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SUPERSTRINGS AND GEOMETRY OF SUPERSPACE*

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ABSTRACT

These lectures present some recent developments in the σ-model approach to the Green-Schwarz superstring. Among the topics included are i) interpretation of the free superstring as a flat superspace σ-model; ii) propagation of the superstring in curved superspace and iii) in the presence of background super Yang-Mills fields. The role of the world-sheet fermionic gauge symmetry needed to ensure consistent coupling to background fields is emphasized.

1. INTRODUCTION

In these lectures we will study some aspects of 2-dimensional non-linear σ-models in relation to superstring theories. Non-linear σ-models have been the focus of attention since the recent emergence[1] of superstring theories as promising candidates for the unification of all known forces, including gravity.[2] One of the main reasons for this is the following. Since consistent superstring theories are formulated only in 10 space-time dimensions, to make contact with phenomenology one must carry through the program of dimensional reduction by which 6 of the 9 space dimensions are arranged to be invisible to us at present energies. One popular way of doing this is to curl up the extra 6 dimensions into a tiny compact space with a length scale of the order of Planck length. To determine the geometrical properties (and other characteristics) of this internal space one must study the solutions of the classical string field theory equations. Since these equations are as yet unknown, one way to learn something about the vacuum solutions is to study string propagation in background gravitational (and

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possibly other) fields under the requirements of conformal invariance, supersymmetry, etc.* This naturally leads one to study non-linear σ-models.[4–6] One of the main results of the recent investigations along these lines is that consistent backgrounds must satisfy classical (point particle) field theory equations of motion.[5] Furthermore, remarkable connections between the symmetries of non-linear σ-models and string theory have been revealed. For example, anomalies in σ-models have been shown to be intimately connected with anomalies in the point particle limit of string field theory.[6] There is therefore enough reason to believe that in the absence of a more appropriate framework, one stands to learn much about the properties of string theory by studying σ-models.

There exist in the literature two different formulations of superstrings which lead to two different kinds of non-linear σ-models when background fields are included. The Neven-Schwarz-Ramond (NSR) formulation[7] of superstring theory possesses manifest Lorentz covariance and world-sheet supersymmetry, but not space-time supersymmetry. On the other hand, the Green-Schwarz (GS) formulation[8] is manifestly space-time supersymmetric, but it cannot be quantized covariantly.[9] Therefore, a manifestly Lorentz covariant and space-time supersymmetric formulation of superstring theory does not exist. One works with the NSR or GS formulation depending on the type of questions one is interested in. It is simpler to study σ-model loop corrections in the NSR formulation. So most of the quantum calculations have been done in this formulation. The GS formulation, on the other hand, is more suited for studying questions of space-time supersymmetry and it has been used for this purpose. In these lectures we shall mostly be interested in the symmetry properties of the superstring in the presence of background fields and, therefore, we shall be using only the GS formulation.

An important aspect of the GS formulation is the presence of world-sheet fermionic gauge symmetry in the theory.[8,10] This gauge symmetry, conventionally called the κ-symmetry, enables one to gauge away the unphysical fermionic degrees of freedom of the superstring. In the physical (light-cone) gauge a global remnant of this symmetry appears as the physical (space-time) supersymmetry of the string. The κ-symmetry is, therefore, crucial for a consistent coupling of the superstring to background fields. As we shall see, ensuring that the non-linear σ-model describing such couplings be κ-symmetric is quite non-trivial and puts restrictions on the kinds of possible backgrounds. In fact, already at the classical level, the background fields are required to satisfy equation of motion of point particle field theory.[11–13]

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* Some information about possible vacuum solutions can also be obtained by studying compactification in the field theory limit.[3]

Because of the critical role that κ-symmetry plays in the GS formulation, this symmetry will be the main focus of attention throughout these lectures. We will begin with an introduction to the GS formulation of the free superstring.[8] For the sake of definiteness (and because of its phenomenological promise) we will restrict our attention to the heterotic superstring.[14] We will study the symmetries of this theory and discuss why κ-symmetry is crucial for the consistency of the theory. The GS superstring has been interpreted as a σ-model defined on flat superspace and the corresponding non-linear σ-models have appropriately been called superspace σ-models.[15] We will discuss the reasons for this interpretation and construct the superspace σ-model for the heterotic superstring using κ-symmetry as our guide. Our treatment will include both background supergravity as well as super Yang-Mills fields. As mentioned earlier, we will find that κ-symmetry constrains the background fields to satisfy point-particle field theory equations of motion. Since we will use the economical language of superfields to construct the superspace σ-model, to see how these constraints come about we will need to known the superspace formulation of 10-dimensional $N = 1$ supergravity[16] coupled to $N = 1$ super Yang-Mills theory.[17] In order to make these lectures as self-contained as possible a discussion of this theory is, therefore, also included. Finally, we will discuss some open problems and questions.

2. FREE HETEROTIC STRING

The basic variables describing the propagation of free heterotic string in the GS formulation are the space-time coordinates $X^a(\xi)$ and the fermionic variables $\theta^\alpha(\xi)$, which constitute a single Majorana-Weyl fermion in 10 dimensions but transform as world-sheet scalars. The action is

$$I = \frac{1}{2\pi\alpha'} \int d^2\xi \left[\frac{1}{2} \sqrt{-g}\, g^{ij}\, V_i^a\, V_j^b\, \eta_{ab} + \mathcal{E}^{ij}\, V_i^a(\partial_j\theta^\alpha\Gamma_{a\alpha\beta}\theta^\beta)\right] \tag{1}$$

where

$$V_i^a = \partial_i X^a + \partial_i\theta^\alpha\Gamma^a_{\alpha\beta}\theta^\beta \quad . \tag{2}$$

Our notation and conventions are as follows: $\xi^i = (\xi^0 = \tau,\ \xi^1 = \sigma)$ are the coordinates of a point on the world-sheet; g^{ij} is the world-sheet metric with signature $(+,-)$ and $g = det\ g_{ij}$; $\mathcal{E}^{ij}$ is the 2-dimensional Levi-Civita symbol ($\mathcal{E}^{01} = +1$); $\eta_{ab}(a,\ b = 0,1,\ldots,9)$ is 10-dimensional Minkowski metric with signature $(+,-,\ldots,-)$; θ^α ($\alpha = 1,2,\ldots,16$) are 16 real anticommuting variables; and α' is the slope parameter. We are using 16×16 dimensional representation for Dirac γ-matrices. In this representation the Dirac algebra is represented by two sets of γ-matrices, $\{\Gamma^a_{\alpha\beta}\}$ and $\{\Gamma^{a\alpha\beta}\}$, which we have chosen to be symmetric. The distinction between these two sets arises because there is no way the fermionic indices can be raised or lowered. The Dirac algebra is

$$\Gamma^a_{\alpha\beta}\Gamma^{b\beta\gamma} + \Gamma^b_{\alpha\beta}\Gamma^{a\beta\gamma} = 2\eta^{ab}\delta^\gamma_\alpha \quad . \tag{3}$$

In addition to the X^a and θ^α the heterotic string has gauge degrees of freedom which may be represented either by 16 left-moving world-sheet and Lorentz scalars or by 32 left-moving world-sheet Majorana-Weyl fermions and Lorentz scalars.[14] We shall work with the fermionic representation. In this representation the contribution of the gauge degrees of freedom to the action is

$$I_{YM} = \frac{1}{2\pi} \int d^2\xi \, \sqrt{-g} \, g^{ij} \, \bar{\psi}^s \rho_i \partial_j \psi^s \tag{4}$$

where 's' is the gauge index and ψ^s transforms either in the fundamental representation of $SO(32)$, for the gauge group $SO(32)/Z_2$, or of the maximal subgroup $SO(16) \times SO(16)$ of the other possible gauge group $E_8 \times E_8$. Also, ρ^i are 2-dimensional Dirac γ-matrices satisfying the algebra

$$\{ \rho^i, \rho^j \} = 2g^{ij} \quad . \tag{5}$$

Moreover, ψ^s is real (Majorana condition) and satisfies the Weyl condition

$$P^{ij}_- \, \rho_j \, \psi^s = 0 \ , \tag{6}$$

where

$$P^{ij}_\pm = \frac{1}{2}\left(g^{ij} \pm \varepsilon^{ij}/\sqrt{-g}\right) \quad . \tag{7}$$

Because of the Majorana-Weyl condition on ψ^s, the connection term in the derivative in (4) does not contribute.

