, a study of the quantum properties of the GS σ-models would then be no more difficult than that of the NSR σ-models (although of course manifest Lorentz covariance would be lost). To investigate this possibility one needs a method for obtaining θ-expansion of curved superspace σ-models. The purpose of this paper is twofold. First, we develop a general method for obtaining normal coordinate expansion of any curved superspace σ-model. This method is essentially an extension to superspace of Mukhi's method [13] for normal coordinate expansion of ordinary bosonic σ-models. The θ-expansion is then obtained as a special case of the superspace normal coordinate expansion. Second, using our expansion rules we argue that in the absence of background fermions the θ-expansion terminates at order θ^2 in the light-cone gauge for the heterotic string [15]. (The argument can be easily generalized to other superstrings as well). We also derive the light-cone gauge-fixed action for the heterotic string from the covariant action given in Ref.[9] involving background metric, torsion and Yang-Mills fields.[♮1]

Throughout this paper we will use the notation and conventions of Refs.[9] and [16]. Consider, then, a curved superspace with points parameterized by $Z^M = (X^m, \theta^\mu)$. For any function of Z a θ-expansion can in principle be obtained simply by Taylor expanding it around $\theta^\mu = 0$. Such an expansion is, however, noncovariant since the variables θ^μ are the coordinates of superspace. This is the same problem that one faces in carrying out background field expansion of ordinary bosonic σ-models and its resolution, as there, is to use normal coordinates. Below we give a supersymmetric version of the ordinary bosonic normal coordinate expansion method which is suitable for background field expansion of GS σ-models.[♮2]

Consider a point P in superspace with coordinates Z^M in some neighborhood of the origin Z_0^M. The point P is connected to the origin by some geodesic $Z^M(t)$. Specifying which geodesic P lives on and at what value of the affine parameter is equivalent to specifying its coordinates. But a geodesic through Z_0^M can be characterized by its tangent vector $y^A = y^M e_M{}^A(Z)$ at Z_0^M. Therefore, one can essentially adopt the components of the tangent vector at the origin and the affine parameter to parametrize the neighborhood of Z_0^M. The connection

♮1 GS σ-models in the light-cone gauge have previously been discussed, though not derived from a covariant action, in Ref. [6].

♮2 For more general applications see the superspace normal coordinate expansion method of Ref. [14].

between Z^M and these coordinates can be readily written down by solving the geodesic equation

$$D_t V^A = 0 \ , \tag{1}$$

where $V^A(t) = \dfrac{dZ^M(t)}{dt}\ e_M{}^A(Z(t))$ is the tangent vector to the geodesic and $D_t = \dfrac{dZ^M(t)}{dt} D_M$ is the covariant derivative along the geodesic, subject to the boundary conditions

$$Z^M \equiv Z^M(t=1) = Z_0^M + \epsilon^M(Z_0, y) \tag{2}$$

$$\begin{aligned} y^A \equiv V^A(t=0) &= \left(\frac{dZ^M(t)}{dt}\,|_{t=0}\right) e_M{}^A(Z_0) \\ &= y^M e_M{}^A(Z_0) \quad . \end{aligned} \tag{3}$$

The result is

$$\begin{aligned} \epsilon^M(Z_0, y) = y^M &- \frac{1}{2!}\, y^L y^N\ \Gamma_{NL}{}^M(Z_0) \\ &- \frac{1}{3!}\, y^L y^N y^P \left[\partial_P \Gamma_{NL}{}^M(Z_0) - 2\Gamma_{PN}{}^Q(Z_0)\Gamma_{QL}{}^M(Z_0)\right] \\ &+ \dots \end{aligned} \tag{4}$$

where

$$\Gamma_{NL}{}^M \equiv \frac{1}{2}\left[D_N e_L{}^A + (-)^{[N][L]} D_L e_N{}^A\right] E_A{}^M \ .$$

In the normal frame, where all geodesics are straight lines, (4) simply reads

$$\bar{\epsilon}^M(Z_0, \bar{y}) = \bar{y}^M \ , \tag{5}$$

where the bar denotes quantities referred to the normal frame.

