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THE QUANTUM LIMIT FOR SUCCESSIVE POSITION MEASUREMENTS*

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ABSTRACT

We apply a formalism for the description of multitime measurements to determine the quantum limit on the precision with which the second of a pair of successive position measurements can be performed on a free mass. The result depends on the position resolution of the measuring device as well as on the time interval between the two measurements, and it spans a range of values whose minimum is smaller than a presently controversial result, the SQL, by a factor of $\sqrt{2}$; the issue is hereby resolved.

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Recently there has been a confluence of interest in the quantum mechanical effects and limitations associated with very small scales and with ultra-high-precision position measurements. Examples include optical communication, laser interferometry in gravitational wave detection, and very-small-scale solid state devices.[1] In particular, the quantum mechanical limitations on the precision of successive position measurements of a free mass are of considerable importance in connection with the detection of gravitational waves. In this connection, a result known as the *standard quantum limit* (SQL) for position measurements [recorded in Eq. (10) below] has been the subject of considerable interest, as well as of some recent controversy.[2] It is the purpose of this letter to apply a formalism developed for the description of multitime measurements[3] to a derivation of the quantum mechanical limitation on the precision with which the second of a successive pair of position measurements can be performed. We shall refer to this result as the *quantum limit*, or the QL [recorded in Eq. (7) below], so as to avoid confusion with the SQL. As will be seen in the following, the crucial feature in the present derivation is the explicit incorporation of the properties (*e.g.,* finite resolutions) of measuring devices in the description of quantum measurements, a point which has served as a guiding principle in the developments that have led to the present formulation.[3,4,5] (Refs. 3, 4 and 5 will be referred to as Papers I, II and III.) It should be pointed out here that the relevance of the finite resolution of the measuring device was recognized by Caves when he attempted to reestablish the SQL after a serious flaw had been pointed out by Yuen; see the Letters cited in Ref. 2.

The SQL states that in two successive position measurements of a free mass m, a time T apart, the variance in the second measurement cannot be reduced below $(T/M)^{1/2}$. On the other hand, the above-mentioned Letter by Yuen maintains that the so-called *contractive* states in fact violate the SQL and reduce the variance in question below the quoted value. The current (and rather unsettled) state of the issue is summarized in Cave's Letter. The underlying difficulty, as may be seen in the cited works, is a lack in the existing literature on measurement theory of a realistic formulation capable of analyzing problems that arise in actual measurements. We believe the present formalism provides an appropriate means for treating such problems. As we shall see in the following, our analysis shows that the QL in fact spans a range of values depending on the position resolution of the measuring device, and that the lower limit of this range is indeed lower than the SQL value by a factor of $\sqrt{2}$. We now turn to the derivation of these results.

Consider two successive measurements of position at times $-T/2$ and $+T/2$ by means of a device whose *resolution* for position measurements is Δx . This particular measurement, as well as the notation used here, is described in Paper III, where it is shown that the state of the free mass m so measured is given by the density matrix

$$\hat{\rho} \;=\; Z^{-1}\,\exp\left\{ -\sum\,\left[\lambda_i^-\,\hat{\pi}_i^x\left(-\tfrac{T}{2}\right) + \lambda_i^+\,\hat{\pi}_i^x\left(+\tfrac{T}{2}\right)\right] \right\} \quad . \tag{1}$$

The objects of our attention are the variances $\delta x^{\pm}$ defined by

$$\left(\delta x^{\pm}\right)^2 \;=\; \mathrm{Tr}\,\hat{\rho}\left[\hat{x}\left(\pm\tfrac{T}{2}\right)\right]^2 - \;\left[\mathrm{Tr}\,\hat{\rho}\,\hat{x}\left(\pm\tfrac{T}{2}\right)\right]^2 \quad , \tag{2}$$

where, as for any operator (that does not explicitly depend upon time),

$$\hat{x}(T) \;=\; \hat{U}^{\dagger}(T)\,\hat{x}\,\hat{U}(T) \quad ,$$

$$\hat{U}(t) \;=\; \exp\left(-\frac{i\,t\,\hat{p}^2}{2m}\right) \quad ,$$

and where the absence of a time argument implies the reference time $t = 0$. The quantum mechanical limitation we are seeking is a lower bound on δx^+, the variance in the second position measurement.

