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LIGHT FERMION MASS GENERATION IN A KALUZA KLEIN THEORY

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ABSTRACT

I study a model of a compactified Kaluza-Klein theory as an SU(2) gauge theory on a flat two dimensional Minkowski space M^2 direct producted with a two sphere S^2. The perturbation theory on M^2 is developed. In particular, induced tree level fermion-gauge field and one loop fermion-higgs Yukawa couplings are computed. The latter calculation generates a mass heirarchy for the massless fermions on M^2.

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1. Introduction

In recent years, Kaluza-Klein unification schemes have resurged. In a simple scenario, a higher dimensional space is compactified to the direct product of flat four dimensional Minkowski space with a two sphere. If the four dimensional metric contains the gauge fields, there is an SO(3) gauge theory on the product space at low energies.[1]

As a model of this low energy theory, I consider an SU(2) gauge theory on the direct product of a flat two dimensional Minkowski space M^2 with a two sphere S^2 and neglect gravitational effects. In this theory, a background vector potential preserving two dimensional Lorentz invariance is allowed. I choose an 't Hooft-Polyakov monopole on S^2[2,3]

$$\overrightarrow{A} = \frac{1}{r}\hat{e}_r \times \overrightarrow{T}$$

where r and $\hat{e}_r$ are the radius of is the outward normal to the sphere. This background gauge field, or any topologically non-trivial one in any compactification scheme, will induce a multiplet of massless chiral fermions and massive gauge bosons and fermions on the uncompactified space. If the compactification scale is very large compared with the weak scale, only the massless fermions can become the observed quarks and leptons. No one has examined in detail how they might obtain the correct masses.

This paper develops the perturbative analysis of this problem on $M^2 \times S^2$. It is shown that chiral symmetry preservation on the total space implies chiral symmetry preservation in the uncompactified space. However, on M^2 the ground state monopole vector potential couples to fermions as a pseudoscalar boson and can act as a Higgs field. Only the massless fermions in the field of the lowest charged ($z = 1$) SU(2) doublet monopole receive tree level Higgs mass insertions. For higher charged ($z \geq 2$) doublet backgrounds, the massless fermions obtain masses from one loop fermion-Higgs Yukawa couplings. All these techniques can be easily generalized to arbitrary M^n. But then one must confront the usual problems of perturbative non-renormalizability in higher dimensions which I will not address.

In the next chapter, I find the vector potential and fermion eigenmodes in the field of a monopole. In chapter three, I find the induced fermion-vector and fermion-pseudo-scalar couplings on M^2 by integrating out the S^2 dependence of the fermion-gauge field coupling on the total space. I verify that chiral symmetry is preserved on M^2 in chapter four and calculate the one loop fermion-Higgs couplings to obtain a mass heirarchy in chapter five.

2. Quantum Fluctuations

2.1 Gauge Bosons

In this section, I will find the fluctuation states about the monopole background. The Yang-Mills action on a curved space

$$S = -\frac{1}{4g^2}\int d^{n+2}z Tr F^{\mu\nu}F_{\mu\nu}$$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

$$A_\mu = A^a_\mu T_a$$

yields the inverse propagator

$$\triangle_{ab}^{\mu\nu} \equiv \frac{\delta^2 S}{\delta A_\mu^a \delta A_\nu^b} = (-D^\alpha D_\alpha g_{\mu\nu} + 2F_{\mu\nu} + R_{\mu\nu})^{ab} \tag{2.1}$$

in the background field gauge

$$D_\alpha^{ab} \delta A_b^\alpha = (\nabla_\alpha - iA_\alpha)^{ab} \delta A_b^\alpha = 0 \tag{2.2}$$

where a, b, are group indices. The variation operator (2.1) is manifestly diagonal for $\mu, \nu = 1, 2$ and will be diagonalized for $\mu, \nu = 3, 4 \equiv \theta, \phi$. All terms with one index in M^2 and one index in S^2 are zero. When acting on its eigenvectors $\triangle_{ab}^{\mu\nu}$ has the form $\triangle_{ab} g_{\mu\nu}$.

The $D_\mu D_\nu$ terms in the numerators of the gauge boson propagators

$$(\triangle_{ab}^{\mu\nu})^{-1} = \triangle_{ab}^{-1} [g_{\mu\nu} - (1-\xi) \frac{D_\mu D_\nu}{D^2}]$$

are zero in the one loop diagrams of chapters four and five by Ward identities. Therefore, I only need to find the eigenvalues and eigenvectors of the variation operator in the background field gauge (2.2) and assume that the boson propagators are proportional to $g_{\mu\nu}$ to obtain gauge invariant results. The relevant Ward identities will be proved in chapters four and five.

Vectors Polarized in M^2

The variation operator for vectors polarized in M^2 is♮1

$$\frac{\delta^2 S}{\delta A_i^a \delta A_j^b} = (\nabla^\alpha - iA^\alpha)(\nabla_\alpha - iA_\alpha)\eta_{ij}$$

♮1 A discussion of collective angular momenta and monopoles has been given by D. Olive.[4]

$$= [k^2 + \frac{(\overrightarrow{L+T})^2 - (\hat{e}_r \cdot T)^2}{r}]\eta_{ij}$$

$$= [k^2 + \frac{\overrightarrow{J_2}^2 - (\hat{e}_r \cdot T)^2}{r}]\eta_{ij} \tag{2.3}$$

$$i, j = 1, 2$$

The conserved monopole charge, $e_r \cdot T$, commutes with $(J_2)_i$. In the adjoint, $(\hat{e}_r \cdot T)$ has eigenvalues $0, \pm 1$. Equation (2.3) is the gauge covariant Laplacian for scalars times $\eta_{\mu\nu}$ whose spectrum is bounded below; the $J_2 = 0$ state has $\hat{e}_r \cdot T = 0$. (If G/H $= S^2 =$ SO(3)/SO(2) with SO(2) generated by $\hat{e}_r \cdot T$, these operators have the form $C_2(G) - C_2(H) = C_2(G/H)$, where C_2 denotes the second Casimir invariant.)[5]

By separation of variables, the fluctuations polarized in the M^2 directions are generalized scalar harmonics $Y_{(\hat{e}_r \cdot T)lm}$ times plane waves times unit vectors

$$A^1_{(\hat{e}_r \cdot T)lm} = \frac{1}{r}\begin{pmatrix} \overrightarrow{\epsilon} \\ 0 \end{pmatrix} e^{i\overrightarrow{k}\cdot\overrightarrow{x}} Y_{(\hat{e}_r \cdot T)lm}(\theta, \phi)$$

where $\overrightarrow{k}$, $\overrightarrow{\epsilon}$ and $\overrightarrow{x}$ are on M^2, θ, ϕ are coordinates on S^2, and lower component is on S^2. In M^2 they represent two infinite towers of very massive gauge bosons

$$m^2 = \frac{l(l+1) - 1}{r^2} \quad l = 1, \ldots, \infty$$

and another tower of very massive gauge bosons with one massless gauge boson

$$m^2 = \frac{l(l+1)}{r^2} \quad l = 0, \ldots, \infty$$

Henceforth l will denote the gauge field angular momentum.

It will far easier to compute in chapters three and four in the Abelian gauge.[6,7] The rotation from the previous notation to the abelian gauge essentially rotates $\hat{e}_r \cdot T$ to $\hat{z} \cdot T$. The rotation matrix is

$$U(\theta,\phi) = e^{i\theta T_y} e^{i\phi T_z}$$

The gauge covariant derivative

$$\overrightarrow{D} = \hat{e}_r \times \overrightarrow{L+T}$$

is transformed to

$$\overrightarrow{D} = \frac{\hat{e}_\theta}{r}\partial_\theta \times \mathbf{1} + \frac{\hat{e}_\phi}{r\sin\theta}(\partial_\phi \times \mathbf{1} + T_z \begin{Bmatrix} cos\theta - 1 \\ cos\theta + 1 \end{Bmatrix} \begin{Bmatrix} NH \\ SH \end{Bmatrix})$$

where NH and SH denote the northern and southern hemispheres. In components,

$$\overrightarrow{D} \equiv \frac{\hat{e}_\theta}{r} \begin{pmatrix} D_\theta^{(+1)} & 0 & 0 \\ 0 & D_\theta^{(0)} & 0 \\ 0 & 0 & D_\theta^{(-1)} \end{pmatrix} + \frac{\hat{e}_\phi}{r\sin\theta} \begin{pmatrix} D_\phi^{(+1)} & 0 & 0 \\ 0 & D_\phi^{(0)} & 0 \\ 0 & 0 & D_\phi^{(-1)} \end{pmatrix}$$

I can now add SU(2) monopoles colinearly, or put a Q in front of T_z to change the charge of one SU(2) monopole. I will henceforth use Q to denote this factor.