Now, a general Taylor expansion of a curved-superspace σ-model action $I[Z]$ about the point Z_0 related to Z by (2) has the form

$$
\begin{aligned}
I[Z] = I[Z_0] &+ \int [d^2\sigma]\epsilon^M(\sigma)\frac{\delta}{\delta Z_0^M(\sigma)}\; I[Z_0] \\
&+ \frac{1}{2!}\int [d^2\sigma']\int [d^2\sigma]\epsilon^N(\sigma')\epsilon^M(\sigma)\frac{\delta}{\delta Z_0^M(\sigma)}\;\frac{\delta}{\delta Z_0^N(\sigma')}\; I[Z_0] \\
&+ \ldots
\end{aligned}
$$

where $\sigma^i = (\sigma^0, \sigma^1)$ are the world-sheet coordinates and $[d^2\sigma]$ is an appropriate invariant measure. The central observation of Ref. [13] is that in the normal frame in which ϵ^M is independent of Z_0, (5), this expansion factors out in the form

$$
\begin{aligned}
I[Z] = I[Z_0] &+ \int [d^2\sigma]\overline{y}^M(\sigma)\;\frac{\delta}{\delta Z_0^M(\sigma)}\; I[Z_0] \\
&+ \frac{1}{2!}\;\int [d^2\sigma']\overline{y}^N(\sigma)\;\frac{\delta}{\delta Z_0^N(\sigma')}\;\int [d^2\sigma]\overline{y}^M(\sigma)\frac{\delta}{\delta Z_0^M(\sigma)}\; I[Z_0] \\
&+ \ldots\quad .
\end{aligned}
\tag{6}
$$

In (6) the operator

$$
\int [d^2\sigma]\overline{y}^M(\sigma)\frac{\delta}{\delta Z_0^M(\sigma)}
$$

can obviously be replaced by the covariant expression

$$
\int\; [d^2\sigma]\overline{y}^M(\sigma)\overline{D}_M(\sigma)
$$

(since at each stage the derivative is acting on a scalar) which may then be referred back to an arbitrary frame,

$$\int [d^2\sigma] y^M(\sigma) D_M(\sigma) \ .$$

The expansion in (6) can then be written in the compact form

$$I[Z] = e^{\Delta[Z_0,y]}\ I[Z_0]\ , \tag{7}$$

where

$$\Delta[Z_0, y] = \int [d^2\sigma] y^A(\sigma) D_A(\sigma) \tag{8}$$

and

$$D_A(\sigma) \equiv E_A{}^N(Z_0(\sigma)) D_N(\sigma)$$

is the functional covariant derivative defined through its action on a vector $T^B(\sigma')$ as

$$D_A(\sigma) T^B(\sigma') = E_A{}^M(Z_0(\sigma))\ \frac{\delta T^B(\sigma')}{\delta Z_0^M(\sigma)}\ +\ \delta^{(2)}(\sigma,\ \sigma')(-)^{[A][C]} T^C(\sigma)\omega_{CA}{}^B(\sigma)\ , \tag{9}$$

where $\delta^{(2)}(\sigma, \sigma')$ is the invariant 2-dimensional δ-function and $\omega_{CA}{}^B$ is the superconnection. All we now need to obtain the normal coordinate expansion of I is the action of Δ on the basic constituents of I, together with the equation

$$\Delta y^A(\sigma) = \int [d^2\sigma'] y^B(\sigma') D_B(\sigma') y^A(\sigma) = 0\ , \tag{10}$$

which simply states that the tangent vector y^A is covariantly constant along the geodesic.

A covariant θ-expansion can be obtained as a special case of the above normal coordinate expansion by choosing the origin Z_0^M and the tangent y^M at the origin to the geodesic linking Z_0^M and $Z^M = (X^m, \theta^\mu)$ in a specific way. Let us make the choices

$$Z_0^M = (X^m,\ 0) \qquad y^M = (0,\ y^\mu)\ , \tag{11}$$

and go to the Wess-Zumino gauge in which the $\theta = 0$ components of the super-veilbein take the following form:

$$e_M{}^A(X) = \begin{pmatrix} e_m{}^a(X) & e_m{}^\alpha(X) \\ e_m{}^a(X) = 0 & e_\mu{}^\alpha(X) = \delta_\mu^\alpha \end{pmatrix} \quad . \tag{12}$$

Then

$$\begin{aligned} y^a &= y^M e_M{}^a(Z_0) \\ &= y^\mu e_\mu{}^a(X) \\ &= 0 \quad , \end{aligned} \tag{13}$$

and

$$\begin{aligned} y^\alpha &= y^M e_M{}^\alpha(Z_0) \\ &= y^\mu e_\mu{}^\alpha(X) \\ &= y^\mu \delta_\mu^\alpha \equiv \theta^\alpha\ . \end{aligned} \tag{14}$$

Thus, if we set $y^a = 0$ and $y^\alpha = \theta^\alpha$ and retain only the $\theta = 0$ components of all the superfields appearing in the expansion in (7), we will end up with a manifestly covariant θ-expansion for the action $I[Z]$.

