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CONFORMAL INVARIANCE OF SUPERSYMMETRIC σ-MODELS ON CALABI-YAU MANIFOLDS*

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ABSTRACT

We analyze the possibility of superstring compactification on Calabi-Yau manifolds in the light of recent work by Grisaru, van de Ven and Zanon, and by Freeman and Pope. It is shown that despite the appearance of non-zero β-function at the four loop order, we can construct a conformally invariant supersymmetric σ-model on a Calabi-Yau manifold. The background metric is not Ricci flat, but is related to the Ricci flat metric through a (non-local) field redefinition.

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It has long been conjectured that $N = 2$ supersymmetric σ-models formulated on Ricci flat Kahler manifolds (otherwise known as Calabi-Yau manifolds) have vanishing β-function to all orders in the perturbation theory[1−3] Recent work by Alvarez-Gaume, Coleman and Ginsparg[3] seemed to prove this result. Recently, however, there has been an explicit computation of the four loop β-function[4] showing that this may not be the case. Similar result has been obtained by general analysis of the possible counterterms[5] , as well as computation of the effective action in the string theory[6,7] , invoking the conjectured relationship between the σ-model β-function and the equations of motion in the string theory[8−14] .

We shall show that despite the apparent non-zero contribution to the four loop β-function, the theory can be made to have a vanishing β-function to all orders in the perturbation theory by suitably defining the procedure for subtracting the ultraviolet divergences in the theory. We begin by showing how the procedure for subtracting the ultraviolet divergences in the theory affects the computation of the β-function. Consider, for example, the N=1 supersymmetric σ-model formulated on Ricci-flat manifolds. This has been shown to have vanishing β-function to three loop order[2,4] . The lagrangian is given by,

$$\int d^2\xi d\theta d\bar{\theta} G_{ij}(\Phi)\bar{D}\Phi^i D\Phi^j \tag{1}$$

where $\Phi^i(\xi,\theta,\bar{\theta})$ are two dimensional superfields, G_{ij} is the background metric and D and $\bar{D}$ are supercovariant derivatives. At each order in the perturbation theory the subtraction of ultraviolet divergences leaves us with the freedom of adding a finite local counterterm to the action. In particular, let us adopt a subtraction scheme which differs from the analysis of Refs.[2,4] by the addition of a finite local counterterm to the lagrangian at two loop order. We take the counterterm to have the form,

$$\int d^2\xi d\theta d\bar{\theta} T_{ij}(\Phi) D\Phi^i \bar{D}\Phi^j \tag{2}$$

where T_{ij} is any tensor. A specific example is,

$$T_{ij} = R_{imnp} R_j^{\ mnp} \tag{3}$$

where R is the Riemann tensor constructed out of the metric G_{ij}. With this new subtraction procedure, the three loop β-function of the theory will no longer vanish for Ricci flat manifolds. The easiest way to see this is to note that the addition of the above counterterm is equivalent to modifying action (1) by the replacement $G_{ij} \to G_{ij} + T_{ij}$. Since the one loop β-function is given by the Ricci tensor, the new β-function will be given by the Ricci tensor calculated with the metric $G_{ij} + T_{ij}$. This is given by,

$$R_{ij}(G) + \tfrac{1}{2}(T_{ij;m}^{\quad\ m} + T^{m}_{m;ij} - T_{im;j}^{\quad\ m} - T_{jm;i}^{\quad\ m}) \tag{4}$$

to linear order in T. The second term in (4) gives a three loop contribution to the β-function if we take T_{ij} to be of the form (3).

Thus we see that the criterion for the vanishing of the β-functions in a theory depends on how we subtract the ultraviolet divergences in the theory. As we have seen, a different procedure for subtracting the ultraviolet divergences corresponds to a field redefinition of the metric, the above result simply reflects the fact that the equation for the vanishing of the β-function takes a different form under a redefinition of the coupling constants of the theory. We shall now show that for N=2 supersymmetric sigma models formulated on Ricci-flat Kahler manifolds, it is always possible to choose a subtraction procedure which makes the β-function vanish to all orders in the perturbation theory. Put in another way, we shall show that although for a given subtraction procedure, the β-function does not vanish for a Ricci flat metric, it is always possible to choose a metric on the Calabi-Yau manifold for which the total β-function is zero. This metric is related to the Ricci flat metric through a (non-local) field redefinition.