Scalar eigenfunctions of $D^\alpha D_\alpha$ in the adjoint are now just the Wu-Yang monopole harmonics[8] Y_{qlM} with $q = 0, \pm Q$. In the northern hemisphere

$$Y_{qlm} = M_{qlm}\frac{(-1)^n}{2^n n!}(1-x)^{\frac{-\alpha}{2}}(1+x)^{\frac{-\beta}{2}} e^{i(m+q)\phi}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]$$

$$= 2^{m-n}[\frac{2l+1}{4\pi}\frac{(l-m)!(l+m)!}{(l-q)!(l+q)!}]^{\frac{1}{2}}\frac{(-1)^n}{n!}(1-x)^{\frac{-\alpha}{2}}(1+x)^{\frac{-\beta}{2}}\times$$

$$\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}] \times e^{i(m+q)\phi}$$

$$= (-1)^n 2^m [\frac{2l+1}{4\pi}\frac{(l-m)!}{(l-q)!}\frac{(l+m)!}{(l+q)!}]^{\frac{1}{2}}(1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}} P_n^{\alpha,\beta} e^{i(m+q)\phi}$$

$$\alpha = -q-m \quad \beta = q-m \quad n = l+m \quad x = cos\theta$$

$P_n^{\alpha,\beta}$ is a Jacobi polynomial. The southern hemisphere harmonics are related to the northern hemisphere ones by by $(Y_{qlm})_{SH} = e^{-i2q\phi} Y_{qlm}$. and obey the same derivative formulas.

The fluctuations on M^2 are Y_{qlm} times plane waves times unit vectors

$$A^1_{qlm} = \frac{1}{r}\begin{pmatrix} \vec{\epsilon} \\ 0 \end{pmatrix} e^{i\vec{k}\cdot\vec{x}} Y_{qlm}(\theta,\phi) \tag{2.4}$$

Vectors Polarized in S^2

For vectors polarized in S^2, defining $T_r \equiv \hat{e}_r \cdot T, T_\theta \equiv \hat{e}_\theta \cdot T$ and $T_\phi \equiv \hat{e}_\phi \cdot T$, the variation operator is[♮2] [9–13]

$$(-D^\alpha D_\alpha g_{\mu\nu} - 2iF_{\mu\nu} + R_{\mu\nu})A^\nu =$$

$$(k^2 - \nabla^\alpha \nabla_\alpha g_{\mu\nu} + R_{\mu\nu})A^\nu$$

$$+\frac{(T^2 - T_r^2)}{r^2} A_\mu$$

$$+2i\frac{\hat{e}_\phi}{r^2}[T^\phi \partial_\theta A_\theta - csc\theta T_\theta(\partial_\phi A_\theta - cot\theta A_\phi)]$$

♮2 Related calculations have been performed by Horvath et al, A. Schellekens, and M. Evans and B. Ovrut.[6]

$$+2i\frac{\hat{e}_\phi}{r^2}[T^\phi(\partial_\theta - cot\theta)A_\phi - T_\theta(csc\theta\partial_\phi A_\phi + \cot\theta A_\theta)]$$

$$-\frac{2icsc\theta T_r A_\phi}{r^2}\hat{e}_\theta + \frac{2isin\theta T_r A_\theta}{r^2}\hat{e}_\phi$$

$$\mu, \nu = \theta, \phi$$

This formula should be compared with

$$[k^2 + \frac{(L+T+S)^2 - (\hat{e}_r \cdot T)^2}{r^2}]\overrightarrow{A} =$$

$$+k^2[\hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi]$$

$$+\frac{\hat{e}_\theta}{r^2}[(\partial_\theta^2 + csc^2\theta\partial_\phi^2)A_\theta + 2\csc\theta\cot\theta(\partial_\phi - i\sin\theta T^\theta A_\phi)$$

$$+\csc^2\theta A_\theta + 2iT^\theta csc\theta\partial_\phi A_\theta - 2iT^\phi\partial_\theta A_\theta + (T_\theta^2 + T_\phi^2)A_\theta$$

$$-2iT_r A_\phi + i\cot\theta T^\phi A_\theta + 2A_\theta)]$$

$$+\frac{\hat{e}_\phi}{r^2}[-(\partial_\theta^2 + \csc^2\theta\partial_\phi^2) - 2csc\theta cot\theta(\partial_\phi - iT^\theta sin\theta)A_\theta$$

$$+csc^2\theta A_\phi - 2iT^\phi\partial_\theta A_\phi + 2iT^\theta csc\theta\partial_\phi A_\phi + (T_\theta^2 + T_\phi^2)A_\phi$$

$$+2iT_r A_\theta + icot\theta T^\phi A_\phi + 2A_\phi]$$

If c and s denote the first and second equations, they are equal under the rescaling

$$\overrightarrow{A^s} = \begin{pmatrix} A_\theta^c \\ A_\phi^c csc\theta \end{pmatrix}$$

$$\frac{\hat{e}_\phi^s}{\sin\theta} = \hat{e}_\phi^c$$

As before, $[\hat{e}_r \cdot T, \overrightarrow{L+T}] = 0 = [\hat{e}_r \cdot T, \overrightarrow{L+T+S}]$.

Once again, I will assume separation of variables and focus on the angular dependence. The solutions of the angular equation will be called SU(2) vector spherical harmonics and denoted by $\overrightarrow{V}_{qlm}$. They are vectors on S^2 but scalars on M^2. Hence, the fluctuations polarized in the S^2 directions are SU(2) vector spherical harmonics times plane waves

$$A^2_{qlm} = \frac{1}{r}\begin{pmatrix} 0 \\ \overrightarrow{V}_{qlm}(\theta,\phi) \end{pmatrix} e^{i\overrightarrow{k}\cdot\overrightarrow{x}} \tag{2.5}$$

In the Abelian gauge, the SU(2) vector spherical harmonics are

$$\overrightarrow{V}_{qlm} = \sum_i Y_{qJ_2(m-i)}\hat{e}_i \langle J_2(M-i)1i|J_2 1lM\rangle$$

with $q = \pm Q, 0$ in exact analogy with the usual vector spherical harmonics

$$\overrightarrow{Y}_{ljm} = \sum_i Y_{J(m-i)}\hat{e}_i \langle J\,M-i\,1i|J1lM\rangle$$

Both of these vectors are in three dimensions and must be specialized to the the sphere by imposing

$$\hat{e}_r \cdot \overrightarrow{V}_{qlm} = 0 = \hat{e}_r \cdot \overrightarrow{Y}_{lm}$$

It will be extremely convenient to rotate these harmonics into a 'spin Abelian gauge'. Recalling the monopole harmonic addition theorem

$$Y_{q_1 l_1 m_1}(\Omega) Y_{q_2 l_2 m_2}(\Omega) = \sum_{l_3} \langle l_1\, q_1\, l_2\, q_2 | l_1\, l_2\, l_3\, (q_1+q_2)\rangle$$

$$\times \langle l_1\, m_1\, l_2\, m_2 | l_1\, l_2\, l_3\, (m_1+m_2)\rangle Y_{(q_1+q_2)l_3(m_1+m_2)}(\Omega)$$

the vector potential is rotated by

$$\Omega = \begin{pmatrix} e^{-i\phi}Y_{11\,-1} & e^{-i\phi}Y_{110} & e^{-i\phi}Y_{111} \\ Y_{01\,-1} & Y_{010} & Y_{011} \\ e^{i\phi}Y_{-11\,-1} & e^{i\phi}Y_{-110} & e^{i\phi}Y_{-111} \end{pmatrix}$$

to

$$\overrightarrow{V}_{qlm} = e^{i\phi}Y_{q-1\,lM}\hat{e}_{-1} + Y_{qlM}\hat{e}_0 + e^{-i\phi}Y_{q+1\,lM}\hat{e}_{+1}$$

for each J_2.

Let's specialize to a $Q = +1$ (or equivalently $Q = -1$) background and find the angular momentum degeneracies. Continuous Wu-Yang harmonics cannot have l less than q. Consequently, $\overrightarrow{V}_{\pm 1lm}$ has $3 \times (2l+1)$ states for $l \geq 2$. At $l = 1$, there are only $2 \times (2l+1)$ because J_2 cannot be zero. At $l = 0$, there are two unstable states with eigenvalues

$$\frac{0 \times (0+1) - 1}{r^2} = -\frac{1}{r^2}$$

They correspond to tachyonic bosons on M^2. (The existence of these two instabilities substantiates a claim by Hosotani and refutes another by Schellekens.[9] [15]) $\overrightarrow{V}_{0lm}$ has $3 \times (2l+1)$ states for $l \neq 0$ and one zero mode. This degeneracy analysis can be done for any Q background.

Since I know that I can impose the $\hat{e}_r \cdot \overrightarrow{V} = 0$ condition before the rotation and still have an eigenfunction of the variation operator, I can also impose it after the rotation. Since

$$\hat{e}_r = -\sqrt{4\pi}\,\overrightarrow{Y}_{010} = \overrightarrow{V}_{000}$$

in spin abelian gauge is

$$\hat{e}_0 = \hat{z}$$

forcing the vector potential to lie on the sphere implies

$$\overrightarrow{V}_{qlm} = e^{i\phi} Y_{q-1\,lM} \hat{e}_{-1} + e^{-i\phi} Y_{q+1\,lM} \hat{e}_{+1}$$

Now there are $2 \times (2l+1)$ states for $l \geq 1$, $q = 0$ and for $l \geq 2$, $q = \pm 1$. For $l = 1$, $q = \pm 1$ there are $(2l+1)$ eigenvectors.