The expressions given in (1) and (4) are manifestly invariant under 2-dimensional general coordinate transformations. It is convenient to partially fix this symmetry by going to the conformal gauge,

$$g^{ij} = \eta^{ij} \, e^{-2\rho} \quad , \tag{8}$$

where $\eta^{ij} = diag(+1, -1)$. In the rest of these lectures we will work in this gauge. We will also set $\alpha' = 1/2$ for convenience. Then (1) and (4) take the form

$$I = \frac{1}{\pi} \int d^2\xi \left[\frac{1}{2} \, \eta^{ij} \, V^a_i \, V^b_j \, \eta_{ab} + \varepsilon^{ij} \, V^a_i (\partial_j \theta^\alpha \Gamma_{a\alpha\beta} \theta^\beta) \right] \ , \tag{9}$$

$$I_{YM} = \frac{1}{2\pi} \int d^2\xi \psi^s \partial_- \psi^s \ , \tag{10}$$

where we have used that ψ^s is real and satisfies (6). We have also used the notation, for any 2-dimensional vector v_i, $v_\pm = v_0 \pm v_1$. In the conformal gauge the total action $I + I_{YM}$, given by the sum of (9) and (10), must, of course, be appended by the Virasoro constraints that follow from the equation of motion for g^{ij} obtained from the sum of (1) and (4). These constraints are

$$V_-^2 = 0, V_+^2 + 2\psi^s\ \partial_+\ \psi^s = 0\ , \tag{11}$$

where $V_\pm^2 = V_\pm^a V_\pm^b \eta_{ab}$.

The action I contains cubic and quartic interactions in X and θ. The dynamics described by it would, therefore, appear to be quite complicated. This is, in fact, not actually the case. The reason is that this action possesses a fermionic gauge symmetry, the κ-symmetry, which enables one to reduce it to a free theory in terms of physical variables. To see how this happens let us first study the symmetrics of this action.

2.1 *Conformal Reparametrizations*

The gauge (8) is invariant under the conformal reparametrizations given by

$$\delta\xi^i = f^i(\xi), \quad \partial_- f^+ = 0 = \partial_+ f^-\ , \tag{12}$$

under which the conformal factor, ρ, transforms as

$$\delta\rho = -f^i\partial_i\rho - \frac{1}{4}(\partial_+ f^+ + \partial_- f^-)\ . \tag{13}$$

I is, therefore, invariant under the conformal transformations

$$\delta X^a = -f^i\partial_i X^a, \quad \delta\theta = -f^i\partial_i\theta^\alpha, \quad \delta\psi^s = -f^i\partial_i\psi^s - \frac{1}{4}\partial_+ f^+\psi^s\ . \tag{14}$$

Although ψ^s is a 2-dimensional spinor in (4) the change in its transformation property in (14) is due to the fact that in going from (4) to (10) in the conformal gauge we have redefined ψ^s to absorb a factor of $e^{\rho/2}$ in it, which gives the additional term in $\delta\psi^s$ in (14).

2.2 *Global $N = 1$ Space-Time Supersymmetry*

Here the transformations are

$$\delta X^a = \mathcal{E}^\alpha \Gamma^a_{\alpha\beta} \theta^\beta \ , \quad \delta\theta^\alpha = \mathcal{E}^\alpha \ . \tag{15}$$

The parameter of transformation $\mathcal{E}^\alpha$ is a constant 10-dimensional Majorana-Weyl spinor. To verify that I is invariant under (15) one notices that V_i^a is trivially invariant and, therefore, so is the first term in (9). The invariance of the second term follows if one uses the Fierz identity

$$\Gamma_{a\alpha\beta}\Gamma^a_{\gamma\delta} + \Gamma_{a\beta\gamma}\Gamma^a_{\alpha\delta} + \Gamma_{a\gamma\alpha}\Gamma^a_{\beta\delta} = 0 \tag{16}$$

2.3 *Fermionic Gauge Symmetry (κ-symmetry)*

In this case the parameter of transformation, $\kappa_{i\alpha}$, is a 10-dimensional Majorana-Weyl fermion and a 2-dimensional vector, satisfying the self-duality condition

$$P_+^{ij}\kappa_{j\alpha} = \kappa^i_\alpha \tag{17}$$

Using the notation $\not{V}_i^{\alpha\beta} = V_i^a \Gamma_a^{\alpha\beta}$, the κ-transformation can be written as

$$\delta X^a = -\delta\theta^\alpha \Gamma^a_{\alpha\beta}\theta^\beta, \quad \delta\theta^\alpha = 2\not{V}_i^{\alpha\beta}\ \kappa^i_\beta \ . \tag{18}$$

To verify that (9) is invariant under (18) one first shows that $\delta V_i^a = 2\partial_i\theta^\alpha\Gamma^a_{\alpha\beta}\delta\theta^\beta$ and then uses (16) to get

$$\delta I = \frac{1}{\pi}\int d^2\xi\ V_-^2\ \partial_+\theta^\alpha\kappa_{+\alpha} \ .$$

This vanishes due to the Virasoro constraint $V_-^2 = 0$, (11). Alternatively, the above variation in I can be cancelled by changing the transformations in (18) by a reparametrization with

$$f^0 = -f^1 = 4\theta^\alpha\kappa_{+\alpha}, \quad \partial_+\kappa_{+\alpha} = 0 \ .$$

Note that this is <u>not</u> a conformal reparametrization (under which I is invariant) since f^- is not necessarily a function of ξ^- only.

2.4 *Gauge-Fixing*

We have already mentioned that the complicated-looking interactions in I are actually a gauge-artifact and that in the physical gauge I reduces to a free theory. Let us now see in more detail how that happens.

The action (9) possesses two local (world-sheet) symmetries – the conformal reparametrizations and κ-symmetry. To quantize this system one needs to gauge-fix these symmetries. This is achieved by the following light-cone gauge conditions:

$$X^+(\xi) = x^+ + p^+\tau, \quad \Gamma^+_{\alpha\beta}\theta^\beta = 0 \ , \tag{19}$$

where x^a and p^a are respectively the center-of-mass coordinate and total momentum of the string and we have used the notation, for any 10-dimensional vector v^a, $v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^9)$. (There is no possibility of confusion between the '$\pm$' of 2 dimensions and 10 dimensions because the reference will always be clear from the context.) The first of (19) fixes the freedom of conformal transformations and the second that of the κ-transformations. It is easy to verify that these two symmetries allow the gauge choice in (19) and that it fixes the gauge completely.