Let us now apply this method to the curved superspace heterotic string action of Ref.[9]. In the conformal gauge the action is given by

$$I[Z] = \int d^2\sigma \big[\frac{1}{2}\eta^{ij}\phi V_i^a V_{ja} + \epsilon^{ij} V_i^D V_j^C B_{CD} + \psi^s \mathcal{D}_-^{st}\psi^t \big] \ . \tag{15}$$

Here

$$V_i^a \equiv \partial_i Z^M e_M{}^a(Z), \quad \mathcal{D}_-^{st}\psi^t \equiv \partial_- \psi^s - A_-^{st}\psi^t$$

and $A_- \equiv V_-^B A_B$ is the projection on the world-sheet of the 10-dimensional super Yang-Mills potential A_B. The subscript '$-$' refers to the world-sheet; $\partial_- \equiv (\partial_0 - \partial_1)$ and $V_-^A \equiv (V_0^A - V_1^A)$. ψ^s are world-sheet fermions representing the gauge degrees of freedom of the heterotic string. Also $\phi(Z)$ is the dilaton superfield and $B_{CD}(Z)$ are the components of the 2-form potential B in terms of which the gauge-invariant 3-form field strength H of the supergravity - super Yang-Mills system [16, 17] is defined by

$$H = dB + c_1\omega_{3YM} \ , \tag{16}$$

where $\omega_{3YM} = tr(AF - \frac{1}{3}\ A^3)$ is the super Yang-Mills Chern-Simons 3-form. If curvature-squared terms are added to the supergravity Bianchi identities (16) gets modified by a Lorentz Chern-Simons piece. We will briefly discuss this case at the end. For now we restrict our attention to (16).

To obtain θ-expansion of the action given in (15) we need to know the action of the operator Δ on an arbitrary superfield and on V_i^A. A straight forward computation gives

$$\Delta F_{PQ\cdots}^{BC\cdots} = y^A D_A F_{PQ\cdots}^{BC\cdots} \quad , \tag{17}$$

$$\Delta V_i^A = D_i y^A + V_i^C y^B T_{BC}{}^A \quad , \tag{18}$$

where $F_{PQ\cdots}^{BC\cdots}$ is an arbitrary tensor and $T_{BC}{}^A$ is the supertorsion tensor. Since (18) involves a new object, $D_i y^A$, we also need to know the operation of Δ on it. This, too, is straightforward to compute and the result is

$$\Delta(D_i y^A) = y^B V_i^D y^C R_{CDB}{}^A \ . \tag{19}$$

The rules given in (10) and (17) - (19), the initial conditions (11) - (14) and the solutions to supergravity Bianchi identities given in Ref. [16] are sufficient to obtain θ-expansion of the action in (15). With background fermions set to zero only even powers of θ contribute to the expansion. Even so one must derive the general expression in each order in θ, before the fermions are set to zero and the conditions (11) - (14) used, since the next higher order contribution is obtained by a further application of Δ. Condition (13) is, however, an exception to this since Δy^a vanishes due to (10) and $\Delta(D_i y^a)$ vanishes if y^a vanishes as can readily be seen from (19). Thus y^a can be set to zero at any stage of the θ-expansion and this considerably simplifies the algebra.

We can now expand (15) to arbitrarily high orders in θ (the expansion of course terminates at the 16th order). The zeroth order contribution is trivially obtained from (15) by simply setting $\theta = 0$ in it. For vanishing background fermions we get

$$I^{(0)} = \int d^2\sigma \Big[\frac{1}{2}\, \eta^{ij} \phi V_i^a(X) V_{ja}(X) + \epsilon^{ij} V_i^d(X) V_j^c(X) B_{cd}(X) + \ \psi^s \mathcal{D}_-^{st} \psi^t \ \Big] \ , \tag{20}$$

where now $V_i^a(X) = \partial_i X^m \ e_m{}^a(X)$ and the superscript '(0)' denotes the order in θ. The second order contribution is also not hard to compute. We only quote the result which can be written in the following compact form:

$$I^{(2)} = \int d^2\sigma \big[-\phi(X) \left(\theta \not{V}_-(X) \tilde{D}_+\theta\right)$$

$$+ \frac{1}{4} \left(\theta \not{V}_-(X)\Gamma^{ab}\theta\right) \psi^s F^{st}_{ab}(x)\psi^t \tag{21}$$

$$- \frac{c_1}{2} \epsilon^{ij} \left(\theta \not{V}_i(X)\Gamma^{ab}\theta\right) tr\left(A_j(X)F_{ab}(X)\right)\big] ,$$

where $F_{ab}(X)$ is the Yang-Mills field strength, $\not{V}_i(X) = V_i^a(X)\Gamma_a$ and

$$(\tilde{D}_+\theta)^\alpha = \partial_+\theta^\alpha + \frac{1}{4} \theta^\beta(\Gamma^{bc})_\beta{}^\alpha V_+^a(X)\tilde{\omega}_{abc}(X) .$$