2.2 FERMIONS

The gamma matrices on $M^2 \times S^2$ may be written as[6] [16]

$$\Gamma^{\mu}(M^2 \times S^2) = \begin{pmatrix} \Gamma^j(M^2) \otimes \mathbf{1} \\ i\gamma_5(M^2) \otimes \Gamma^s(S^2) \end{pmatrix} \equiv \begin{pmatrix} \Gamma^j(M^2) \otimes \mathbf{1} \\ i\gamma_5(M^2) \otimes \sigma^s \end{pmatrix}$$

with $j = 1, 2$ and $s = 3, 4$. The Dirac Operator on $M^2 \times S^2$ is[6] [17,18]

$$D_{S^2}(A) = k_j \Gamma^j(M^2) + i\gamma_5(M^2) e_{\theta s} \Gamma^s(S^2)[\frac{1}{r}\partial_\theta - igA_\theta]$$

$$+ i\gamma_5(M^2) e_{\phi s} \Gamma^s(S^2)[\frac{1}{r\sin\theta}\partial_\phi - igA_\phi] - i\gamma_5(M^2) e_{rs} \Gamma^s(S^2)\frac{1}{r}$$

$$\equiv k_j \Gamma^j(M^2) + i\gamma_5(M^2)\hat{e}_\theta \cdot \overrightarrow{\sigma}[\frac{1}{r}\partial_\theta - igA_\theta]$$

$$+ i\gamma_5(M^2)\hat{e}_\phi \cdot \overrightarrow{\sigma}[\frac{1}{r\sin\theta}\partial_\phi - igA_\phi] - i\gamma_5(M^2)\hat{e}_r \cdot \overrightarrow{\sigma}\frac{1}{r}$$

It will be convenient to rewrite this operator such that the rotational symmetry is obscured. Let's rotate to spin abelian gauge again by taking $\hat{e}_r$ to $\hat{z}$. The Dirac operator becomes

$$k_j\Gamma^j(M^2) + \sigma_3\sigma_1[\frac{1}{r}(\partial_\theta + \frac{1}{2}cot\theta)] + \sigma_3\sigma_2[\frac{1}{rsin\theta}\partial_\phi - igA_\phi^{mon}] - ig\sigma_s\delta A^s$$

Now let's find the solutions for $\delta A_s = 0$. Let $B = \pm z$ and z for fermions in a doublet be the analogues of $q = 0, \pm Q$ and Q for gauge bosons in the adjoint. By separation of variables, the solutions are direct products of two two-component spinors

$$\Psi_{BJM}(z) = \overline{u}_{BJ}(x) \otimes \psi_{BJM}(y) \tag{2.6}$$

$z\epsilon M^2 \times S^2$, $x\epsilon M^2$, $y\epsilon S^2$. Using the monopole harmonic addition theorem, the eigenspinors are rotated by $e^{i\frac{\theta}{2}\sigma_2}e^{i\frac{\phi}{2}\sigma_3}$ from

$$(\psi_{BJM})_{NH} = \sum_\mu Y_{BL(M-\mu)}(\theta,\phi)\,\chi_\mu\langle L\,(M-\mu)\frac{1}{2}\,\mu|L\frac{1}{2}JM\rangle$$

to

$$(\psi^\pm_{BJM})_{NH} = \frac{1}{\sqrt{2}r}\begin{pmatrix} e^{\frac{i\phi}{2}}Y_{B-\frac{1}{2}JM}(\theta,\phi) \\ \pm e^{\frac{-i\phi}{2}}Y_{B+\frac{1}{2}JM}(\theta,\phi)\end{pmatrix}$$

$$\equiv \frac{1}{\sqrt{2}r}\begin{pmatrix} e^{\frac{i\phi}{2}}Y_{B-\frac{1}{2}JM}(\theta,\phi) \\ \frac{E}{|E|}e^{\frac{-i\phi}{2}}Y_{B+\frac{1}{2}JM}(\theta,\phi)\end{pmatrix}$$

for B greater than zero, and

$$= \frac{1}{\sqrt{2}r}\begin{pmatrix} \frac{E}{|E|}e^{\frac{i\phi}{2}}Y_{B-\frac{1}{2}JM}(\theta,\phi) \\ e^{\frac{-i\phi}{2}}Y_{B+\frac{1}{2}JM}(\theta,\phi)\end{pmatrix}$$

for B less than zero, with eigenvalues

$$E = \pm\frac{\sqrt{J(J+1)+\frac{1}{4}-B^2}}{r} = \pm\frac{\sqrt{(J+\frac{1}{2})^2-B^2}}{r}$$

and

$$(\psi^{\pm}_{BJM})_{SH} = e^{-i2q\phi}(\psi^{\pm}_{BJM})_{NH}$$

Once again B is an isospin index. For fermions in an isodoublet, $B = \pm z$ and z is the charge of the SU(2) background. Positive and negative z's have zero modes with zero upper and lower components, respectively. Henceforth, fermion angular momenta will be labeled J or J_i.

Fermions in an arbitrary isospin background can be written as direct products of doublets or triplets. For example, a background of

$$\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \tag{2.7}$$

in a quadruplet is equivalent to doublet monopoles of charge $z = 3$ and $z = 1$ or to four abelian monopoles of charge $B = \pm\frac{3}{2}$ and $B = \pm\frac{1}{2}$.

3. Induced Tree Level Yukawa Couplings

3.1 Decomposition of the Interaction

Fluctuations depending on θ and ϕ induce effective coupling constants on M^2 through integrals over θ and ϕ of the fermion-gauge field interaction on $M^2 \times S^2$.[19] For gauge fields in an isotriplet and fermions in an isodoublet, the decompostion (2.6) implies

$$L_I = -ig \int d^4z \overline{\Psi}_{AJ_2M_2}(z)\ \not{A}_{qlm} \tau^q_{AC} \Psi_{CJ_1M_1}(z)$$

$$= -ig \int d^4 z A^i_{qlm}(z) \tau^q_{AC} \overline{u}_{AJ_2}(x) \Gamma^i (M^2) u_{CJ_1}(x) \psi^\dagger_{AJ_2M_2}(y) \psi_{CJ_1M_1}(y)$$

$$-ig \int d^4 z A^s_{qlm}(z) \tau^q_{AC} \overline{u}_{AJ_2}(x) i\gamma_5 (M^2) u_{CJ_1}(x) \psi^\dagger_{AJ_2M_2}(y) \sigma^s \psi_{CJ_1M_1}(y)$$

For our fluctuations (2.4) and (2.5) these reduce to

$$-i\frac{g}{r} \int d^2 y Y_{qlm}(y) \psi^\dagger_{AJ_2M_2}(y) \psi_{CJ_1M_1}(y) \times \int d^2 x \overline{u}_{AJ_2}(x) \hat{\epsilon} \cdot \overrightarrow{\Gamma} (M^2) e^{ikx} u_{CJ_1}(x)$$

$$-i\frac{g}{r} \int d^2 y \psi^\dagger_{AJ_2M_2}(y) V^s_{qlm} \sigma^s \psi_{CJ_1M_1}(y) \times \int d^2 x \overline{u}_{AJ_2}(x) i\gamma_5 (M^2) u_{CJ_1}(x)$$

$$\equiv -ig_{AC} \int d^2 x \overline{u}_{AJ_2}(x) \hat{\epsilon} \cdot \overrightarrow{\Gamma} (M^2) e^{ikx} u_{CJ_1}(x)$$

$$-iM_{AC} \int d^2 x \overline{u}_{AJ_2}(x) i\gamma_5 (M^2) u_{CJ_1}(x)$$

I am taking the gauge fields to be in an isotriplet and the fermions in an isodoublet. However, using the decompositions similar to (2.7), the fermions can be in any isopsin representation and in the presence of a monopole of arbitrary charge . The first term will induce an effective two dimensional vector interaction while the second will induce an effective pseudo-scalar interaction.

3.2 Effective Vector Coupling Constants

The vector coupling constant matrix only picks up contributions from A^0_{qlm} and A^1_{qlm}. The triple overlap integral I need to evaluate is

$$g_{AC} = \frac{g}{r} \int d^2 y Y_{qlm}(y) \psi^\dagger_{AJ_2M_2}(y) \psi_{CJ_1M_1}(y)$$

The Dirac equation was solved for a background

$$\overline{\psi}\,\not{A}\psi = -z(1-\cos\theta)\overline{\psi}_{AJM}\tau^0_{AC}\sigma_1\psi_{CJM}$$

with $A_{+Q} = 0 = A_{-Q}$. It is convenient to make the definitions

$$g_{AC} = \frac{g}{r}\int d^2y \begin{pmatrix} \frac{1}{2}\psi^\dagger_{+z_2J_2M_2}Y_{0lm}\psi_{+z_1J_1M_1} & \psi^\dagger_{+z_2J_2M_2}Y_{-Qlm}\psi_{-z_1J_1M_1} \\ \psi^\dagger_{-z_2J_2M_2}Y_{+Qlm}\psi_{+z_1J_1M_1} & \frac{-1}{2}\psi^\dagger_{-z_2J_2M_2}Y_{0lm}\psi_{-z_1J_1M_1} \end{pmatrix}$$

$$= \begin{pmatrix} g_0^{(++)} & 0 \\ 0 & g_0^{(--)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ g_+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & g_- \\ 0 & 0 \end{pmatrix}$$

Note that $|z_1|$ and $|z_2|$ are not necessarily equal. So, explicitly,

$$g_+ = \frac{g}{2r}\int d^2yY_{+Qlm}\psi^\dagger_{-z_2J_2M_2}\psi_{+z_1J_1M_1}$$

$$= \frac{g}{2r^3}\int d^2yY_{+Qlm}[\frac{E_2}{|E_2|}Y_{z_2+\frac{1}{2}J_2\,-M_2}Y_{z_1-\frac{1}{2}J_1M_1}$$

$$-\frac{E_1}{|E_1|}Y_{-z_2-\frac{1}{2}J_2\,-M_2}Y_{z_1+\frac{1}{2}J_1M_1}] \times (-1)^{-z_2-\frac{1}{2}+M_2}$$

with similar expressions for $g_-, g_0^{(++)}$ and $g_0^{(--)}$.