In the light-cone gauge I takes the following simple form:

$$I^{(l.c.)} = -\frac{1}{2\pi}\int d^2\xi \left[\eta^{ij}\ \partial_i X^{\hat{a}}\partial_j X^{\hat{a}} + S^\alpha\Gamma^-_{\alpha\beta}\partial_+ S^\beta\right], \quad S^\alpha = \sqrt{2p^+}\ \theta^\alpha \ . \tag{20}$$

(The index $\hat{a}$ takes on only the transverse values 1, 2, ..., 8.) We see that the interactions have disappeared! This miracle happens because $\partial_i\theta^\alpha\Gamma^a_{\alpha\beta}\theta^\beta$ vanishes unless the index 'a' takes the value '$-$'. (This is because

$$\begin{aligned}\partial_i\theta^\alpha\Gamma^a_{\alpha\beta}\theta^\beta &= \frac{1}{2}\partial_i\theta^\alpha\Gamma^a_{\alpha\beta}(\Gamma^+\Gamma^- + \Gamma^-\Gamma^+)^\beta{}_\gamma\theta^\gamma \\ &= \frac{1}{2}\partial_i\theta^\alpha\Gamma^a_{\alpha\beta}(\Gamma^+\Gamma^-)^\beta{}_\gamma\theta^\gamma \quad (\text{ because of (19)}) \\ &= 0 \ , \text{ except when '}a = -\text{' .)}\end{aligned}$$

The light-cone action $I^{(l.c.)}$ possesses a global supersymmetry which shows up in the spectrum of the theory. The possibility of such a symmetry arises because the light-cone gauge condition $\Gamma^+_{\alpha\beta}\theta^\beta = 0$ cuts down the number of θ variables by half (because $\Gamma^{+\alpha\beta}\Gamma_{+\beta\gamma} = 0$ and Γ^+ is a matrix with eight-dimensional kernal) making the number of physical fermionic variables equal to the number of physical (transverse) coordinates of the string. This is the reason why κ-symmetry is so crucial for the consistency of the GS superstring. The global supersymmetry of $I^{(l.c.)}$ can actually be seen to be a combination of the global $N = 1$ supersymmetry of (9) and the κ-symmetry. The transformations are given by

$$\delta X^{\hat{a}} = \frac{1}{\sqrt{p^+}}\,\mathcal{E}^\alpha\Gamma^{\hat{a}}_{\alpha\beta}S^\beta, \ \ \delta S^\alpha = \frac{1}{2\sqrt{p^+}}\,\partial_- X^{\hat{a}}(\Gamma^{\hat{a}}\Gamma^+)^\alpha{}_\beta\mathcal{E}^\beta \ , \quad \Gamma^-_{\alpha\beta}\mathcal{E}^\beta = 0 \ . \tag{21}$$

There is, in fact, another combination that also survives:

$$\delta X^{\hat{a}} = 0, \quad \delta S^{\alpha} = \delta^{\alpha}, \quad \Gamma^{+}_{\alpha\beta}\delta^{\beta} = 0 \ . \tag{22}$$

But this has no physical consequences for the present problem.

It is now straightforward to quantize the above system and obtain the spectrum. At the massless level the spectrum of this theory is precisely that of $N = 1$ supergravity coupled to super Yang-Mills in 10 dimensions. We shall not pursue the details of quantization of (20) here but proceed on to discuss the σ-model interpretation of the GS superstring. The interested reader is invited to see the papers by Gross, Harvey, Martinec and Rohm[14] for details of light-cone quantization of the heterotic string.

3. SUPERSPACE σ–MODELS

In 10 dimensions the superspace which admits of an $N = 1$ supersymmetry has 10 commuting (bosonic, X^a) and 16 anticommuting (fermionic, θ^α) directions. Let us denote the generators of supertranslations on this superspace by $(P_a,\ Q_\alpha)$. They satisfy the algebra

$$[P_a, P_b] = 0 = [P_a, Q_\alpha], \ \{Q_\alpha, Q_\beta\} = -2i\Gamma^a_{\alpha\beta}P_a \tag{23}$$

and the normalization conditions

$$tr(P_aP_b) = \eta_{ab}, \ \ tr(P_aQ_\alpha Q_\beta) = -i\Gamma_{a\alpha\beta} \ . \tag{24}$$

Consider now an element h of supertranslations, which may be written as

$$h = e^{iX^aP_a - \theta^\alpha Q_\alpha} \tag{25}$$

Using (23) it is easy to show that

$$h^{-1}\partial_i h = iV_i^{\ a}P_a - \partial_i\theta^\alpha Q_\alpha \tag{26}$$

with $V_i^{\ a}$ defined as in (2). We may, therefore, rewrite the first term in I as

$$-\frac{1}{2\pi}\int d^2\xi \ \eta^{ij} \ tr(h^{-1}\partial_i h)(h^{-1}\partial_j h) \tag{27}$$

Since the superspace coordinates (X^a, θ^α) parametrize elements of the supertranslation group ($\sim$ superPoincaré group/Lorentz group), (27) may be viewed as the action for a σ-model defined on flat superspace. The second term in I can then be understood as the corresponding Wess-Zumino term:[15]

$$-\frac{1}{\pi}\int d^3\xi\ \mathcal{E}^{ijk}\ tr(h^{-1}\partial_i h)(h^{-1}\partial_j h)(h^{-1}\partial_k h) \tag{28}$$

The existence of the Wess-Zumino term in this model can be traced to the fact that even flat superspace has torsion, which is reflected in the (anti)commutation relations (23). The coefficient of this term is, however, not quantized in the present case since the integrand in (28) can be written as a total derivative, and so this term gives a local action on the world-sheet.[15,18] Its value (relative to the first term) is, nevertheless, fixed by the requirement of κ-symmetry as we have already seen. In this respect κ-symmetry is like the higher (gauge) symmetry which ordinary bosonic σ-models develop at the infrared-stable fixed point where the coefficient of the Wess-Zumino term has a specific value relative to the kinetic energy term.[19]

The above discussion clarifies in what sense the GS superstring may be regarded as a superspace σ-model. So far, we have been discussing only the free superstring, i.e. flat superspace σ-model. To generalize this model to curved superspace we need some tools from superspace differential geometry. We also need to know the superspace formulation of supergravity and superYang-Mills theories. The next section is devoted to building the necessary technology.

4. SUPERSPACE DIFFERENTIAL GEOMETRY

Let us consider a superspace with points parametrized in local coordinates by $Z^M = (X^m, \theta^\mu)$, where X^m are ordinary bosonic world coordinates and θ^μ are anticommuting fermionic world coordinates. At each point in superspace we introduce a set of basis one-forms $\{e^A\}$:

$$e^A = dZ^M\ e_M{}^A\ , \tag{29}$$

where $e_M{}^A$ is the superveilbein. We shall denote its inverse by $E_A{}^M$:

$$e_M{}^A\ E_A{}^N = \delta_M{}^N, \quad E_A{}^M\ e_M{}^B = \delta_A{}^B\ . \tag{30}$$

The tangent-space indices $A, B, ...$ can either be bosonic $a, b, ...$ or fermionic $\alpha, \beta, ...$ (We will always contract indices diagonally as in (29) and (30). One is, of course, free to make a different choice.) A basis for p-forms is constructed from the set $\{e^A\}$, in the usual way, by forming wedge products, except that the wedge product is now graded, i.e.

$$e^A e^B = -(-)^{[A][B]}\ e^B e^A\ , \tag{31}$$

where $[a] = 0$ and $[\alpha] = 1$. We have omitted an explicit wedge symbol in (31) for ease of notation.