The connection $\tilde{\omega}_{abc}(X)$ is defined by

$$\tilde{\omega}_{abc}(X) = \omega_{abc}(X) + T_{abc}(X) - (\phi(X))^{-1}\eta_{a[b}D_{c]}\phi(X) \quad . \tag{22}$$

In this equation $\omega_{abc}(X)$ is the $\theta = 0$ component of the superconnection appearing in the Bianchi identies of Ref. (16). Also the bracket [ ] denotes antisymmetrization normalized to unit weight. In obtaining (21) we have used the gauge for the superfield $A_B(Z)$ in which $A_\alpha(X) = 0$ and $D_{[\beta}A_{\alpha]}(X) = 0$. Such a gauge choice is allowed by the (super) gauge symmetry of (15), under which $A_B(Z)$ transforms as $\delta A_B(Z) = \not{D}_B\Lambda(Z)$, where $\Lambda(Z)$ is a scalar superfield.[♮3]

The last term in (21) comes from the Chern-Simons piece in (16). The appearance of this term may at first seem surprising, especially since the zeroth order action $I^{(0)}$ in (20) is gauge-invariant due to the anomalous transformation law of B_{cd} implied by (16). But in fact its presence is easily explained. The point

♮3 Note that this does not fix a gauge for the x-space gauge invariance of the theory.

is that in the total action $I^{(0)} + I^{(2)}$ the effective gauge field that couples to the chiral fermions ψ^s is

$$\tilde{A}_- = A_- - \frac{1}{4}(\theta \not{V}_- \Gamma^{ab}\theta)F_{ab}$$

and not just A_-. As a result the ψ-anomaly, which is given by

$$-\frac{1}{8\pi}\int d^2\sigma \epsilon^{ij} tr(\partial_i \Lambda \tilde{A}_j) \ ,$$

contains a piece equal to

$$\frac{1}{32\pi}\int d^2\sigma \epsilon^{ij} tr\left[\partial_i \Lambda \ (\theta \ \not{V}_j \Gamma^{ab}\theta) \ F_{ab}\right]$$

in addition to the one that is cancelled by the anomalous variation of the B_{cd} term in $I^{(0)}$. This extra anomaly precisely cancels the variation of the non-gauge- invariant piece in (21) under the gauge transformation $\delta A_i = \not{D}_i \Lambda$, if we use that $c_1 = -\ 1/16\pi$ [9]. It may be remarked here that a non-gauge-invariant counterterm, similar to the one under discussion, was also required in Ref. [3] for one-loop gauge invariance of the heterotic σ-model in the NSR formulation. In the present case this term arises naturally as a consequence of having ensured gauge invariance and κ-symmetry of the original superfield action (15).

Calculation of the 4th and higher order contributions to the θ-expansion becomes increasingly laborious with increasing order (since terms involving an increasing number of fermionic derivatives on background superfields appear and calculating these in terms of physical fields is rather tedious), but no subtleties are involved. In the light-cone gauge, however, there is a dramatic simplification – the expansion terminates at order θ^2. In the following we present an argument to show that in the light-cone gauge and with background fermions set to zero $I^{(0)} + I^{(2)}$ is all that remains of the superfield action (15). To simplify matters a little bit we will present the argument only for the pure supergravity case. An identical argument can be given when background super Yang-Mills fields are also present.

Our starting point is the following expansion for the (2n)th order contribution to the θ-expansion of (15):

$$I^{2n} = \int d^2\sigma \sum_{q=0}^{2n-1} \sum_{p=0}^{q} \frac{1}{2n(2n-q-1)!(q-p)!p!}$$
$$\left[(\Delta^{2n-q-1}\phi)\,(\Delta^{q-p+1}V_+^a)\,(\Delta^p V_{-a})\right. \tag{23}$$
$$\left. + \frac{1}{2}\,(\Delta^p(y\Gamma_a\Gamma_b\lambda))\,(\Delta^{2n-q-1}V_-^a)\left(\Delta^{q-p}\,V_+^b\right)\right] \quad .$$

Equation (23) can be obtained by applying the operator $\frac{1}{(2n)!}\,\Delta^{2n-1}$ to

$$I^{(1)} = \Delta I = \int d^2\sigma[\phi(\Delta V_+^a)\,V_{-a} + \frac{1}{2}(y\Gamma_a\Gamma_b\lambda)V_+^a V_-^b] \, ,$$

before setting the background fermions to zero, and using the Leibnitz rule. In (23) the conditions (11), (12) and (14) and the condition of vanishing background fermions must be used only after all the Δ operations have been performed.