To arrive at the desired limit, we find it expedient to consider a unitary transformation implemented by

$$\hat{V} \;=\; \exp\left(\frac{i\,m\,\hat{x}^2}{2T}\right)\,\exp\left(\frac{i\,T\,\hat{p}^2}{4m}\right) \quad . \tag{3}$$

The transformed state $\hat{\rho}_V = \hat{V}\,\hat{\rho}\,\hat{V}^\dagger$ is then found to be

$$\hat{\rho}_V \;=\; Z^{-1}\,\exp\left\{-\sum_i \left[\,\lambda_i^-\,\hat{\pi}_i^x + \lambda_i^+\,\hat{\pi}_i^q\right]\right\} \quad , \tag{4}$$

where we have used $\hat{V}\,\hat{x}\,(-\frac{T}{2})\,\hat{V}^\dagger \;=\; \hat{x}$, and $\hat{V}\,\hat{x}\,(+\frac{T}{2})\,\hat{V}^\dagger \;=\; (\frac{T}{m})\hat{p}$. The latter quantity, $\frac{T}{m}\,\hat{p}$, is just what we have called $\hat{q}$ in Eq. (4), so that $\hat{\pi}_i^q$ are in fact projection operators for momentum bins of size $\Delta p \;=\; \frac{m}{T}\,\Delta x$. Therefore, Eq. (4) describes a *canonical measurement* of state (*cf.* Paper III) accomplished by means of a position measurement with resolution Δx and a momentum measurement with resolution $\Delta p = \frac{m}{T}\,\Delta x$. Moreover, in the V representation, the variances $\delta x^\pm$ appear as

$$\begin{aligned}\left(\delta x^+\right)^2 \;&=\; \left(\tfrac{T}{m}\right)^2\left[\mathrm{Tr}\;\hat{\rho}_V\,\hat{p}^2 - (\mathrm{Tr}\;\hat{\rho}_V\,\hat{p})^2\right] \quad , \\ \left(\delta x^-\right)^2 \;&=\; \left[\mathrm{Tr}\;\hat{\rho}_V\,\hat{x}^2 - (\mathrm{Tr}\;\hat{\rho}_V\,\hat{x})^2\right] \quad .\end{aligned} \tag{5}$$

In other words, $\delta x^\pm$ are respectively equal to $\frac{T}{m}\,\delta p$ and δx in the new representation. Our task is thus reduced to finding the minimum of δp and the state that realizes that minimum.

The last-mentioned minimum was in fact considered in Paper III, where we found that the symmetries possessed by the optimal state (*i.e.*, the state that realizes the said minimum) imply the equalities $\lambda_i^+ \,=\, \lambda_i^-$. But these equalities in turn imply that $\delta x^+ = \delta x^-$, as can be seen from Eqs. (1) and (2), so that we have the result that $(\delta x^+)^2 = (\delta x^+)(\delta x^-) = \frac{T}{m}\,(\delta x)(\delta p)$. In other words, instead of the minimum of $\delta p \;=\; \frac{m}{T}\delta x^+$, we may equivalently look for the minimum value of the variance product $(\delta x)(\delta p)$ in the V representation (where position and momentum are measured with resolutions Δx and $\frac{m}{T}\,\Delta x$, respectively).

We considered this last problem in Paper II, where we found that the variance product $(\delta x)(\delta p)$ has a universal lower bound U_{inf} which is a function of the dimensionless quantity $k = \frac{1}{2\pi}\,(\Delta x)(\Delta p)$. Moreover, we found that[7]

$$
\begin{aligned}
U_{inf}(k) &\simeq \tfrac{1}{2} + \tfrac{\pi}{6}\,k + O(k^2) \quad , \qquad k \ll 1 \quad , \\
U_{inf}(k) &\simeq \tfrac{\pi}{6}\,k + \ldots. \quad , \qquad k \gg 1 \quad .
\end{aligned}
\tag{6}
$$

In particular, in the limit of $k = 0$ one has $U_{inf}(0) = 1/2$, which is the standard Heisenberg result. On the other hand, the behavior for $k \gg 1$ is a purely classical result arising from finite resolutions; recall the definition of k given above.