To evaluate these integrals, I use a formula of Edmonds.[20] If $D^j_{m_1',m}(\alpha\beta\gamma)$ is an angular momentum matrix element

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} D^{j_1}_{m_1',m_1}(\alpha\beta\gamma)D^{j_2}_{m_2',m_2}(\alpha\beta\gamma)D^{j_3}_{m_3',m_3}(\alpha\beta\gamma)d\alpha sin\beta d\beta d\gamma$$

$$= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

A bit of algebra shows that this implies

$$= \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} \prod_{i=1}^{3} \{ \frac{(l_i + m_i)!(l_i - m_i)!}{(l_i + q_i)!(l_i - q_i)!} \}^{\frac{1}{2}} 2^{m_i} (1-x)^{\frac{\alpha_i}{2}} (1+x)^{\frac{\beta_i}{2}} P_{n_i}^{(\alpha_i, \beta_i)}(x)$$

$$\times e^{-im_i\phi} e^{iq_i\gamma} d\phi sin\theta d\theta d\gamma$$

$$= \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

The last integral is non-zero if and only if the exponentials go away. This means that all of the m's and q's must separately add to zero on the left hand side. So, for $m_1 + m_2 + m_3 = 0 = q_1 + q_2 + q_3$,[14]

$$\int d^2 y Y_{q_1 l_1 m_1} Y_{q_2 l_2 m_2} Y_{q_3 l_3 m_3}$$

$$= (-1)^{(l_1+l_2+l_3)} \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$$

The $e^{-i2q\phi}$ coming from switching hemispheres will not matter. The minus one factor for us will always be $(-1)^{J_1+J_2+l}$. It will henceforth be implicitly assumed in all our coupling constants and will eventually cancel out of the one loop calculations below.

Using our overlap formula

$$g_+ = \frac{g}{2r} \sqrt{\frac{(2J_1+1)(2J_2+1)(2l+1)}{4\pi}} \begin{pmatrix} J_1 & J_2 & l \\ M_1 & -M_2 & M \end{pmatrix} \times$$

$$\{ \frac{E_2}{|E_2|} \begin{pmatrix} J_1 & J_2 & l \\ z - \frac{1}{2} & z_2 + \frac{1}{2} & Q \end{pmatrix} + \frac{E_1}{|E_1|} \begin{pmatrix} J_1 & J_2 & l \\ z_1 + \frac{1}{2} & z_2 - \frac{1}{2} & Q \end{pmatrix} \} (-1)^{-z_2 - \frac{1}{2} + M_2}$$

$$g_- = \frac{g}{2r}\sqrt{\frac{(2J_1+1)(2J_2+1)(2l+1)}{4\pi}} \begin{pmatrix} J_1 & J_2 & l \\ M_1 & -M_2 & M \end{pmatrix} \times$$

$$\{\frac{E_2}{|E_2|}\begin{pmatrix} J_1 & J_2 & l \\ -z_1-\frac{1}{2} & -z_2+\frac{1}{2} & -Q \end{pmatrix} + \frac{E_1}{|E_1|}\begin{pmatrix} J_1 & J_2 & l \\ -z_1+\frac{1}{2} & -z_2-\frac{1}{2} & -Q \end{pmatrix}\}$$

$$\times(-1)^{-z_2-\frac{1}{2}+M_2}$$

$$g_0^{(++)} = \frac{g}{4r}\sqrt{\frac{(2J_1+1)(2J_2+1)(2l+1)}{4\pi}} \begin{pmatrix} J_1 & J_2 & l \\ M_1 & -M_2 & M \end{pmatrix} \times$$

$$\{\begin{pmatrix} J_1 & J_2 & l \\ z_1-\frac{1}{2} & -z_2+\frac{1}{2} & 0 \end{pmatrix} - \frac{E_1}{|E_1|}\frac{E_2}{|E_2|}\begin{pmatrix} J_1 & J_2 & l \\ z_1+\frac{1}{2} & -z_2-\frac{1}{2} & 0 \end{pmatrix}\}(-1)^{-z_2-\frac{1}{2}+M_2}$$

$$g_0^{(--)} = \frac{-g}{4r}\sqrt{\frac{(2J_1+1)(2J_2+1)(2l+1)}{4\pi}} \begin{pmatrix} J_1 & J_2 & l \\ M_1 & -M_2 & M \end{pmatrix} \times$$

$$\{\begin{pmatrix} J_1 & J_2 & l \\ -z_1-\frac{1}{2} & z_2+\frac{1}{2} & 0 \end{pmatrix} - \frac{E_1}{|E_1|}\frac{E_2}{|E_2|}\begin{pmatrix} J_1 & J_2 & l \\ -z_1+\frac{1}{2} & z_2-\frac{1}{2} & 0 \end{pmatrix}\}(-1)^{-z_2-\frac{1}{2}+M_2}$$

Note that I have used $Y^*_{zlm} = (-1)^{z+m}Y_{-z\,l-m}$. The $g_\pm$ are non-zero if and only if $Q = -2z$. This is the same constraint required for the interaction to be gauge invariant. On the other hand, the g_0's are potentially non-zero for any value of Q and $2q$.

If I only want to know how the light particles couple amongst themselves, then I set $J_1 = |z_1| - \frac{1}{2}$,$J_2 = |z_2| - \frac{1}{2}$. Since there are no light gauge bosons with

$\pm Q$ charge

$$g_+(light)(z_1, z_2, M_1, M_2)(A_1) = 0$$

$$g_-(light)(z_1, z_2, M_1, M_2)(A_1) = 0$$

But

$$g_0^{(++)}(light)(z_1, z_2, M_1, M_2)(A_1) = \frac{g|z_1|}{4r\sqrt{4\pi}} = \frac{|z_2|}{4r\sqrt{4\pi}}$$

$$g_0^{(--)}(light)(z_1, z_2, M_1, M_2)(A_1) = -\frac{g|z_1|}{4r\sqrt{4\pi}} = -\frac{|z_2|}{4r\sqrt{4\pi}}$$

The U(1) (massless) gauge boson does indeed couple to the light (massless) fermions. This light boson is the zero mode polarized in M^2 with $\hat{e}_r \cdot T = 0$.

3.3 Effective Pseudo Scalar Coupling Constants

By the Ward identity, I do not need to find the explicit solutions to the background field gauge. Thus, I can consider the two independent spatial fluctuations of A^2_{qlm} simultaneously and the corresponding induced pseudoscalar coupling constant.

The coupling constant matrix is

$$M_{AC} = \frac{g}{r} \int d^2y \psi^\dagger_{AJ_2M_2}(y) V^s_{qlm}(y) \sigma_s \psi_{CJ_1M_1}(y)$$

It is convenient to make the definitions

$$M_{AC} = \frac{g}{r} \int d^2y \begin{pmatrix} \frac{1}{2}\psi^\dagger_{+z_2J_2M_2} V^s_{0lm} \sigma_s \psi_{+z_1J_1M_1} & \psi^\dagger_{+z_2J_2M_2} V^s_{+Qlm} \sigma_s \psi_{-z_1J_1M_1} \\ \psi^\dagger_{-z_2J_2M_2} V^s_{-Qlm} \sigma_s \psi_{+z_1J_1M_1} & -\frac{1}{2}\psi^\dagger_{-z_2J_2M_2} V^s_{0lm} \sigma_s \psi_{-z_1J_1M_1} \end{pmatrix}$$

$$= \begin{pmatrix} M_0^{(++)} & 0 \\ 0 & M_0^{(--)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ M_+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & M_- \\ 0 & 0 \end{pmatrix}$$

Explicitly, using the previous solutions to the Dirac equation and the vector potential fluctuations

$$M_+ = \int d^2y \frac{g}{2r}(-1)^{-z_2-\frac{1}{2}+M_2}[\frac{E_1}{|E_1|}\frac{E_2}{|E_2|}Y_{z_2+\frac{1}{2}J_2-M_2}Y_{Q-1\,lM}Y_{z_1+\frac{1}{2}J_1M_1}$$

$$+Y_{z_2-\frac{1}{2}J_2-M_2}Y_{Q+1\,lm}Y_{z_1-\frac{1}{2}J_1M_1}]$$

with similar expressions for $M_-, M_0^{(++)}$ and $M_0^{(--)}$.