One additional result that we will need is

$$\theta^\alpha\theta^\beta = \frac{1}{96}\,(\Gamma_{abc})^{\alpha\beta}\left(\theta\Gamma^{abc}\theta\right) \quad . \tag{24}$$

This follows from the Majorana-Weyl nature of θ^α and the properties of the 16 $\times$ 16 dimensional representation that we are using for the γ-matrices. Now, the light-cone gauge-fixing is achieved by imposing the conditions

$$\Gamma^{+}_{\alpha\beta}\theta^{\beta} = 0 \qquad x^{+}(\sigma) = x^{+} + p^{+}\sigma^{0} \quad . \tag{25}$$

We will additionally assume that the background fields depend only on transverse directions and that tensors have only transverse components nonvanishing. In the light-cone gauge (24) reduces to

$$\theta^{\alpha}\theta^{\beta} = \frac{1}{32}\,(\Gamma^{+}\Gamma_{\tilde{a}\tilde{b}})^{\alpha\beta}(\theta\Gamma^{-}\Gamma^{\tilde{a}\tilde{b}}\theta) \quad , \tag{26}$$

where a tilde on an index indicates that it takes transverse values only. Moreover, in this gauge the only θ^2 contribution that is non zero is the bilinear

$$Q^{\tilde{a}\tilde{b}} \equiv (\theta\Gamma^{-}\Gamma^{\tilde{a}\tilde{b}}\theta) \ . \tag{27}$$

An immediate consequence of (26) is that the following is true:

$$Q^{\tilde{a}\tilde{b}}Q^{\tilde{c}\tilde{d}} = 0 \ . \tag{28}$$

At this stage it may seem that because of (28) the θ-expansion terminates at the 2nd order. An examination of our expansion rules, however, makes it clear that this need not necessarily be the case since these rules also involve covariant derivatives on θ, $D_i\theta$, and bilinears of these with θ do not satisfy equations similar to (28). The following more detailed analysis is, therefore, needed.

To evaluate (23) for arbitrary n we need to compute $\Delta^{2m-1}\ (y\Gamma_a\Gamma_b\lambda)$ for $m = 1,\ 2,\ \ldots$ and $\Delta^{2m}\phi$ and $\Delta^{2m}V_i^{\ a}$ for $m\ =\ 0,\ 1,\ 2,\ \ldots$. In the light-cone gauge the first two of these are especially simple to analyse. Each of them is

$(2m)$th order in θ and by using (26) we can write the θ's as m bilinears of the form given in (27) (since no $D_i\theta$'s can appear here). It then follows from (28) that they must both vanish for $m \geq 2$. Actually things are even simpler since $\Delta^2\phi$ also vanishes. This is true because the indices of the θ-bilinear in $\Delta^2\phi$ must be contracted with some background tensor (which has only transverse components) and because $\theta\Gamma^{\tilde{a}\tilde{b}\tilde{c}}\theta = 0$ as a consequence of the light-cone condition on θ. Thus we have

$$
\begin{aligned}
&\Delta^{2m}\phi = 0\ , \quad m \geq 1 \\
&\Delta^{2m-1}(y\Gamma_a\Gamma_b\lambda) = 0\ , \quad m \geq 2 \quad .
\end{aligned}
\tag{29}
$$

The result is that the double sum in (23) reduces to the following single sum:

$$
\begin{aligned}
I^{(2m)} = \int d^2\sigma \sum_{p=0}^{2n-1} \frac{1}{2n(2n-p-1)!p!} &\left[\phi \left(\Delta^{2n-p}\, V_+^a\right)\left(\Delta^p\, V_{-a}\right)\right. \\
&\left. + \frac{p}{2}\Delta(y\Gamma_a\Gamma_b\lambda)\left(\Delta^{2n-p-1}V_-^a\right)\left(\Delta^{p-1}\, V_+^b\right)\right] \quad .
\end{aligned}
\tag{30}
$$

So, we now only need to analyse the quantity $\Delta^{2m}V_i^a$. From our θ-expansion rules it is clear that the general expression for $\Delta^{2m}V_i^a$ must be of the form

$$
\Delta^{2m}V_i^a = W^{ab}V_{ib} + W^a{}_\beta D_i\theta^\beta \quad . \tag{31}
$$

Now since θ^α is the only fermion around, we must also have

$$
W^a{}_\beta = \theta^\alpha W^a_{\alpha\beta} \quad . \tag{32}
$$

Moreover, the most general γ-matrix structure of $W^a_{\alpha\beta}$ is

$$
W^a_{\alpha\beta} = S^{ab}\Gamma_{b\alpha\beta} + M^{abcd}\left(\Gamma_{bcd}\right)_{\alpha\beta} + G^{abcdef}\left(\Gamma_{bcdef}\right)_{\alpha\beta} \quad . \tag{33}
$$