Under supergeneral coordinate transformations the superveilbeins transform as follows:

$$Z \to Z' \equiv Z'(Z), \quad e_M{}^A(Z) \to e'_M{}^A(Z') = \partial'_M Z^N e_N{}^A(Z) \ . \tag{32}$$

For infinitesimal transformations,

$$\delta Z = \mathcal{E}(Z), \quad \delta e_M{}^A = -\mathcal{E}^N \partial_N e_M{}^A - \partial_M \mathcal{E}^N e_N{}^A \ . \tag{33}$$

The transformation property (32) of the superveilbeins is designed so that the one-form e^A transforms as a scalar i.e.

$$Z \to Z', \quad e^A(Z) \to e'^A(Z') = e^A(Z) \ . \tag{34}$$

Vectors transform under the tangent-space group as

$$\delta V^A = V^B L_B{}^A, \quad \delta V_A = -L_A{}^B V_B \ . \tag{35}$$

It is clear that there is some freedom in the choice of the tangent-space group. With the most general possible choice (super Lorentz group) one is lead to an analogue of Riemannian geometry in superspace. However, in this formulation it is not so straightforward to make contact with the ordinary formulation of supergravity theories. For this reason we make a more restrictive choice of the tangent-space group, namely, we choose it to be the ordinary Lorentz group.* This choice implies that the Lie-algebra-valued matrices $L_A{}^B$ must satisfy

$$L_\alpha{}^a = 0 = L_a{}^\alpha, \quad L_\alpha{}^\beta = \frac{1}{4} L_{ab} (\Gamma^{ab})_\alpha{}^\beta \ , \tag{36}$$

as must all other Lie-algebra-valued tensors. (Here $\Gamma^{ab\cdots}$ is a totally antisymmetrized product of Dirac γ-matrices normalized to unit weight.) As a result of (36) a general vector V^A splits into two irreducible pieces, namely, an ordinary (bosonic) vector V^a and a spinor V^α which transform as usual under the Lorentz group.

In curved superspace we also introduce the covariant exterior derivative $D = dZ^M D_M = e^A D_A$, which may be defined by its action on vector-valued p-forms:

$$DV^A = dV^A + V^B \omega_B{}^A, \quad DV_A = dV_A - (-)^p \omega_A{}^B V_B \ , \tag{37}$$

where $\omega_A{}^B = dZ^M \omega_{MA}{}^B = e^D \omega_{DA}{}^B$ is the superconnection one-form. The operator $d = dZ^M \partial_M$ is the exterior derivative. It satisfies $d^2 = 0$ and obeys

* This is the Wess-Zumino formulation of supergravity in superspace.[20]

Leibnitz rule with the following sign convection given by its action on the product $\Omega_1\Omega_2$, where Ω_2 is a p-form:

$$d(\Omega_1\Omega_2) = \Omega_1(d\Omega_2) + (-)^p(d\Omega_1)\Omega_2 \quad . \tag{38}$$

Under a supergeneral coordinate transformation the superconnection one-form $\omega_A{}^B$ transforms as a scalar,

$$Z \to Z' \equiv Z'(Z), \;\; \omega_A{}^B(Z) \to \omega'_A{}^B(Z') = \omega_A{}^B(Z) \ , \tag{39}$$

while under the tangent-space group it transforms as

$$\delta\omega_A{}^B = \omega_A{}^C \, L_C{}^B - L_A{}^C\omega_C{}^B - d\omega_A{}^B \quad . \tag{40}$$

As a result of (34) and (39) our formalism will be manifestly supergeneral coordinate invariant (and, therefore, also supersymmetric) if we work with objects that carry no world indices. This is the plan that one follows in formulating supergravity in superspace.

From the superconnection and superveilbein one can construct two basic geometrical quantities, namely, the supertorsion two form T^A and the supercurvature two-form $R_A{}^B$, as in the ordinary bosonic case:

$$De^A \equiv T^A \ , \tag{41}$$

$$d\omega_A{}^B + \omega_A{}^C\omega_C{}^B \equiv R_A{}^B \quad . \tag{42}$$

In components,

$$T^A \equiv \frac{1}{2!}\, dZ^M \, dZ^N \, T_{NM}{}^A = \frac{1}{2!}\, e^C \, e^B \, T_{BC}{}^A \ , \tag{43}$$

$$R_A{}^B \equiv \frac{1}{2!}\, dZ^M \, dZ^N \, R_{NMA}{}^B = \frac{1}{2!}\, e^C \, e^D \, R_{DCA}{}^B \ . \tag{44}$$

As a result of our choice for the tangent-space group $R_A{}^B$ and $\omega_A{}^B$ satisfy relations similar to (36).

An immediate consequence of the definitions of T^A and $R_A{}^B$ is that they must satisfy consistency conditions, called Bianchi identities. These can be obtained from (41) and (42) by using $d^2 = 0$ and are

$$DT^A = e^B \, R_B{}^A \ , \tag{45}$$

$$DR_A{}^B = 0 \ . \tag{46}$$

It turns out that because of our choice of the tangent-space group the Bianchi identities (45) and (46) are not all independent.[21] One can show that, in fact, (46) is identically satisfied by virtue of (45). Thus (45) is the only independent Bianchi identity. In components it is

$$D_{[A}T_{BC)}{}^D + T_{[AB}{}^E T_{\hat{E}C)}{}^D = R_{[ABC)}{}^D \ , \tag{47}$$

where [ ) represents graded antisymmetrization normalized to unit weight (e.g. $P_{[AB)} = \frac{1}{2!}\,\{P_{AB} - (-)^{[A][B]}P_{BA}\}$). Indices with a caret are excluded from this operation.

So far our discussion has been general, applicable to superspace formulation of supergravity in any dimensions. We would now like to specialize to $N = 1$ supergravity in 10 dimensions. We must then consider a superspace of 10 bosonic and 16 real fermionic coordinates, i.e. a (10,16) superspace. Also, the tangent-space group is just the 10-dimensional Lorentz-group, i.e. $SO(1,9)$. The degrees of freedom of $N = 1$ supergravity in 10 dimensions consist of a graviton, an antisymmetric (rank two) field, a scalar field (dilation), a gravitino and a 'spin-$\frac{1}{2}$' field. In order to accommodate each degree of freedom of the theory in the $\theta = 0$ component of some superfield we need to introduce a two-form field B, in addition to the geometrical objects already introduced:

$$B \equiv \frac{1}{2!}\,dZ^M dZ^N B_{NM} = \frac{1}{2!}\,e^D e^C B_{CD} \tag{48}$$

The $\theta = 0$ component of B_{mn} is just the antisymmetric field mentioned above. (The rest of the degrees of freedom of the theory can be accommodated in the $\theta = 0$ components of the superveilbein $e_M{}^A$ in an appropriate gauge, e.g. the Wess-Zumino gauge). The way we shall introduce B is by constructing a closed three-form H in superspace:

$$\begin{aligned} H &\equiv \frac{1}{3!}\,dZ^N\,dZ^M dZ^L H_{LMN} = \frac{1}{3!}\,e^C\,e^B\,e^A\,H_{ABC} \ , \\ dH &= 0 \ . \end{aligned} \tag{49}$$

Because H is closed it can be written, at least locally, as $H = dB$, where B is the desired object. Thus H may be regarded as the field strength for the potential

B and (49) as the Bianchi identity following from the definition of H in terms of B. In components (49) reads

$$D_{[A}H_{BCD)} + \frac{3}{2}\, T_{[AB}{}^{E}\, H_{\hat{E}CD)} = 0 \quad . \tag{50}$$