Once again the overlap formula gives

$$M_+ = \frac{g}{2r}\sqrt{\frac{(2J_1+1)(2J_2+1)(2l+1)}{4\pi}}\begin{pmatrix} J_1 & J_2 & l \\ M_1 & -M_2 & M \end{pmatrix} \times$$

$$\{\frac{E_2}{|E_2|}\frac{E_1}{|E_1|}\begin{pmatrix} J_1 & J_2 & l \\ z_1+\frac{1}{2} & z_2+\frac{1}{2} & Q-1 \end{pmatrix} + \begin{pmatrix} J_1 & J_2 & l \\ z_1-\frac{1}{2} & z_2-\frac{1}{2} & Q+1 \end{pmatrix}\}(-1)^{-z_1-\frac{1}{2}+M_2}$$

$$M_- = \frac{g}{2r}\sqrt{\frac{(2J_1+1)(2J_2+1)(2l+1)}{4\pi}}\begin{pmatrix} J_1 & J_2 & l \\ M_1 & -M_2 & M \end{pmatrix} \times$$

$$\{\frac{E_2}{|E_2|}\frac{E_1}{|E_1|}\begin{pmatrix} J_1 & J_2 & l \\ -z_1+\frac{1}{2} & -z_2+\frac{1}{2} & -Q-1 \end{pmatrix}$$

$$+\begin{pmatrix} J_1 & J_2 & l \\ -z_1-\frac{1}{2} & -z_2-\frac{1}{2} & -Q+1 \end{pmatrix}\}(-1)^{+z-\frac{1}{2}+M_2}$$

$$M_0^{(++)} = \frac{g}{4r}\sqrt{\frac{(2J_1+1)(2J_2+1)(2l+1)}{4\pi}}\begin{pmatrix} J_1 & J_2 & l \\ M_1 & -M_2 & M \end{pmatrix} \times$$

$$\{\frac{E_1}{|E_1|}\begin{pmatrix} J_1 & J_2 & l \\ z_1+\frac{1}{2} & -z_2+\frac{1}{2} & -1 \end{pmatrix} - \frac{E_2}{|E_2|}\begin{pmatrix} J_1 & J_2 & l \\ z_1-\frac{1}{2} & -z_2-\frac{1}{2} & 1 \end{pmatrix}(-1)^{+z_2-\frac{1}{2}+M_2}$$

$$M_0^{(--)} = \frac{-g}{4r}\sqrt{\frac{(2J_1+1)(2J_2+1)(2l+1)}{4\pi}}\begin{pmatrix} J_1 & J_2 & l \\ M_1 & -M_2 & M \end{pmatrix} \times$$

$$\{\frac{E_1}{|E_1|}\begin{pmatrix} J_1 & J_2 & l \\ -z_1+\frac{1}{2} & z_2+\frac{1}{2} & -1 \end{pmatrix} - \frac{E_2}{|E_2|}\begin{pmatrix} J_1 & J_2 & l \\ -z_1-\frac{1}{2} & z_2-\frac{1}{2} & +1 \end{pmatrix}(-1)^{-z_2-\frac{1}{2}+M_2}$$

4. One Loop Chiral Symmetry Preservation on M^2

4.1 Positive Chirality Spinors and Gamma Matrix Structure

Chiral symmetry preservation on $M^2 \times S^2$ implies its preservation on M^2. Positive chirality spinors on $M^2 \times S^2$ have the form

$$\Psi_{(B=+z)JM} = u^{(+)}_{+zJ}(x) \otimes \psi^{(+)}_{+zJM}(y) \pm u^{(-)}_{+zJ}(x) \otimes \psi^{(-)}_{+zJM}(y)$$

$$= u^{(+)}_{+zJ}(x) \otimes \frac{1}{r}\begin{pmatrix} e^{\frac{i\phi}{2}} Y_{z-\frac{1}{2}JM}(\theta,\phi) \\ 0 \end{pmatrix} \pm u^{(-)}_{+zJ}(x) \otimes \frac{1}{r}\begin{pmatrix} 0 \\ e^{\frac{-i\phi}{2}} Y_{z+\frac{1}{2}JM}(\theta,\phi) \end{pmatrix}$$

and

$$\Psi_{(B=-z)JM} = u^{(+)}_{-zJ}(x) \otimes \psi^{(+)}_{-zJM}(y) \pm u^{(-)}_{-zJ}(x) \otimes \psi^{(-)}_{-zJM}(y)$$

$$= u^{(+)}_{-zJ}(x) \otimes \frac{1}{r}\begin{pmatrix} e^{\frac{i\phi}{2}} Y_{-z-\frac{1}{2}JM}(\theta,\phi) \\ 0 \end{pmatrix} \pm u^{(-)}_{-zJ}(x) \otimes \frac{1}{r}\begin{pmatrix} 0 \\ e^{\frac{-i\phi}{2}} Y_{-z+\frac{1}{2}JM}(\theta,\phi) \end{pmatrix}$$

where (+) and (-) superscripts denote positive and negative chirality spinors on the subspaces, and

$$\Gamma_5(M^2 \times S^2) \equiv \gamma_5(M^2) \otimes \sigma_3$$

The $2z$ zero modes for $B = \pm z$ are

$$\Psi_{(B=+z)(z-\frac{1}{2})M} = u^{(+)}_{+zJ}(x) \otimes \frac{1}{r}\begin{pmatrix} e^{\frac{i\phi}{2}} Y_{z-\frac{1}{2}(z-\frac{1}{2})M}(\theta,\phi) \\ 0 \end{pmatrix}$$

$$\Psi_{(B=-z)JM} = u^{(-)}_{-zJ}(x) \otimes \frac{1}{r}\begin{pmatrix} 0 \\ e^{\frac{-i\phi}{2}} Y_{-z+\frac{1}{2}(z-\frac{1}{2})M}(\theta,\phi) \end{pmatrix}$$

By looking at the gamma matrix structure of the interaction, on can see that g_{AC} vertices leave the chirality on both subspaces unchanged while the M_{AC} vertices flip both chiralities. Hence, the zero mode massless fermions cannot obtain mass corrections from self energy diagrams.

As a check that no mass counterterms are gernerated for the massless fermions on M^2, I now calculate the one loop self energy diagrams for a $z = \frac{1}{2}$ background. In general, the graphs for the $\pm M$ states of the $2z$ massless multiplet are equal by the symmetry relating the 3-j symbols with the signs in the bottom row reversed.

4.2 A Ward Identity

Consider the diagram in figure 1(a). Let's introduce the notation

$$|Y^{(q)}_j\rangle = \Psi_{qjm}(z)$$

$$\langle Y^{(q)}_j| = \Psi^{\dagger}_{qjm}(z)\Gamma^0(M^2 \times S^2)$$

Then for incoming and outgoing zero modes, second order perturbation theory equates the first diagram to

$$\sum_{j=1}^{\infty}\langle Y_0^{(\frac{1}{2})}|A^*_{0lm}\Gamma_\mu|Y_j^{(\frac{1}{2})}\rangle\frac{1}{\not{D}_j^{(\frac{1}{2})}}\langle Y_j^{(\frac{1}{2})}|A_{0lm}\Gamma_\nu|Y_0^{(\frac{1}{2})}\rangle$$

Now operate on this diagram with $D^{(0)\mu}$ and symmetrize the boson legs

$$\sum_{j=1}^{\infty}\langle Y_0^{(\frac{1}{2})}|D^{(0)\mu}A^*_{0lm}\Gamma_\mu|Y_j^{(\frac{1}{2})}\rangle\frac{1}{\not{D}_j^{(\frac{1}{2})}}\langle Y_j^{\frac{1}{2}}|A_{0lm}\Gamma_\nu|Y_0^{(\frac{1}{2})}\rangle$$

$$+\sum_{j=1}^{\infty}\langle Y_0^{(\frac{1}{2})}|A^*_{0lm}\Gamma_\nu|Y_j^{\frac{1}{2}}\rangle\frac{1}{\not{D}_j^{(\frac{1}{2})}}\langle Y_j^{\frac{1}{2}}|D_\mu^{(0)}A_{0lm}\Gamma_\mu|Y_0^{(\frac{1}{2})}\rangle$$

Next integrate each vertex by parts and note that

$$\not{D}^{(\frac{1}{2})}|Y_j^{(\frac{1}{2})}\rangle = \frac{\sqrt{j(j+1)-\frac{1}{4}}}{r} \equiv E|Y_j^{(\frac{1}{2})}\rangle$$

$$\langle D_\mu^{(\frac{1}{2})}Y_j^{(\frac{1}{2})}|\Gamma^\mu = -E\langle Y_j^{(\frac{1}{2})}|$$

We then get that $D^{(0)\mu}$ acting on the symmetric sum of diagrams is

$$-\langle Y_0^{(\frac{1}{2})}|A^*_{0lm}A_{0lm}\Gamma_\nu|Y_0^{(\frac{1}{2})}\rangle + \langle Y_0^{(\frac{1}{2})}|A^*_{0lm}\Gamma_\nu A_{0lm}|Y_0^{(\frac{1}{2})}\rangle = 0$$

Similarly, it's easy to show that the symmetric divergence of the diagram of figure 1(b) is zero. This proves that the results of the next section are gauge invariant, since the $D_\mu D_\nu$ terms in the boson propagators are automatically zero. A similar identity will be derived for the one loop fermion-higgs Yukawa couplings calculated below.