Substituting (32) and (33) in (31) we get:

$$\begin{aligned}\Delta^{2m} V_i^a = W^{ab} V_{ib} + S^{ab} \left(\theta \Gamma_b D_i \theta\right) + M^{abcd} \left(\theta \Gamma_{bcd} D_i \theta\right) \\ + G^{abcdef} \left(\theta \Gamma_{bcdef} D_i \theta\right) \quad .\end{aligned} \tag{34}$$

Thus the terms involving $D_i\theta$ that appear in $\Delta^{2m} V_i^a$ have been explicitly separated out. The argument leading to (29) can now be applied to the tensor coefficients W, S, M and G appearing in (34) to conclude that each of these can only be at most of order θ^2. The index structure of the last three terms in this equation is, however, such that potential θ^4 terms actually vanish. To see this observe that the bilinear $(\theta\Gamma_{ab\ldots} D_i\theta)$ vanishes in the light-cone gauge unless exactly one of the indices a, $b, \ldots$ takes the value '+' and all the rest are transverse, namely, only the bilinear $\left(\theta\Gamma_+ \Gamma_{\tilde{b}\tilde{c}\ldots} D_i\theta\right) = \left(\theta\Gamma^- \Gamma_{\tilde{b}\tilde{c}\ldots} D_i\theta\right)$ is nonvanishing. Then (34) reduces to

$$\begin{aligned}\Delta^{2m} V_i^a = W^{ab} V_{ib} + S^{a+} \left(\theta \Gamma^- D_i \theta\right) + 3M^{a+\tilde{c}\tilde{d}} \left(\theta \Gamma^- \Gamma_{\tilde{c}\tilde{d}} D_i \theta\right) \\ + 5G^{a+\tilde{c}\tilde{d}\tilde{e}\tilde{f}} \left(\theta \Gamma^- \Gamma_{\tilde{c}\tilde{d}\tilde{e}\tilde{f}} D_i \theta\right) \quad .\end{aligned} \tag{35}$$

Remembering now that the background tensors have only transverse components we conclude that the last three terms in (35) contribute only if $S^{a+} = \eta^{a+}\tilde{S}$, $M^{a+\tilde{c}\tilde{d}} = \eta^{a+}\tilde{M}^{\tilde{c}\tilde{d}}$ and $G^{a+\tilde{c}\tilde{d}\tilde{e}\tilde{f}} = \eta^{a+}\tilde{G}^{\tilde{c}\tilde{d}\tilde{e}\tilde{f}}$.

Equation (35) then becomes

$$\begin{aligned}\Delta^{2m} V_i^a = W^{ab} V_{ib} + \eta^{a+}\tilde{S} \left(\theta \Gamma^- D_i \theta\right) + 3\tilde{M}^{\tilde{c}\tilde{d}} \left(\theta \Gamma^- \Gamma_{\tilde{c}\tilde{d}} D_i \theta\right) \\ + 5\tilde{G}^{\tilde{c}\tilde{d}\tilde{e}\tilde{f}} \left(\theta \Gamma^- \Gamma_{\tilde{c}\tilde{d}\tilde{e}\tilde{f}} D_i \theta\right) \quad .\end{aligned} \tag{36}$$

An argument similar to the one leading to (29) now enables one to conclude that $\tilde{S}$, $\tilde{M}^{\tilde{c}\tilde{d}}$ and $\tilde{G}^{\tilde{c}\tilde{d}\tilde{e}\tilde{f}}$ must be order <u>zero</u> in θ. Consequently $\Delta^{2m} V_i^a = 0$ for $m \geq 2$.

In fact the following more specific statement is true and can be easily verified from (36):

$$\Delta^{2m} V_i^{+} = 0 \ , \qquad m \geq 1 \quad , \tag{37}$$

$$\Delta^{2m} V_i^{-} = \Delta^{2m} V_i^{\tilde{a}} = 0 \ , \quad m \geq 2 \quad . \tag{38}$$

Using these results, (28) and the fact that $\Delta(y\Gamma_a\Gamma_b\lambda)$ vanishes unless either one of the two indices a, b is '+' and the other is transverse, it is now straightforward to see that $I^{(2n)} = 0$ for $n \geq 2$. We omit the details but would like to point out that the specific structure of (23), which is of course governed by the covariant superfield action (15), is primarily responsible for this result. In other words it is possible to construct actions in the light-cone gauge with nonzero θ^4 terms, but these cannot be derived from a Lorentz-covariant and κ-invariant action like (15).