The theoretical tools and the various superfields introduced above are necessary for a superspace formulation of $N = 1$ supergravity in 10 dimensions. However, the number of ordinary (i.e. x-space) fields that we have introduced is far greater than the number of dynamical fields required to describe this theory. (Remember, each of the superfields $T_{AB}{}^{C}$, H_{ABC} and $R_{ABC}{}^{D}$ can be written as a polynomial in θ, the maximum power of θ being sixteen. The coefficient of each power in θ is an independent x-space field.) This, in fact, is a basic feature of superspace formulation of all supersymmetric theories. It is, therefore, necessary to impose constraints on some of the superfields to eliminate the redundant x-space fields. Actually it is sufficient to impose constraints on $T_{AB}{}^{C}$ and H_{ABC} since $R_{ABC}{}^{D}$ can be related to $T_{AB}{}^{C}$ using the Bianchi identity (45). Once constraints are imposed on $T_{AB}{}^{C}$ and H_{ABC} (47) and (50) are no longer identically satisfied. In fact, for an appropriate set of constraints some of these equations determine all the unconstrained superfields in terms of the dynamical (x-space) fields (which are undetermined) and the rest provide equations of motion for them. The present formulation is, therefore, an on-shell formulation of supergravity theories.

No systematic procedure for determining an appropriate set of constraints in the general case exists. One just proceeds by trial and error. Some simplification can, however, always be made. To illustrate this point, and to motivate the set of constraints we shall use, let us rewrite the definition of $T_{AB}{}^{C}$ given in (41) and (43) in the following form:

$$T_{AB}{}^{C} = -2\; D_{[A}E_{B)}{}^{M}\; e_{M}{}^{C} \quad . \tag{51}$$

Let us now look at one particular component, say, $T_{a\alpha}{}^{\beta}$:

$$\begin{aligned} T_{a\alpha}{}^{\beta} &= -2D_{[a}E_{\alpha)}{}^{M}e_{M}{}^{\beta} \\ &= -2E_{[a}E_{\alpha)}{}^{M}e_{M}{}^{\beta} + \omega_{a\alpha}{}^{\beta} \ , \end{aligned} \tag{52}$$

where $E_a \equiv E_a^M\partial_M$. Also, $T_{a\alpha}{}^{\beta}$ has the following general decomposition in terms of irreducibles of $SO(1,9)$:

$$T_{a\alpha}{}^{\beta} = K_a\delta_{\alpha}{}^{\beta} + K_{abc}(\Gamma^{bc})_{\alpha}{}^{\beta} + K_{abcde}(\Gamma^{bcde})_{\alpha}{}^{\beta} \quad . \tag{53}$$

(Note that since the fermionic indices cannot be raised or lowered, the irreducible decomposition of $T_{a\alpha}{}^{\beta}$ can only involve even number of γ-matrices. Also, since we are working with 16×16 dimensional representation for γ-matrices we may restrict ourselves to a maximum of four γ-matrices.) From (52) and (53) we see that K_{abc} can be set to zero, without loss of generality, by redefining the component ω_{abc} of superconnection (since $\omega_{a\alpha}{}^{\beta} = \frac{1}{4}\omega_{abc}(\Gamma^{bc})_{\alpha}{}^{\beta}$). Thus we may write

$$T_{a\alpha}{}^{\beta} = K_a\ \delta_{\alpha}{}^{\beta} + K_{abcde}(\Gamma^{bcde})_{\alpha}{}^{\beta}\ . \tag{54}$$

We emphasize that (54) is quite general and does not constitute a constraint on $T_{a\alpha}{}^{\beta}$. The redefinition of ω_{abc} required to bring (53) in the form (54), however, shifts the component $T_{abc} \equiv T_{ab}{}^{d}\ \eta_{dc}$ of the supertorsion. Thus, we could alternatively have set T_{abc} to zero and decided to work with (53). But if we use (54) T_{abc} is, in general, not zero. One can similarly redefine away parts of the other components of the supertorsion. In this way one is left with a certain minimal set of superfields (K_a, K_{abcde}, etc). Any further restrictions on this set constitute what we have called constraints. That such restrictions must be imposed is clear since the number of x-space fields is still too large. Moreover, it is also clear from the above discussion that one can impose different, but equivalent, sets of constraints. For the theory under discussion there do indeed exist two different sets of constraints in the literature.[11,16] One can, however, show that they are equivalent. For our purposes we shall use the following set:

$$\begin{aligned} &T_{\alpha\beta}{}^{a} = 2\Gamma^{a}_{\alpha\beta}, \qquad T_{a\alpha}{}^{b} = 0 = -T_{\alpha a}{}^{\gamma}, \qquad T_{\alpha\beta}{}^{\gamma} = 0\ , \\ &T_{a\alpha}{}^{\beta} = (\Gamma_a\psi)_{\alpha}{}^{\beta} = -T_{\alpha a}{}^{\beta}\ , \end{aligned} \tag{55}$$

with $T_{ab}{}^{\gamma}$, T_{abc} and $\psi^{\alpha\beta}$ unconstrained. In addition we will impose the following constraint on H_{ABC}:

$$H_{\alpha\beta\gamma} = 0 \tag{56}$$

Using (55) and (56) one can now solve for all the unconstrained components in terms of the physical (x-phase) fields and obtain equations of motion for the latter. For details we refer the reader to the literature.[11,16,17] Here we only list solutions for the components of H_{ABC}:

$$H_{a\beta\gamma} = \phi\ \Gamma_{a\beta\gamma}, \quad H_{ab\gamma} = -\frac{1}{2}(\Gamma_{ab})_{\gamma}{}^{\delta}\lambda_{\delta}, \quad H_{abc} = -\frac{3}{2}\phi\ T_{abc}\ , \tag{57}$$

where $\lambda_{\alpha} \equiv D_{\alpha}\phi$ and ϕ is a scalar superfield. The $\theta = 0$ components of ϕ and λ_{α} are respectively the dilation and the 'spin-1/2' degrees of freedom of 10-dimensional $N = 1$ supergravity theory.

Since we shall also be discussing superstring propagation in curved superspace in the presence of background super Yang-Mills fields, we also need to introduce here the necessary formalism. Let us denote by A the (Lie-algebra-valued in the gauge group) one-form potential of super Yang-Mills theory. The corresponding two form field strength is defined by

$$dA + A^2 \equiv F \quad . \tag{58}$$

In components

$$F \equiv \frac{1}{2!}\, dZ^M dZ^N F_{NM} = \frac{1}{2!} e^A\ e^B\ F_{BA} \quad . \tag{59}$$