The action for the N=2 supersymmetric σ-model is given by,

$$\int d^2\xi d^2\theta d^2\bar{\theta} K(\Phi,\bar{\Phi}) \tag{5}$$

where Φ^i and $\bar{\Phi}^i$ are N=2 superfields, and K is the Kahler potential of the target manifold. Since it is always possible to carry out the computation in the N=2 superfield formulation, all the ultraviolet divergent terms have the form of Eq.(5) and their effect may be summarized in a single β-function β_K.* Let us define,

$$G_{i\bar{j}}(\Phi) = \partial_i \partial_{\bar{j}} K(\Phi,\bar{\Phi}) \tag{6}$$

as the metric on the Kahler manifold. The one loop contribution to β_K is then given by[1–4]

$$c\ Tr\ell nG \tag{7}$$

where c is a numerical constant. Let us denote the contribution to β_K from two and higher loop graphs as $\Delta\beta_K$. As was pointed out before, $\Delta\beta_K$ will depend on the procedure of subtracting the ultraviolet divergences, but once we choose a certain scheme, there is a unique expression for $\Delta\beta_K$. We now prove the following lemma:

For any Calabi-Yau manifold with Ricci flat metric $\tilde{G}_{i\bar{j}}$ and the corresponding Kahler potential $\tilde{K}$, there exists a Kahler potential,

$$K(\Phi,\bar{\Phi}) = \tilde{K}(\Phi,\bar{\Phi}) - \delta K(\Phi,\bar{\Phi}) \tag{8}$$

such that,

$$\beta_K = c\ Tr\ell nG + \Delta\beta_K = c\ Tr\ell n\tilde{G} \tag{9}$$

* In order that the theory is conformally invariant, it is not necessary for β_K to vanish, it is enough that the β-function for the metric $\partial_i\partial_{\bar{j}}\beta_K$ vanishes.

So that $\partial_i \partial_{\bar{j}} \beta_K$ vanishes. Here,

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = \tilde{G}_{i\bar{j}} - \delta G_{i\bar{j}} \tag{10}$$

$$\delta G_{i\bar{j}} = \partial_i \partial_{\bar{j}} \delta K \tag{11}$$

Proof: Eq.(9) may be written as,

$$\Delta \beta_K = -c \; Tr \; \ell n(I - \tilde{G}^{-1} \delta G) \tag{12}$$

which may be written as,

$$\begin{aligned} \tilde{G}^{i\bar{j}} \partial_i \partial_{\bar{j}} \delta K &= c^{-1} \Delta \beta_K - \sum_{n=2}^{\infty} \frac{1}{n} Tr\big((\tilde{G}^{-1} \delta G)^n\big) \\ &= c^{-1} \Delta \beta_{\tilde{K}} + (c^{-1} \Delta \beta_K - c^{-1} \Delta \beta_{\tilde{K}}) - \sum_{n=2}^{\infty} \frac{1}{n} Tr\big((\tilde{G}^{-1} \delta G)^n\big) \end{aligned} \tag{13}$$

where $\Delta \beta_{\tilde{K}}$ is obtained from $\Delta \beta_K$ by replacing K by $\tilde{K}$ everywhere in the expression for the latter. This equation may be solved iteratively for δK. To lowest order the right hand side may be replaced by $c^{-1} \Delta \beta_{\tilde{K}}$. (Since $K = \tilde{K}$ to lowest order, $\Delta \beta_{\tilde{K}} - \Delta \beta_K$ is of higher order). The left hand side is just the Laplacian acting on δK, and hence the above equation always has a solution in a local coordinate patch. This may be seen by repeated application of the Hodge decomposition theorem on $\Delta \beta_{\tilde{K}}$. This value of δK may then be substituted on the right hand side of (13) to get iterative solutions for δK.