From the expressions given in (20) and (21) we can now write down the light-cone gauge-fixed action for the heterotic string in background fields. With the rescaling $\theta^\alpha \to \theta^\alpha/\sqrt{p^+}$ the action we get is

$$\begin{aligned} I^{(l.c.)} = \int d^2\sigma \Big[& \frac{1}{2}\, \eta^{ij} \phi V_i^{\tilde{a}} V_{j\tilde{a}} + \epsilon^{ij}\ V_i^{\tilde{d}} V_j^{\tilde{c}} B_{\tilde{c}\tilde{d}} \\ & + \psi^s \mathcal{D}_-^{st} \psi^t - \phi(\theta\Gamma^- \tilde{D}_+ \theta) + \frac{1}{4}(\theta\Gamma^-\Gamma^{\tilde{a}\tilde{b}}\theta)\psi^s F_{\tilde{a}\tilde{b}}^{st}\psi^t \\ & - \frac{c_1}{4}\ (\theta\Gamma^-\Gamma^{\tilde{a}\tilde{b}}\theta) tr(A_+ - A_-) F_{\tilde{a}\tilde{b}} \Big] \quad , \end{aligned} \tag{39}$$

where $V_i^{\tilde{a}} = \partial_i X^{\tilde{m}}\ e_{\tilde{m}}\ {}^{\tilde{a}}(X)$ and the connection appearing in $\tilde{D}_+$ is given in (22), restricted to transverse indices only. To determine how this connection is related to the ordinary connection and torsion, we use the definition of supertorsion,

$T_{AB}{}^{C} = -2D_{[A}E_{B]}{}^{M}e_{M}{}^{C}$, and the solution for H_{abc} of the supergravity Bianchi identities given in Ref. [16]. Using the $\theta = 0$ components of these equations in (22) we get

$$\tilde{\omega}_{abc} = \Omega_{abc} - \phi^{-1}\eta_{a[b}D_{c]}\phi - \phi^{-1}H_{abc} \ , \tag{40}$$

where

$$\Omega_{abc} \equiv E_{[b}{}^{n}E_{c]}{}^{m}\partial_{n}e_{ma} + E_{[a}{}^{n}E_{c]}{}^{m}\partial_{n}e_{mb} - E_{[a}{}^{n}E_{b]}{}^{m}\partial_{n}e_{mc} \ , \tag{41}$$

is the torsionless connection. Equation (40) can be simplified by the rescaling $e_{m}{}^{a} \rightarrow \phi^{-1/2}e_{m}{}^{a}$. The second term then disappears and in terms of the rescaled veilbeins the connection appearing in $\tilde{D}_{+}$ becomes

$$\tilde{\omega}_{abc} = \Omega_{abc} - H_{abc} \ , \tag{42}$$

where, in terms of B, H_{abc} is defined by (16) and is given by

$$\begin{aligned} H_{abc} &= E_{a}{}^{\ell}E_{b}{}^{m}E_{c}{}^{n}H_{\ell m n} \quad , \\ H_{\ell m n} &= 3\ \partial_{[\ell}B_{mn]} + 6c_{1}(\omega_{3YM})_{[\ell m n]} \quad . \end{aligned} \tag{43}$$

The rescaling of the veilbeins also has the effect that the factor of ϕ in the first term in (39) disappears while the last two terms get multiplied by ϕ due to the presence of $F_{\tilde{a}\tilde{b}} = E_{\tilde{a}}{}^{\tilde{m}}E_{\tilde{b}}{}^{\tilde{n}}F_{\tilde{m}\tilde{n}}$ in them. As a result all the θ^{2} terms in (39) have a factor of ϕ after the rescaling and so it can be absorbed in a redefinition of θ. We then end up with the following expression for the light-cone gauge-fixed action

$$I^{(1.c.)} = \int d^2\sigma \; [\frac{1}{2}\; \eta^{ij} V_i^{\tilde{a}} V_{j\tilde{a}} + \epsilon^{ij} V_i^{\tilde{d}} V_j^{\tilde{c}} B_{\tilde{c}\tilde{d}} + \psi^s \mathcal{D}_-^{st} \psi^t$$

$$- (\theta\Gamma^- \tilde{D}_+ \theta) + \frac{1}{4}\; (\theta\Gamma^-\Gamma^{\tilde{a}\tilde{b}}\theta)\psi^s F^{st}_{\tilde{a}\tilde{b}} \psi^t \qquad (44)$$

$$- \frac{c_1}{4}\; (\theta\Gamma^-\Gamma^{\tilde{a}\tilde{b}}\theta) tr((A_+ - A_-)F_{\tilde{a}\tilde{b}})] \quad .$$

The connection now appearing in $\tilde{D}_+$ is given by (42) and (43), restricted to transverse directions only. It may be remarked here that precisely such a connection has been previously advocated [3, 6] for the gauge invariance of the NSR σ-model action. The analysis carried out here suggests that the κ-symmetry and gauge-invariance of the superfield action (15) are responsible for the appearance of this specific connection in (44).