Analogous to (45), (46) and (49) it satisfies a Bianchi identity given by

$$\mathcal{D}F = dF + [F, A] = 0 \ , \tag{60}$$

where $\mathcal{D}$ is the gauge- and Lorentz-covariant exterior derivative. In components (60) reads

$$\mathcal{D}_{[A}F_{BC)} + T_{[AB}{}^{D}F_{\hat{D}C)} = 0 \ . \tag{61}$$

One must also impose constraints on F_{AB} since it has many more x-space fields than are needed to describe 10-dimensional $N=1$ super Yang-Mills theory. The appropriate constraint in this case is

$$F_{\alpha\beta} = 0 \quad . \tag{62}$$

It turns out that the theory obtained by solving the Bianchi identities (47), (50) and (61) under the constraints (55), (56) and (62) does not describe 10-dimensional $N=1$ supergravity coupled to super Yang-Mills theory. The modification needed to rectify this is simple and aesthetically very beautiful. One is only required to change the Bianchi identity satisfied by H to

$$dH = c_1\ tr\ F^2 \ , \tag{63}$$

where the trace is taken over the gauge group indices and c_1 is, a priori, and arbitrary constant (of length dimensions four). This modified Bianchi identity for H together with (47) and the constraints (55), (56) and (62) completely specifies the coupling of supergravity to super Yang-Mills.[17] Moreover, since $tr\ F^2 = d\omega_{3YM}$, where

$$\omega_{3YM} = tr\ \left(AF - \frac{1}{3}A^3 \right) \tag{64}$$

is the super Yang-Mills Chern-Simmons three-form, we can still construct a two-form potential B which, however, now satisfies

$$dB = H - c_1\ \omega_{3YM} \ . \tag{65}$$

As a result B is no longer gauge-invariant (since H is, by definition, so). In fact, under the gauge transformation $\delta_\wedge A = \mathcal{D}\wedge$, it transforms as

$$\delta_\wedge B = -c_1 \; tr(\wedge dA) \; . \tag{66}$$

Modifications similar to (63), (65) and (66) are also needed in the x-space version of this theory given by Chapline and Manton.[22]

Solving (47), (61) and (63) under the constraints (55), (56) and (62) does indeed give coupled supergravity – super Yang-Mills equations of motion for the physical fields. We refer the reader to the literature[17] for details and quote here only the solutions for the components of H_{ABC}:

$$\begin{aligned} H_{a\beta\gamma} &= \phi\Gamma_{a\beta\gamma}, \\ H_{ab\gamma} &= -\frac{1}{2}(\Gamma_{ab})_\gamma{}^\delta\lambda_\delta, \\ H_{abc} &= -\frac{3}{2}\phi T_{abc} + \frac{c_1}{4}(\Gamma_{abc})_{\alpha\beta}\; tr(\chi^\alpha\chi^\beta) \; , \end{aligned} \tag{67}$$

where χ^α is the super Yang-Mills fermion in terms of which

$$F_{a\alpha} = F_{a\alpha\beta}\chi^\beta = -F_{\alpha a} \tag{68}$$

solves one of the equations in (61).

With this rather lengthy but necessary discussion of the 'background' material we are now ready to discuss superstring propagation in background fields.

5. SUPERSTRING IN CURVED SUPERSPACE

We have already emphasized the important role that κ-symmetry plays in the GS formulation of the superstring. To consistently couple the superstring to background fields we must ensure that the resulting curved superspace σ-model possesses κ-symmetry. As we shall see below this requires that the constraints given in (55) and (56) be satisfied. Since these constraints imply supergravity equations of motion via the Bianchi identities, we then have that supergravity equations of motion for the background fields ensure a consistent coupling of these to the superstring. Let us now see in detail how this comes about.

To write down the curved superspace σ-model and its κ-symmetry we begin by rewriting the flat superspace action (9) and its κ-symmetry (18) in a fashion which will make the transition to curved superspace almost obvious. (For the moment we shall ignore the gauge degrees of freedom. These will be reinstated in the next section when we introduce background super Yang-Mills fields.)

In flat superspace the superconnection ${\omega_A}^B = 0$ and the only nonvanishing component of torsion is ${T_{\alpha\beta}}^a = 2\Gamma^a_{\alpha\beta}$. From the definition of torsion (41) we then have

$$de^a = e^\alpha e^\beta \Gamma^a_{\beta\alpha}, \quad de^\alpha = 0 \ . \tag{69}$$

These equations are solved by

$$e^a = dX^a + d\theta^\alpha \Gamma^a_{\alpha\beta}\theta^\beta, \quad e^\alpha = d\theta^\alpha \ . \tag{70}$$

Thus the flat superveilbein has the following form:

$${e_M}^A = \begin{pmatrix} {e_m}^a = {\delta_m}^a & {e_m}^\alpha = 0 \\ {e_\mu}^a = \delta^\alpha_\mu \Gamma^a_{\alpha\beta}\theta^\beta & {e_\mu}^\alpha = \delta^\alpha_\mu \end{pmatrix} \tag{71}$$

We can, therefore, rewrite V^a_i and $\partial_i\theta^\alpha$ as follows:

$$\begin{aligned} V^a_i &= \partial_i X^a + \partial_i\theta^\alpha \Gamma^a_{\alpha\beta}\theta^\beta = \partial_i Z^M {e_M}^a \ , \\ \partial_i\theta^\alpha &= \partial_i Z^M {e_M}^\alpha \equiv V^\alpha_i \ . \end{aligned} \tag{72}$$

Consider now the three-form H

$$H = \frac{1}{2} e^\alpha \ e^\beta \ e^a \Gamma_{a\beta\alpha}$$

Using (69) and the Fierz identity (16) we can readily verify that H is closed, i.e. $dH = 0$. Moreover, H can also be shown to be exact, i.e.

$$H = dB, \quad B = -\frac{1}{2} e^\alpha \ e^a \Gamma_{a\alpha\beta}\theta^\beta \ . \tag{73}$$

The only nonvanishing components of B are $B_{\alpha a} = -B_{a\alpha} = 1/2 \ \Gamma_{a\alpha\beta}\theta^\beta$. Using (71), (72) and (73) we can recast the Wess-Zumino term in (9) in the following form:

$$\begin{aligned} &\mathcal{E}^{ij} \ V^a_i \ \partial_j\theta^\alpha \ \Gamma^a_{\alpha\beta} \ \theta^\beta \\ &= \mathcal{E}^{ij} \ V^a_i \ V^\alpha_j (2B_{\alpha a}) \\ &= \mathcal{E}^{ij} \ (V^a_i \ V^\alpha_j B_{\alpha a} + V^\alpha_i \ V^a_j B_{a\alpha}) \\ &= \mathcal{E}^{ij} \ V^A_i \ V^C_j \ B_{CA} \\ &= \mathcal{E}^{ij} \ \partial_i Z^N \ \partial_j Z^M \ B_{MN} \ . \end{aligned} \tag{74}$$

Hence the flat superspace action (9) can be rewritten as

$$I=\frac{1}{\pi}\int d^2\xi\left[\frac{1}{2}\eta^{ij}\ V_i^a\ V_j^b\ \eta_{ab}+i\mathcal{E}^{ij}\ \partial_i Z^N\ \partial_j Z^M\ B_{MN}\right] \tag{75}$$

with $V_i^A=\partial_i Z^N e_N{}^A$. Also, using (71) we can rewrite the κ-symmetry (18) as

$$\delta Z^M\ e_M{}^a=0, \qquad \delta Z^M e_M{}^\alpha=2\not{V}_i^{\alpha\beta}\ \kappa_\beta^i\ . \tag{76}$$

The flat superspace action (75) and its κ-symmetry (76) are written in a form in which they can be readily extended to curved superspace. Since all the world indices $(M,N,...)$ are contracted these formulae are manifestly invariant under supergeneral coordinate tranformations (provided the backgrounds are also tranformed appropriately). However, a näive extension of (75) to curved superspace (by simply interpreting $e_M{}^A$ to describe curved superspace with B_{MN} the corresponding antisymmetric field) does not possess the κ-symmetry (76). The subtlety is associated with the dilation superfield ϕ and the way it enters in the solutions of the Bianchi identity (50) for the various components of H_{ABC}, listed in (57). It turns out that the correct action to use is

$$I^{(Bgd)}=\frac{1}{\pi}\int d^2\xi\left[\eta^{ij}\ \phi\ V_i^a\ V_j^b\ \eta_{ab}+\mathcal{E}^{ij}\ \partial_i Z^N\ \partial_j Z^M\ B_{MN}\right] \tag{77}$$