4.3 Self Energy Diagrams

For fluctuations polarized in the M^2 directions, conservation of the z component of isopspin only permits two diagrams. (See figure 2). In this case, $z_1 = \frac{1}{2} = z_2$. Therefore, either the incoming or outgoing g_+ or g_- in figure 2 (b) is zero. But the diagram in figure 2 (a) is non-zero. The clebsches in $g_0^{(++)}$ enforce $J_2 = l$. Although the incoming and outgoing couplings interchange J_1 and J_2, the minus signs from switching the columns of the 3-j symbols will cancel to give the same expression for each vertex. So summing over the positive internal fermion eigenvalues, the negative internal fermion eigenvalues, and the internal fermion zero mode, we get the contribution to the self energy

$$\Sigma(p, A^1) =$$

$$\frac{2}{4\pi}\frac{g^2}{16r}\int \frac{d^2k}{(2\pi)^2}\sum_{l=1}^{\infty}(2l+1)\gamma_\mu \frac{i}{\not{k} + i\gamma_5\sqrt{l(l+1)}}\gamma_\mu \frac{1}{(k+p)^2 + l(l+1)}$$

$$+\frac{2}{4\pi}\frac{g^2}{16r}\int \frac{d^2k}{(2\pi)^2}\sum_{l=1}^{\infty}(2l+1)\gamma_\mu \frac{i}{\not{k} - i\gamma_5\sqrt{l(l+1)}}\gamma_\mu \frac{1}{(k+p)^2 + l(l+1)}$$

$$+\frac{2}{4\pi}\frac{g^2}{16r}\int \frac{d^2k}{(2\pi)^2}\gamma_\mu \frac{i}{\not{k}}\gamma_\mu \frac{1}{(k+p)^2}$$

where p is the external M^2 momentum. Bringing the gamma matrices into the numerator of the first two terms causes the mass terms proportional to γ_5 to cancel. Then

$$\Sigma(p, A^1) = \frac{1}{\pi}\frac{g^2}{16r}\int_0^1 dx \int \frac{d^2k}{(2\pi)^2}\sum_{l=1}^{\infty}(2l+1)\frac{\gamma_\mu(\not{k} - \not{p}x)\gamma_\mu}{[k^2 + l(l+1) + p^2x(1-x)]^2}$$

$$+\frac{1}{2\pi}\frac{g^2}{16r}\int_0^1 dx\int\frac{d^2k}{(2\pi)^2}\frac{\gamma_\mu(-\not{p}x)\gamma_\mu}{[k^2+p^2x(1-x)]^2}$$

Similarly, the two fluctuations polarized on S^2 give zero M_+ and M_- but non-zero $M_0^{(++)}$. Neglecting the tachyons, and taking the special degeneracies at $l=1$ into account,

$$\Sigma(p,A^2_{qlm}) = -\frac{1}{2\pi}\frac{g^2}{16r}\int_0^1 dx\int\frac{d^2k}{(2\pi)^2}\sum_{l=1}^{\infty}(2l+1)\frac{\gamma_\mu(\not{k}-\not{p}x)\gamma^\mu}{[k^2+l(l+1)+p^2x(1-x)]^2}$$

$$+\frac{1}{4\pi}\frac{g^2}{16r}\int_0^1 dx\int\frac{d^2k}{(2\pi)^2}\sum_{l=1}^{\infty}(2l+1)\frac{\gamma_\mu(\not{k}-\not{p}x)\gamma^\mu}{[k^2+2+p^2x(1-x)]^2}$$

In this graph, two $\frac{E}{|E|}$ factors from the couplings cancelled on the internal fermion line.

5. One Loop Yukawa Couplings and Mass Heirarchies

5.1 A Pseudoscalar Higgs

A slight extension of the results of chapter three shows that a vector potential of the form

$$A^2_{qlm} = \frac{1}{r}\begin{pmatrix} 0 \\ \vec{V}_{qlm}(\theta,\phi)\end{pmatrix}\phi_q(x)$$

induces a pseudoscalar coupling for $\phi_q(x)$ on M^2 for each q. More explicitly, the

Yang-Mills Lagrangian on $M^2 \times S^2$ induces the Lagrangian for the triplet

$$L(\phi_q(x)) = -\frac{1}{2}Tr(D_\mu\phi)^\dagger(D^\mu\phi) + \overline{u}_a\gamma_5 u_c\psi_a^\dagger(\delta A_{qlm}^2)^s\sigma_s\tau_{ac}^m\psi_c$$

$$\equiv -\frac{1}{2}Tr(D_\mu\phi)^\dagger(D^\mu\phi) + \overline{u}_a\gamma_5 u_c\psi_a^\dagger(V_{qlm})^s\sigma_s\tau_{ac}^q\psi_c\phi_q(x)$$

It is reasonable for this scalar to obtain a vacuum expectation value. The scale of this vacuum expectation value must be well below the compactification scale since we are neglecting gravity. For simplicity, I will give a vev to the x dependent part of the ground state hedgehog

$$\langle \overrightarrow{A}_{weak}^{q=+1} \rangle = \langle \overrightarrow{A}_{100}^{2} \rangle = \frac{\langle \phi_{+1}(x) \rangle}{r}\begin{pmatrix} 0 \\ \hat{e}_r \times \overrightarrow{T} \end{pmatrix}$$

$$= \frac{\langle \phi_{+1}(x) \rangle}{r}\begin{pmatrix} 0 \\ \overrightarrow{V}_{+100} \end{pmatrix}$$

implying

$$\langle (\delta A_{100}^2)^s\sigma_s \rangle = \frac{\langle \phi_+(x) \rangle}{r}\begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}$$

We now have gauge boson and fermion mass insertions on M^2

$$M_A^2 \equiv \frac{-g^2}{2r^2}A^{q*}A^k\langle \phi_c^{\alpha*} \rangle\langle \phi_c^{\alpha} \rangle\epsilon_{qca}\epsilon_{kac}$$

$$= \frac{g^2}{r^2}|\langle \phi_+(x) \rangle|^2 \int d^2y A^{-*}A^- = \frac{g^2}{r^2}|\langle \phi_+(x) \rangle|^2$$

$$M_{\phi_+}^{AC} = \frac{g}{2r}|\langle \phi_+(x) \rangle| \int d^2y [\frac{E_2}{|E_2|}Y_{A+\frac{1}{2}J_2M_2}^{*}Y_{C-\frac{1}{2}J_1M_1}$$

$$+\frac{E_1}{|E_1|}Y^*_{A-\frac{1}{2}J_2M_2}Y_{C+\frac{1}{2}J_1M_1}]$$

for A, C greater than zero and

$$M^{AC}_{\phi_+} = \frac{g}{2r}|\langle\phi_+(x)\rangle| \int d^2y[Y^*_{A+\frac{1}{2}J_2M_2}Y_{C-\frac{1}{2}J_1M_1}$$

$$+\frac{E_1}{|E_1|}\frac{E_2}{|E_2|}Y^*_{A-\frac{1}{2}J_2M_2}Y_{C+\frac{1}{2}J_1M_1}]$$

for A less than zero and C greater than zero.

Since opposite chirality zero modes have opposite charge, $M^{AC}_{\phi_+}$ can only generate tree level masses for the massless fermions with $A = -\frac{1}{2}, C = \frac{1}{2}$ To obtain a family mass splitting for higher charged fermions, we must go to one loop.

5.2 Another Ward Identity

Consider the diagrams in figure 3. The graphs (a), (b), and (c) give the three respective expressions:

$$a_{\mu\nu} \equiv \sum_{j=1}^{\infty}\langle Y_1^{(-\frac{3}{2})}|A_{-1lm}\Gamma_\mu|Y_j^{(-\frac{1}{2})}\rangle\frac{1}{\not{D}_j^{(-\frac{1}{2})}}\langle Y_j^{(-\frac{1}{2})}|\phi_+|Y_j^{(\frac{1}{2})}\rangle\frac{1}{\not{D}_j^{(\frac{1}{2})}}$$

$$\times\langle Y_j^{(\frac{1}{2})}|A_{-1lm}\Gamma_\nu|Y_1^{(\frac{3}{2})}\rangle$$

$$b_{\mu\nu} \equiv \sum_{j=1}^{\infty}\langle Y_1^{(-\frac{3}{2})}|A_{-1lm}\Gamma_\mu|Y_j^{(-\frac{1}{2})}\rangle\frac{1}{\not{D}_j^{(-\frac{1}{2})}}\langle Y_j^{(\frac{1}{2})}|A_{-1lm}\Gamma_\nu|Y_1^{(\frac{1}{2})}\rangle\frac{1}{\not{D}_1^{(\frac{1}{2})}}$$

$$\times\langle Y_1^{(\frac{1}{2})}|\phi_+|Y_1^{(\frac{3}{2})}\rangle$$

$$c_{\mu\nu} \equiv \sum_{j=1}^{\infty} \langle Y_1^{(-\frac{3}{2})} | \phi_+ | Y_1^{(-\frac{1}{2})} \rangle \frac{1}{\not{D}_j^{(-\frac{1}{2})}} \langle Y_1^{(-\frac{1}{2})} | A_{-1lm} \Gamma_\mu | Y_j^{(\frac{1}{2})} \rangle \frac{1}{\not{D}_j^{(\frac{1}{2})}}$$

$$\times \langle Y_j^{(\frac{1}{2})} | A_{-1lm} \Gamma_\nu | Y_1^{(\frac{3}{2})} \rangle$$

In analogy with the previous case, one can show that

$$D^{(-1)\nu}[a_{\mu\nu} + a_{\nu\mu}] = 0$$

$$D^{(-1)\nu}[b_{\mu\nu} + c_{\nu\mu}] = 0$$

$$D^{(-1)\nu}[b_{\nu\mu} + c_{\mu\nu}] = 0$$

These three equations imply that the one loop results of the next section are gauge invariant.

5.3 Fermion-Higgs Yukawa Couplings

The simplest one loop diagram changing M^2 chirality is shown in figure 4. I have now put the fermions in an isospin $\frac{3}{2}$ multiplet. The external massless fermions are now in a triplet. The couplings I need are

$$\sqrt{3}\psi^\dagger_{\frac{1}{2}} A^- \psi_{\frac{3}{2}}$$

$$\sqrt{3}\psi^\dagger_{-\frac{3}{2}} A^{-*} \psi_{-\frac{1}{2}}$$

The explicit calculation of the one loop diagrams is straightforward but tedious. See figure 5. Since the fermions have half integer charge and integer angular momenta the gauge bosons have integer charge and integer angular momenta.