In order to show that $G_{i\bar{j}}$ calculated this way is an admissible metric on the Calabi-Yau manifold, we must show that the metric is globally defined, i.e. when we calculate $G_{i\bar{j}}$ in two different coordinate patches, they are related to each other in the overlapping region by the standard transformation law,

$$G'_{i\bar{j}}(z', \bar{z}') = \frac{\partial z^k}{\partial z'^i} \frac{\partial \bar{z}^\ell}{\partial \bar{z}'^j} G_{k\bar{\ell}}(z, \bar{z}) \tag{14}$$

In order to prove the above result we have to assume a specific property of $\Delta \beta_{\tilde{K}}$, namely, that it is a globally defined scalar field on the manifold. This is certainly the case for the four loop contribution to $\Delta\beta_{\tilde{K}}$ calculated in Refs.[4-6], where it turns out to be the Euler density. More generally, because of general covariance, the β-function $(\beta_{\tilde{G}})_{i\bar{j}} = \partial_i\partial_{\bar{j}}\beta_{\tilde{K}}$ for the metric always has a covariant form, and so is a globally defined tensor on the manifold. This does not, by itself, imply that $\beta_{\tilde{K}}$ should be a globally defined scalar field. The simplest example is the one loop contribution to the β-function, Eq.(7) does not transform like a scalar field when we move from one coordinate patch to another, although the corresponding $\beta_{\tilde{G}}$, which is proportional to the Ricci tensor, does transform like a tensor. We shall now sketch a proof showing that this is not the case for higher order contribution $\Delta\beta_{\tilde{K}}$.

In order that $\partial_i\partial_{\bar{j}}\Delta\beta_K$ is a globally defined tensor for any Kahler manifold with Kahler potential K, $\Delta\beta_K$ must differ on two different coordinate patches by a function of the form $f(z) + g(\bar{z})$ where f and g are holomorphic and antiholomorphic functions of the coordinates respectively. From the analysis of Ref.4 it is clear that $\Delta\beta_K$ can be expressed as a product of terms of the form $\partial_{i_1}...\partial_{i_n}\partial_{\bar{j}_1}...\partial_{\bar{j}_m}K$ with the various lower indices contracted with each other by the inverse metric $G^{i\bar{j}}$. Under a change in the coordinate system from $(z,\bar{z})$ to $(z',\bar{z}')$ $\partial_{i_1}...\partial_{\bar{j}_m}K$ transforms into $\frac{\partial z'^{k_1}}{\partial z^{i_1}}...\frac{\partial \bar{z}'^{l_m}}{\partial \bar{z}^{j_m}}\partial'_{k_1}...\partial'_{\bar{l}_m}K$, plus terms involving $\frac{\partial^r z'^i}{\partial z^{i_1}...\partial z^{i_r}}$ or its complex conjugate $(r \geq 2)$. We shall call the first term the homogeneous term, the other terms, which represent the fact that $\partial_{i_1}...\partial_{\bar{j}_m}K$ does not transform like a tensor, will be called the inhomogeneous terms. In the transformation law of $\Delta\beta_K$, the homogeneous terms cancel due to the fact that all the lower indices are contracted with upper indices. The contribution from the inhomogeneous terms contain at least one factor of $\frac{\partial^r z'^i}{\partial z^{i_1}...\partial z^{i_r}}$ or its complex conjugate $(r \geq 2)$. Since each term in the expression must contain equal number of upper and lower indices and since the terms mentioned above have more number of lower indices than upper indices, the final expression must contain factors of $G^{i\bar{j}}$. For a general Kahler potential, such an expression cannot be expressed as a sum of holomorphic and antiholomorphic functions. As a result, the only way this result can be compatible with our analysis before is that the contribution from the inhomogeneous terms to $\Delta\beta'_K - \Delta\beta_K$ vanishes identically. This proves that $\Delta\beta_K$ is a globally defined scalar field for any Kahler manifold with Kahler potential K.

Using the Hodge decomposition theorem we may now express $\Delta\beta_{\tilde{K}}$ as

$$c^{-1}\Delta\beta_{\tilde{K}} = a_0 + \delta a_1 \tag{15}$$

where a_0 is a globally defined harmonic zero form, a_1 is a globally defined one form and δ is the adjoint of the exterior derivative d. Using the Hodge decomposition theorem again on a_1 we may write,

$$a_1 = b_1 + db_0 + \delta b_2 \tag{16}$$

where b_1 is a harmonic one form, and b_0 and b_2 are two globally defined zero and two forms respectively. Substituting (16) in (15) and using the fact that,

$$\delta^2 b_2 = \delta b_0 = \delta b_1 = db_1 = 0,$$

we get,

$$c^{-1}\Delta\beta_{\tilde{K}} = a_0 + (\delta d + d\delta)b_0 = a + \tilde{\nabla} b_0 \tag{17}$$

where $\tilde{\nabla}$ is the Laplacian operator on the Calabi-Yau manifold. To lowest order Eq.(13) may then be written as,

$$\tilde{\nabla}\delta K = a_0 + \tilde{\nabla} b_0 \tag{18}$$

Since the only harmonic zero form in a connected manifold is a constant function, a_0 in Eq.(18) is just a constant. A solution to Eq.(18) is then given by,

$$\delta K = \frac{a_0}{N}\tilde{K} + b_0 \tag{19}$$

since,

$$\tilde{\nabla}\tilde{K} = \tilde{G}^{i\bar{j}}\tilde{G}_{\bar{j}i} = N \tag{20}$$

in a manifold of complex dimension N.