To summarize, in this paper we have developed a simple procedure for obtaining manifestly covariant θ-expansion of any curved superspace σ-model. On the basis of our expansion rules we have argued that for the heterotic string in background fields the expansion terminates at the 2nd order in the light-cone gauge (for vanishing background fermions). It is perhaps worth mentioning that in obtaining the θ-expansion we have explicitly used the constraints of supergravity - super Yang-Mills theory and the solutions to the Bianchi identities. This is not a limitation of our procedure since κ-symmetry of the σ-model anyway requires these conditions. But what it does signify is that the action (44) is consistent only if the background fields satisfy classical equations of motion, since we have used the κ-symmetry to go to the light-cone gauge. Also, in analogy with what happens in the case of the free superstring we expect the action (44) to possess some physical supersymmetries which must be a combination of the κ-symmetry and the $N = 1$ supersymmetry of the background. Of course, since background fermions have been set to zero in (44), the requirement of $N = 1$ supersymmetry (and hence that of physical supersymmetries of the action (44)) will impose constraints on the background.[♮4] We must also point out that (44) has a Lorentz anomaly. One expects that this problem will be cured by including curvature-squared terms in the supergravity Bianchi identities. A detailed calculation of the solutions of these modified Bianchi identities is underway. It would be interesting to see whether the resulting modification in (44) actually gets rid of the Lorentz anomaly. It would also be interesting to know precisely what connection appears in the corvariant derivative $\tilde{D}_+$ since this question has important implications for compactified solutions of string theory [6]. Work in these directions is in progress.

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♮4 In Ref. [5], for example, such constraints have been analysed and shown to require backgrounds of SU(3) holonomy. See also Ref. [6].

REFERENCES

[1] V. G. Knizhnik and A. Zamoldchikov, Nucl. Phys. B247 (1984) 83; E. Fradkin and A. Tseytlin, Phys. Lett. 158B (1985) 316, and Nucl. Phys. B261 (1985) 1; D. Nemeschansky and S. Yankielowicz, Phys. Rev. Lett. 54 (1985) 620; S. Jain, R. Shankar and S. Wadia, Phys. Rev. D32 (1985) 2713; E. Beryshoeff, S. Randjbar-Daemi, A. Salam, H. Sarmedi and E. Sezgin, ICTP Preprint.

[2] A. Sen, Phys. Rev. Lett. 55 (1985) 1846 and Phys. Rev. D32 (1985) 2102; C. G. Callan, E. Martinec, M. J. Perry and D. Friedan, Nucl. Phys. B262, 593 (1985); T. Banks, D. Nemeschansky and A. Sen SLAC-PUB-3885; C. G. Callan, J. R. Klebanov, M. J. Perry, Princeton Preprint (1986); M. T. Grisaru, A. Van de Ven, D. Zanon HUTP-861A027; D. Nemeschansky and A. Sen, SLAC-PUB-3925.

[3] A. Sen Phys. Lett. 166B (1986) 300 and SLAC-PUB-3908.

[4] C. M. Hull and E. Witten, Phys. Lett. 160B(1985) 398.

[5] C. M. Hull, Nucl. Phys. B267 (1986) 266 and DAMTP preprint (1986); C. M. Hull, and P. K. Townsend, DAMTP preprint (1985).

[6] A. Strominger Santa Barbara preprint NSF-ITP-86-16; C. M. Hull, DAMTP preprint (1986).

[7] E. Witten, Nucl. Phys. B266 (1986) 245.

[8] M. T. Grisaru, P. Howe, L. Mezincescu, B. Nilsson and P. K. Townsend, Phys. Lett. 162B (1985) 116.

[9] J. J. Atick, A. Dhar and B. Ratra, Phys. Lett. 169B (1986) 54.

[10] A. Neveu and J. H. Schwarz, Nucl. Phys. B31 (1971) 86; P. Ramond, Phys. Rev. D3 (1971) 2415.

[11] M. B. Green and J. H. Schwarz, Phys. Lett. 136B (1984) 367.

[12] M. B. Green and J. H. Schwarz, Nucl. Phys. B243 (1984) 285; I. Bengtsson and M. Cederwall, Goteborg preprint 84-21 (1984).

[13] S. Mukhi, Nucl. Phys. B264 (1986) 640.

[14] I. McArthur, Class. Quantum. Grav. 1(1984) 233.

[15] D. J. Gross, J. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54, (1985) 502; Nucl. Phys. B256 (1985) 253 and Nucl. Phys. B267 (1986) 75.

[16] J. J. Atick, A. Dhar and B. Ratra, Phys. Rev. D33 (1986) 2824.

[17] R. Kallosh and B. Nilsson Phys. Lett. 167B (1986) 46. B. Nilsson and A. K. Tollstein Phys. Lett. 171B 1986 212; S. Gates, H. Nishino, Universityof Maryland preprint; S. W. McDowell and M. J. Rokowski, Yale University preprint.

[18] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46.