The external zero modes have $J = 1$ and will be labeled by their z-component of angular momenta $m_E = 0, \pm 1$. I leave the internal fermion angular momenta at J and let the boson momenta vary over $J, J \pm 1$. For simplicity, I will completely ignore the two tachyonic gauge bosons at $l = 0$.

The outgoing and incoming vertices are related

$$(g_+)_{out} = (g_+)_{in}(-1)^{2l+1}$$

where l is the boson momenta. Although the process reverses charge, it conserves the z-component of angular momenta. Also since the incoming and outgoing flavor states have the same sign of the z-angular momenta the g vertices will not induce $\frac{E}{|E|}$ factors on the propagators but the M vertices will. All annoying factors of $(-1)^m$ cancel.

The diagrams for $m_E = \pm 1$, as are all diagrams with the opposite external z angular momenta, are equal by the reversal of the sign the bottom row of the 3-j symbols. *Coincidentally*, for the case at hand, they equal the $m_E = 0$ diagrams because the sums over internal m make the squared 3j symbols coming from the conservation of isospin equal. Diagrams with different magnitudes of external z-angular momenta are generally *not* equal for a higher charged background (more families).

Keeping these facts in mind, let's now calculate the diagrams with an internal fermion insertion. The insertion itself on these diagrams with two M vertices gives zero. It is

$$M_{\phi_+} = \frac{g}{2r}|\langle\phi_+(x)\rangle| \int d^2\Omega Y^*_{-1J_2M_2} Y_{1J_1M_1} \frac{E_1}{|E_1|}\frac{E_2}{|E_2|} = 0$$

So consider the graphs with two g vertices. The insertion is

$$M_{\phi_+} = \frac{g}{2r}|\langle\phi_+(x)\rangle| \int d^2\Omega Y^*_{0J_2M_2} Y_{0J_1M_1} = \frac{g}{2r}|\langle\phi_+(x)\rangle|\delta_{J_2J_1}\delta_{M_1M_2}$$

If p is the external Minkowski momentum, figure 5(a) for $m_E = 0$,$l = j$,is

$$M(p, l = J, m_E = 0) =$$

$$\frac{-9}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \int \frac{d^2k}{(2\pi)^2} \sum_{J=1}^{\infty} \sum_{m=-J}^{+J} \int_0^1 dx \frac{4(k^2 - p^2)}{(k^2 + l(l+1) + p^2x(1-x) - x)^4}$$

$$\times\{\begin{pmatrix} J & J & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} J & J & 1 \\ -m & -m & 0 \end{pmatrix}\}^2 \times (2J+1)^2$$

For $m_E = \pm 1$

$$M(p, l = J, m_E = \pm 1) =$$

$$\frac{9}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \int \frac{d^2k}{(2\pi)^2} \sum_{J=1}^{\infty} \sum_{m=-J}^{+J} \int_0^1 dx \frac{4(k^2 - p^2)}{(k^2 + l(l+1) + p^2x(1-x) - x)^4}$$

$$\times\{\begin{pmatrix} J & J & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} J & J & 1 \\ -m & -m-1 & 1 \end{pmatrix}\}^2 \times (2J+1)^2$$

All of three of these terms equal

$$= -\frac{6}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \int \frac{d^2k}{(2\pi)^2} \sum_{J=1}^{\infty} \int_0^1 dx (2J+1) \frac{k^2 - p^2}{(k^2 + J(J+1) + p^2x(1-x) - x)^4}$$

At p=0, this is

$$\frac{1}{(4\pi)^2}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \sum_{J=1}^{\infty}(2J+1)\{\frac{1}{J(J+1)-1} - \frac{1}{J(J+1)}\}$$

This expression is finite.

Similarly,the graph in figure 5(b) for $l = J+1$, $m_E = 0$, is

$$M(p, l = J+1, m_E = 0) =$$

$$\frac{-9}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \int \frac{d^2k}{(2\pi)^2} \sum_{J=1}^{\infty} \sum_{m=-J}^{+J} \int_0^1 dx \frac{4(k^2-p^2)}{(k^2+l(l+1)+p^2x(1-x)+(2J+1)x)^4}$$

$$\times\{\begin{pmatrix} J & J+1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} J & J & 1 \\ m & -m & 0 \end{pmatrix}\}^2 \times (2J+1)(2J+3)$$

For $m_E = \pm 1$

$$M(p, l = J+1, m_E = \pm 1) =$$

$$\frac{9}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \int \frac{d^2k}{(2\pi)^2} \sum_{J=1}^{\infty} \sum_{m=-J}^{+J} \int_0^1 dx \frac{4(k^2-p^2)}{(k^2+l(l+1)+p^2x(1-x)+(2J+1)x)^4}$$

$$\times\{\begin{pmatrix} J & J+1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} J & J & 1 \\ m & -m-1 & 1 \end{pmatrix}\}^2 \times (2J+1)(2J+3)$$

At p=0, all of three of these terms equal

$$= -\frac{6}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \int \frac{d^2k}{(2\pi)^2} \sum_{J=1}^{\infty} \int_0^1 dx(2J+1) \frac{k^2-p^2}{(k^2+J(J+1)+p^2x(1-x)-x)^4}$$

$$= \frac{1}{(4\pi)^2}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \sum_{J=1}^{\infty}(J+2)\{\frac{1}{J(J+3)-1} - \frac{1}{J(J+1)}\}$$

Also for part (c) of figure 5 with $l = J-1$, $m_E = 0$,

$$M(p, l = J-1, m_E = 0) =$$

$$\frac{-9}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3\int\frac{d^2k}{(2\pi)^2}\sum_{J=1}^{\infty}\sum_{m=-J}^{+J}\int_0^1 dx\frac{4(k^2-p^2)}{(k^2+l(l+1)+p^2x(1-x)-(2J+1)x)^4}$$

$$\times\{\begin{pmatrix} J & J-1 & 1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix} J & J-1 & 1\\ m & -m & 0\end{pmatrix}\}^2\times(2J+1)(2J-1)$$

For $m_E = \pm 1$

$$M(p, l = J-1, m_E \pm 1) =$$

$$\frac{9}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3\int\frac{d^2k}{(2\pi)^2}\sum_{J=1}^{\infty}\sum_{m=-J}^{+J}\int_0^1 dx\frac{4(k^2-p^2)}{(k^2+l(l+1)+p^2x(1-x)-(2J+1)x)^4}$$

$$\times\{\begin{pmatrix} J & J-1 & 1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix} J & J-1 & 1\\ m & -m-1 & 1\end{pmatrix}\}^2\times(2J+1)(2J-1)$$

At p=0, all three of these terms equal

$$= -\frac{6}{4\pi}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3\int\frac{d^2k}{(2\pi)^2}\sum_{J=1}^{\infty}\int_0^1 dx(2J+1)\frac{k^2-p^2}{(k^2+J(J+1)+p^2x(1-x)-x)^4}$$

$$= \frac{1}{(4\pi)^2}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3\sum_{J=2}^{\infty}(-1))(J-1)\{\frac{1}{J(J-1)-1}-\frac{1}{J(J+1)}\}$$

Putting all of these contributions together and summing for the first few hundred J gives a mass for each fermion of

$$Mass(\overrightarrow{A}^1_{qlm}) \equiv \sum_{l=J-1}^{J+1} M(p=0, l, m_E = (0, \pm 1) =$$

$$\frac{-1.07}{(4\pi)^2}\frac{g^5}{r}|\langle\phi_+(x)\rangle|^3 \tag{5.1}$$

for $\overrightarrow{A}^1_{qlm}$ gauge boson loops.

There are two other generic graphs (figure 6) to this order. They are similar in structure to the self energy diagrams I considered earlier in this chapter. Both sets of graphs with two g vertices and two M have factors of

$$\frac{2\,\not{p}}{p^2+2}=[\frac{1}{\not{p}+i\gamma_5\sqrt{2}}+\frac{1}{\not{p}-i\gamma_5\sqrt{2}}]$$

from the internal fermion propagator attached to the Higgs insertion. Thus they are zero at zero external momenta and (5.1) is the total mass correction for each massless fermion in the triplet.

6. Conclusions

I have presented a formalism for calculating loop corrections in two generic dimenionally reduced theories with non-trivial vector potential vacuum expectation values before reduction. These techniques may be useful in any Kaluza-Klein theory where the holonomy group of the extra dimensions can contain SU(2) or SO(3) monopoles in a subgroup with the corresponding gauge group in M^n. In these theories, it should be possible to calculate the mass splitting between families. For example, with a fermion monopole background of charge 4, one should find four pairs of particles with four distinct masses, corresponding to $m_E = \pm 7/2, \pm 5/2, \pm 3/2, \pm 1/2$ when coupled to a pseudoscalar Higgs.