This action possesses the κ-symmetry (76) if the constraints (55) and (56) are satisfied. To verify this statement one proceeds as follows. We have,

$$\begin{aligned}
\delta V_i^a &= \partial_i(\delta Z^M)\ e_M{}^a+\delta_i Z^M\ \delta e_M{}^a\\
&= \partial_i(\delta Z^M\ e_M{}^a)-\delta Z^M\ \partial_i e_M{}^a+\partial_i Z^M\ \delta e_M{}^a\\
&= \partial_i Z^N\ \delta Z^M\ \partial_{[M}e_{N)}{}^a\\
&= -(\delta Z^M e_M{}^\alpha)(V_i^b\ \omega_{\alpha b}{}^a+2V_i^\beta\ \Gamma^a_{\beta\alpha})
\end{aligned} \tag{78}$$

where we have used (51) and (76). Also, using (57) and (76) we get

$$\begin{aligned}
\delta(\mathcal{E}^{ij}\ \partial_i Z^N\ \partial_j Z^M\ B_{MN}) &= \mathcal{E}^{ij}\ \partial_i Z^N\ \partial_j Z^M\ \delta Z^L\{3\ \partial_{[L}\ B_{MN)}\}\\
&\quad +\text{surface terms}(S.T.)\\
&= \mathcal{E}^{ij}\ \partial_i Z^N\ \partial_j Z^M\ \delta Z^L\ H_{LMN}+S.T.\\
&= \mathcal{E}^{ij}\ (\delta Z^M\ e_M{}^\alpha)\\
&= \{\ -2\phi\ \not{V}_{i\alpha\beta}\ V_j^\beta+\frac{1}{2}(\not{V}_i\not{V}_j)_\alpha^{\ \beta}\ \lambda_\beta\ \}\ +S.T.
\end{aligned} \tag{79}$$

Finally, we also have

$$\begin{aligned} \delta\phi &= \delta Z^M \, \partial_M \phi \\ &= (\delta Z^M \, e_M{}^\alpha) \, \lambda_\alpha \end{aligned} \tag{80}$$

which follows from the definition of λ_α and (76). Using (78), (79) and (80) one can show that

$$\delta I^{(Bgd)} = \frac{1}{\pi} \int d^2\xi \; V_-^2 \; V_+^\alpha \; \kappa_{+\alpha}$$

which vanishes because of the Virasoro constraint $V_-^2 = 0$.

We emphasize that we need to use the constraints (55) and (56) to ensure that $I^{(Bgd)}$ has κ-symmetry. This result (which requires the background fields to satisfy equations of motion of supergravity for a consistent coupling to the superstring), true already at the classical level, may be contrasted with the NSR formulation of the superstring in which it is the requirement of quantum conformal invariance that implies equations of motion for the background fields. Of course, since we have used the Virasoro constraint $V_-^2 = 0$ to prove κ-symmetry, we must ensure that $I^{(Bgd)}$ has no conformal anomaly. No quantum calculations have as yet been carried out with $I^{(Bgd)}$, but we expect that the requirement of (quantum) conformal invariance will not imply any new conditions on the background fields. In a sense, then, κ-symmetry encodes both the physical supersymmetry as well as the conformal invariance of the superstring.

6. INTRODUCING BACKGROUND SUPER YANG-MILLS FIELDS

So far we have ignored the gauge degrees of freedom of the superstring. We would now like to reinstate these and extend the analysis of the previous section to the case in which background super Yang-Mills fields are also present. It is of interest to do so since vacuum configurations of string field theory having nonvanishing Yang-Mills fields seem to be phenomenologically promising.[3]

The gauge part of the free heterotic superstring is given in (10), which we rewrite here for convenience.

$$I_{YM} = \frac{1}{2\pi} \int d^2\xi \; \psi^s \; \partial_- \; \psi^s \tag{81}$$

To couple background super Yang-Mills fields to the superstring one obvious change that we must make is to replace the ordinary derivative in (81) by the gauge – covariant derivative. This gives

$$I_{YM}^{(Bgd)} = \frac{1}{2\pi} \int d^2\xi \ \psi^s (\mathcal{D}_- \psi)^s \ , \tag{82}$$

where

$$(\mathcal{D}_- \psi)^s = \partial_- \psi^s - (A_-)^{st} \ \psi^t \ , \tag{83}$$

$A_- = \partial_- Z^M A_M$ being the projection on the world-sheet of the 10-dimensional super Yang-Mills potential A_M. Since A_- transforms under a κ-transformation, the gauge part of the superstring action is no longer trivially κ-invariant. For correct coupling to background super Yang-Mills fields we must, therefore, check whether (82) is κ-invariant or not. To do so we need to find out how A_- transforms under (76). We have,

$$\begin{aligned} \delta A_- &= \partial_-(\delta Z^M) \ A_M + \partial_- Z^M \ \delta A_M \\ &= \partial_-(\delta Z^M \ A_M) - \delta Z^M \ \partial_- A_M + \partial_- Z^M \delta A_M \\ &= \partial_-(\delta Z^M \ A_M) + \partial_- Z_N \ \delta Z^M \ \partial_{[M} A_{N)} \\ &= \mathcal{D}_- \wedge_\kappa + \partial_- Z^N \delta Z^M \ F_{MN} \end{aligned} \tag{84}$$

where

$$\mathcal{D}_- \wedge_\kappa = \partial_- \wedge_\kappa + [\wedge, A_-], \qquad \wedge_\kappa \equiv \delta Z^M A_M \ . \tag{85}$$

Now, using the super Yang-Mills constraint (62), the solution (68) for $F_{a\alpha}$ and the Virasoro constraint $V_{-}^{2} = 0$, we can show that

$$\delta A_- = \mathcal{D}_- \wedge_\kappa \quad . \tag{86}$$

Thus, the κ variation of A_- is a field-dependent gauge transformation. Therefore, contrary to what happens in the free case, in the presence of background super Yang-Mills fields the superstring action possesses κ-symmetry only if the gauge fermions ψ^s transform by a gauge transformation:

$$\delta \psi^s = (\wedge_\kappa \psi)^s \quad . \tag{87}$$

With this transformation rule for ψ^s $I_{YM}^{(Bgd)}$ is κ-invariant. One might, therefore, conclude that the total action $I^{(Bgd)} + I_{YM}^{(Bgd)}$ is admissible as a proper description of the superstring propagating in curved superspace in the presence of background super Yang-Mills fields. Further reflection, however, shows that there are serious problems with the above system. As discussed below these problems and their resolution are intimately connected with the properties of the field theory limit of the superstring.

A puzzling aspect of the above analysis is that in proving the κ-invariance of $I_{YM}^{(Bgd)}$ we implicitly assumed that $I^{(Bgd)}$ is separately κ-invariant, which is true only if we assume that H satisfies the Bianchi identity (49). However, as mentioned before the constraints (55), (56) and (62) and the Bianchi identities (45), (49) and (60) do not imply coupled supergravity – super Yang-Mills equations of motion. What we have above is, therefore, somewhat surprising since, in analogy with the pure supergravity case, one would have expected to have used the coupled equations of motion to ensure the κ-symmetry of the superstring. The resolution of this puzzle is deeply connected with the other, more serious, problem that the above system has. The gauge fermions ψ^s are 2-dimensional chiral fermions and the action $I_{YM}^{(Bgd)}$ involves chiral gauge couplings. So the (background) gauge symmetry of this action has an anomaly. As a consequence of this and the fact that the κ-transformation of ψ^s is a gauge transformation, the κ-symmetry is also anomalous. The resolution of this problem lies in adopting the modified Bianchi identity (63) for H. This is satisfying since it also resolves the puzzle mentioned above.