Thus,

$$\delta G_{i\bar{j}} = \frac{a_0}{N}\partial_i\partial_{\bar{j}}\tilde{K} + \partial_i\partial_{\bar{j}}b_0 = \frac{a_0}{N}\tilde{G}_{i\bar{j}} + D_iD_{\bar{j}}b_0 \tag{21}$$

is a globally defined tensor, since b_0 is a globally defined scalar field and a_0 is a constant. Thus $G_{i\bar{j}}$ is also a globally defined tensor.

Substituting the value of $\delta G_{i\bar{j}}$ given in (21) we may show that the right hand side of Eq.(13) is a globally defined scalar field. We may then repeat the arguments presented above to show that the next order contribution to δG obtained by solving Eq.(13) is a globally defined tensor field.

Thus our analysis shows that given a Calabi-Yau manifold, we can always construct a conformally invariant σ-model. The metric of the σ-model is not the Ricci flat metric, but is related to it by a non-local field redefinition. The new metric is a globally defined tensor on the manifold, and hence is a valid choice of the metric.

Our result may be viewed in a somewhat different way by writing Eq.(5) as,

$$\int \tilde{K}(\Phi,\bar{\Phi})d^2\theta d^2\bar{\theta}d^2\xi - \int \delta K(\Phi,\bar{\Phi})d^2\theta d^2\bar{\theta}d^2\xi \tag{22}$$

In the above equation we may interprete $\tilde{K}$ as the background Kahler potential and δK as the finite local counterterms added in higher orders in the σ-model perturbation theory. In this particular renormalization scheme, the N=2 supersymmetric non-linear σ-model, formulated on Ricci flat Kahler manifolds, gives us a vanishing β-function to all orders in the perturbation theory. As we shall see, this particular way of interpreting our result may be useful for some application to the superstring theory.

Finally we discuss the implication of our result for the superstring theory. For convenience, we start from a specific renormalization scheme in which the calculated β-function of the σ-model is identical to the equation of motion of the metric derived from the string effective action, which in turn is calculated from the scattering amplitude involving the massless particles in the string theory. We call this metric the physical metric. Since the scattering amplitude remains unchanged under a local redefinition of the metric,[6] this prescription defines the physical metric only up to a local field redefinition (which does not involve the inverse of the Laplacian operator). But this is enough for our discussion.* Our result tells us that given a Calabi-Yau manifold, we can always find a background vacuum expectation value of the physical metric which will satisfy the equations of motion of the string theory. This background metric is obtained by solving Eq.(13) with the $\Delta\beta_K$ calculated in this particular renormalization scheme.

For some purposes (e.g. the study of the four dimensional effective field theory obtained after compactification) we do not need to know what the physical metric looks like in terms of the Ricci flat metric. In that case it may be useful to use the modified renormalization scheme that we have proposed to analyze the system. We take the Calabi-Yau metric as the background metric, and add finite local counterterms in each order in the perturbation theory in order to have a vanishing β-function. The result is a two dimensional conformally invariant field theory. We may then calculate the particle spectrum and the interaction in the effective four dimensional theory by identifying operators of conformal dimension (1,1) as vertex operators[15] and calculating their correlation functions in the two dimensional conformal field theory obtained this way. In this scheme we say that N=2 non-linear σ-models, formulated on Ricci flat manifolds, have vanishing β-function to all orders in the perturbation theory. It is in this scheme that Witten's general argument[16] showing the vanishing of the β-function on Ricci flat Kahler manifolds works. This argument was based on a study of the effective four dimensional field theory, and does not specify the renormalization scheme in which the proof should work.

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* A specific example of such a renormalization scheme is the calculation by Grisaru et. al., the β-function calculated by them agrees with the string S-matrix calculation, and hence in their renormalization scheme the background metric can be identified with the physical metric.

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