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FIGURE CAPTIONS

1. Diagrams for the Ward identity of chapter four
2. Self energy diagrams for a $z = +\frac{1}{2}$ background
3. Diagrams for the Ward identity of chapter five
4. A correction to the fermion-Higgs Yukawa couplings
5. Three specific corrections for each of the three incoming massless fermions
6. Two more sets of fermion-Higgs coupling corrections

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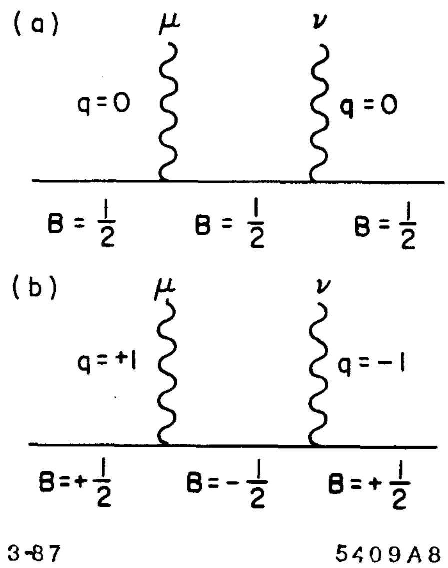

Fig. 1

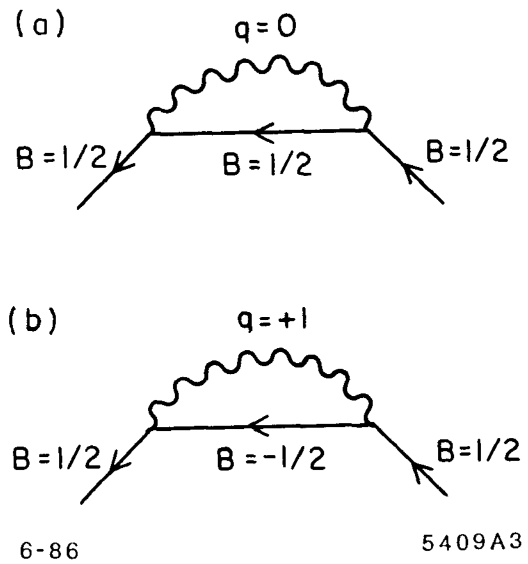

Fig. 2

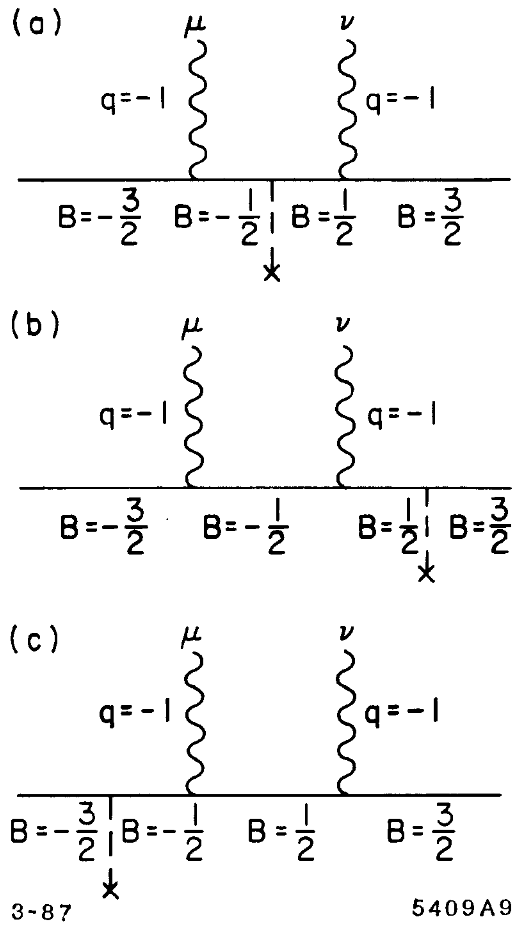
(a)
μ
ν
q=-1
q=-1
B=-3/2
B=-1/2
B=1/2
B=3/2
(b)
μ
ν
q=-1
q=-1
B=-3/2
B=-1/2
B=1/2
B=3/2
(c)
μ
ν
q=-1
q=-1
B=-3/2
B=-1/2
B=1/2
B=3/2
3-87
5409A9

Fig. 3

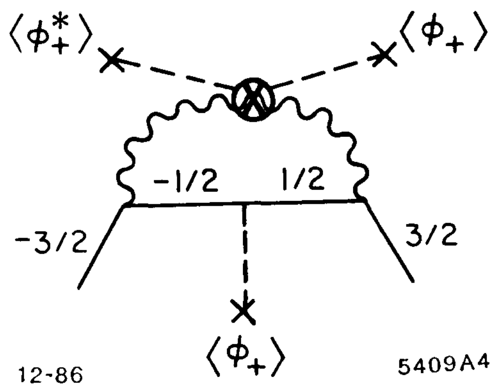
$\langle \phi_+^* \rangle$
$\langle \phi_+ \rangle$
$-1/2$
$1/2$
$-3/2$
$3/2$
$\langle \phi_+ \rangle$
12-86
5409A4

Fig. 4

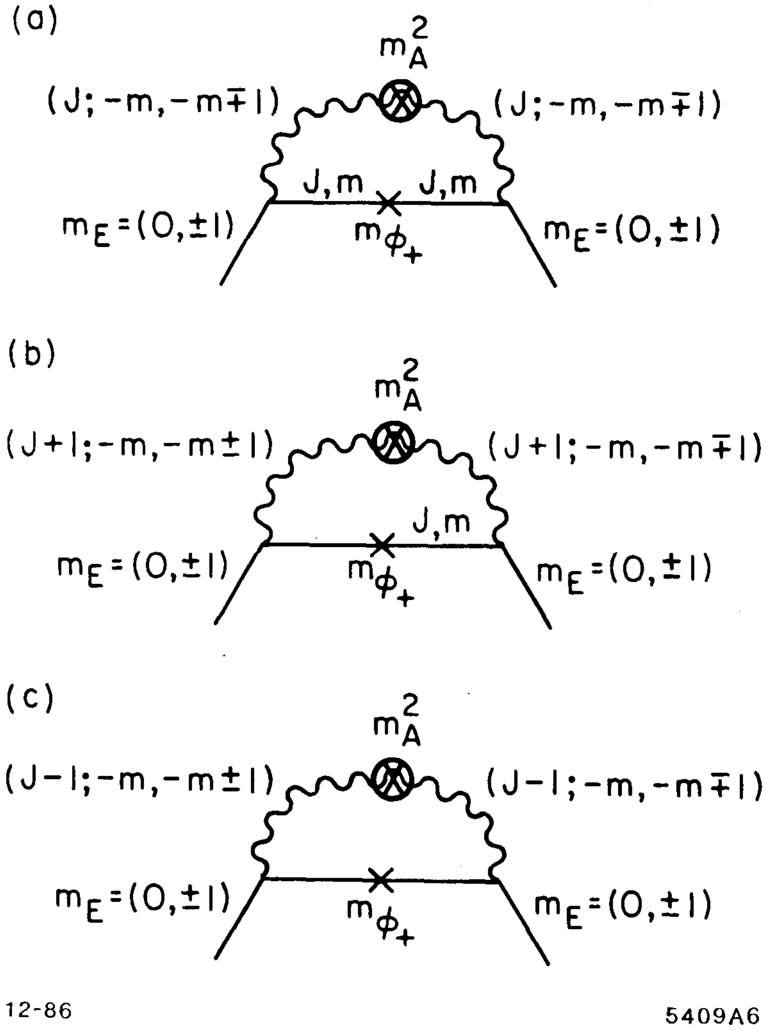
(a)
m_A^2
(J; −m, −m∓1)
(J; −m, −m∓1)
J, m
J, m
m_E=(0,±1)
m_{ϕ_+}
m_E=(0,±1)
(b)
m_A^2
(J+1; −m, −m±1)
(J+1; −m, −m∓1)
J, m
m_E=(0,±1)
m_{ϕ_+}
m_E=(0,±1)
(c)
m_A^2
(J−1; −m, −m±1)
(J−1; −m, −m∓1)
m_E=(0,±1)
m_{ϕ_+}
m_E=(0,±1)
12-86
5409A6

Fig. 5

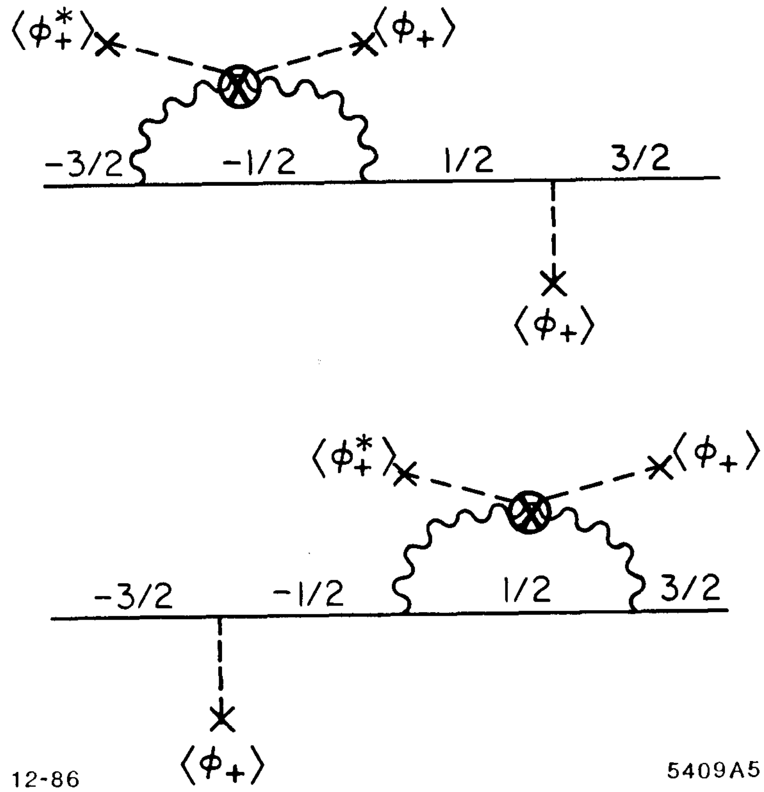

Fig. 6