One result of using (63) instead of (49) is that the two-form potential B is no longer gauge-invariant but transforms as in (66). This has the welcome feature, familiar from a similar phenomenon that occurs in purely bosonic σ-models,[6] of removing the gauge anomaly from $I_{YM}^{(Bgd)}$. Less obvious is the fact that it also eliminates the anomaly in the κ-symmetry, as we shall now show.

We start by integrating out the gauge fermions from $I_{YM}^{(Bgd)}$. In perturbation theory the resulting effective action, $I_{YM}^{(Eff)}$, can be written as the sum of n-point functions all of which, except for $n = 2$, are finite by power counting. Thus, although the gauge field couples to ψ^s only through A_-, the effective action acquires a dependence on A_+ through the regularization needed for the 2-point function. Now, the gauge-variation of $I_{YM}^{(Eff)}$ can be shown to be equal to the anomaly

$$\delta_\wedge I_{YM}^{(Eff)} = \frac{1}{8\pi} \int d^2\xi \; \mathcal{E}^{ij} \; tr(\wedge \partial_i A_j) \tag{88}$$

Since the nonlocal part of the effective action is a functional of A_- only, the effective action must contain a local piece equal to $-\frac{1}{16\pi}\int d^2\xi \; tr(A_+A_-)$ in order to reproduce (88). We may, therefore, write the result of integrating ψ^s from $I_{YM}^{(Bgd)}$ as

$$I_{YM}^{(Eff)} = -\frac{1}{16\pi} \int d^2\xi \; tr(A_+A_-) + G[A_-] \; . \tag{89}$$

Before proceeding further we remark here that cancellation of the anomaly in (88) by the change in $I^{(Bgd)}$ (induced by the anomalous transformation law of B, (66)) fixes the arbitrary constant c_1 to be $-1/16$. More precisely, if we restore the ten-dimensional gauge coupling constant g_{10}, Newton's constant k_{10} and the slope parameter α' in the action then the anomaly cancellation gives the heterotic string relation $g_{10}/k_{10} \sim 1/\sqrt{\alpha'}$. It is interesting that this result follows from purely σ-model considerations.

We are now ready to show that $I^{(Bgd)} + I^{(Eff)}_{YM}$ is invariant under (76). The proof rests on the observation made earlier in (86), namely, that the κ-variation of A_- is just a gauge transformation. This is crucial for the present analysis since it allows us to compute the κ-variation of (89) without knowing the exact form of G. In fact, using (88), (89), (86) and the expression for κ-variation of A_+,

$$\delta A_+ = \mathcal{D}_+ \wedge_\kappa + \chi^\alpha (\not{V}_+ \not{V}_-)_\alpha^\beta \, \kappa_{+\beta} \,, \tag{90}$$

which can be obtained as for A_-, we get

$$\delta I^{(Eff)}_{YM} = \frac{1}{8\pi} \int d^2\xi \; \mathcal{E}^{ij} \; tr(\wedge_\kappa \, \partial_i A_j) - \frac{1}{16\pi} \int d^2\xi \; tr(\chi^\alpha (\not{V}_+ \not{V}_-)_\alpha^\beta \; \kappa_{+\beta} \; A_-) \tag{91}$$

To find the κ-variation of the total action $I^{(Bgd)} + I^{(Eff)}_{YM}$ we must add to (91) the κ-variation of $I^{(Bgd)}$. The κ-variation of the B-term is

$$\begin{aligned}
&\delta(\mathcal{E}^{ij} \; \partial_i Z^N \; \partial_j Z^M \; B_{MN}) \\
&= \mathcal{E}^{ij} \; \partial_i Z^N \; \partial_j Z^M \; \delta Z^L \{ \; 3 \; \delta_{[L} B_{MN)} \} \; + S.T. \\
&= \mathcal{E}^{ij} \; \partial_i Z^N \; \partial_j Z^M \; \delta Z^L \{ \; H_{LMN} - 6c_1 (\omega_{3YM})_{[LMN)} \} \; + S.T.
\end{aligned}$$

where in the last equality we have used (65). The κ-variation of the first term in $I^{(Bgd)}$ combines with the H-term above and vanishes as in the pure supergravity case. The ω_{3YM} piece contributes

$$\frac{2c_1}{\pi} \int d^2\xi \; \mathcal{E}^{ij} \; tr\{ \; \wedge_\kappa \; \partial_i A_j + \chi^\alpha (\not{V}_i \not{V}_-)_\alpha^\beta \; \kappa_{+\beta} A_j \} \quad . \tag{92}$$

Using $c_1 = -1/16$ and the Virasoro constraint $V_-^2 = 0$ one can verify that the sum of (91) and (92) vanishes.

We thus have now an action which consistently couples the superstring to background super Yang-Mills fields in curved superspace.

7. OUTLOOK

There are several problems and questions that have been left unresolved by the analysis of the last few sections. Among them a relatively more serious one is the fact that the model described by $I^{(Bgd)}$ is expected to have a Lorentz anomaly. The reason for this is that in the physical (light-cone) gauge the θ's are 2-dimensional chiral fermions and so in the presence of background fields they have chiral couplings to the spin connection. In the case of purely bosonic σ-models a similar problem arises[6] which is resolved by ascribing an anomalous transformation law to the antisymmetric field under local Lorentz rotations, much like the modification we needed in the presence of background super Yang-Mills fields. It has been suggested that a supersymmetric extension of this might work in the present case also.[12] In effect, what this means is that we must further modify the Bianchi identity satisfied by H to read

$$dH = c_1 \, tr \, F^2 + c_2 \, tr \, R^2 \tag{93}$$

where c_2 is, a priori, an arbitrary parameter and the trace in the second term is over tangent space indices. A detailed analysis of the resulting supergravity theory for the background fields has as yet not been carried out, but there are indications that the equations of motion of this theory cannot be obtained in a closed form but only as infinite expansions in the parameter c_2.[17] So the whole thing starts looking more and more like the GS mechanism[1] for the anomaly cancellation in the field theory limit of the superstring. It is, therefore, tempting to speculate that (93) describes some sort of field theory limit of the superstring theory, but that it is really so is far from clear. Moreover, it has not yet been verified that (93) actually removes the Lorentz anomaly from $I^{(Bgd)}$ and that this Bianchi identity is consistent with κ-symmetry. A thorough investigation of this problem is clearly required. One expects that this will also fix c_2 in the same manner as c_1 was fixed by the cancellation of the gauge anomaly.

Another aspect of the superspace σ-models that remains poorly understood is the nature of κ-symmetry. It is true that we have learnt to efficiently use κ-symmetry as a guiding principle for building consistent superspace σ-models. Nevertheless, a deeper understanding of its nature and its connection with the symmetries of the 10-dimensional field theory is clearly desirable.

Finally, there is still the problem of finding a manifestly covariant and supersymmetric description of the superstring. Such a description is clearly needed for a manifestly covariant and supersymmetric formulation of superstring field theory. Some progress in this direction has been claimed in Ref. 23 but this formulation appears to have problems.[24] Clearly much work remains to be done. We hope that these lectures have sufficiently aroused your interest to actively pursue answers to some of the questions and problems mentioned above.

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