We are now in a position to assemble the above information. First, we have from $\left(\delta x^+\right)^2 = \frac{T}{m}\,(\delta x)(\delta p)$ the statement that $\left(\delta x^+\right)^2 \geq \frac{T}{m}\,U_{inf}(k)$. Next, using the definition of k we find $k = \frac{m}{2\pi T}\,(\Delta x)^2$, and from this the result that

$$
\delta x^+ \geq \ell_0\,\left\{2U_{inf}\left(\frac{m}{2\pi T}(\Delta x)^2\right)\right\}^{1/2} \quad , \quad (\mathrm{QL}) \quad , \tag{7}
$$

where we have defined $\ell_0 \equiv (T/2m)^{1/2}$. Equation (7) is a statement of the quantum limit (QL) for successive position measurements. Using the limiting behavior of U_{inf} given in Eq. (6), we obtain from Eq. (7)

$$
\delta x^+ \geq \ell_0\,\left[1 + \tfrac{1}{24}\,\left(\tfrac{\Delta x}{\ell_0}\right)^2\right] \,, \quad \tfrac{\Delta x}{\ell_0} \ll 1 \quad , \tag{8}
$$

for high-resolution and/or long-duration measurements, and

$$
\delta x^+ \geq \tfrac{1}{\sqrt{12}}\,\Delta x \,, \quad \tfrac{\Delta x}{\ell_0} \gg 1 \quad , \tag{9}
$$

for low-resolution and/or short-duration measurements. Note that Eq. (8) is a classical result,[8] as is the second member of Eq. (6), and it merely reflects the fact that the finite bin size Δx induces a minimum in the variance δx which cannot be reduced below $\frac{1}{\sqrt{12}}\,\Delta x$ (corresponding to a uniform spatial distribution confined to a single bin).

For comparison, we note that the SQL gives

$$\delta x^+ \geq \sqrt{2}\,\ell_0 \quad , \qquad (\text{SQL}) \quad , \tag{10}$$

a value which is intermediate between the absolute minimum ℓ_0 seen in Eq. (8) and the classical result given in Eq. (9). One can see from Eqs. (7)–(9) that the two important scales in the problem are the position resolution Δx and the natural quantum scale of the problem, ℓ_0. For example, for sufficiently long measurement times T, ℓ_0 can be made arbitrarily large (for a fixed mass m) so as to render the required resolution for achieving the absolute limit a relatively easy task. Physically, this corresponds to the fact that for such long measurement times, the spread in the spatial distribution is enhanced to such a degree as to make the finite bin size Δx inconsequential.

As pointed out in the introductory remarks, the current discussion on the SQL arose in connection with gravitational wave detectors using laser interferometry. The most optimistic estimates of Δx for these devices place it at or about ℓ_0, i.e., $\Delta x \gtrsim \ell_0$. On the basis of Eq. (8), then, one would expect that $\delta x^+ \gtrsim \ell_0$ for such resolutions. However, it should be remembered that the estimated optimal resolution $\Delta x \gtrsim \ell_0$ is subject to a number of conditions,[9] among which is a stringent requirement on the measurement time T (which must be matched to the interferometer parameters so as to minimize thermal noise), and also that the presently achievable resolutions actually correspond more closely to the limit given in Eq. (9).

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References

1. As examples of recent literature, we mention C. Caves *et al., Rev. Mod. Phys.* **52**, 341 (1980); J. Tucker and M. J. Feldman, *ibid.* **57**, 1055 (1985); *"Quantum Optics, Experimental Graviation and Measurement Theory,* eds. P. Meystre and M.O. Scully (Plenum, New York, 1983); Y. Srivastava, A. Widom and M. Freidman, *Phys. Rev. Lett.* **55**, 2246 (1985).
2. V. B. Braginsky and Yu. I. Vorontsov, *Usp. Fiz. Nauk* **114**, 41 (1974) [*Sov. Phys. Usp.* **17**, 644 (1975)]; C. M. Caves *et al., Rev. Mod. Phys.* **52**, 341 (1980); H. Yuen, *Phys. Rev. Lett.* **51**, 719 (1983) and **52**, 1730 (1984); C. Caves, *ibid.* **54**, 2465 (1985).
3. H. Partovi, *Phys. Rev. Lett.* **50**, 1883 (1983).
4. R. Blankenbecler and H. Partovi, *Phys. Rev. Lett.* **54**, 373 (1985).
5. H. Partovi and R. Blankenbecler, SLAC-PUB-3903, March (1986).
6. This time reversal property should be compared to those of the contractive states of Yuen, Ref. 2 above, and to Fig. 1 found therein.
7. We have not succeeded in finding a nonperturbative method of calculating $U_{inf}(k)$ for all k; see Ref. 4 above.
8. To restore Planck's constant to Eqs. (7)–(9), simply replace T by $\hbar T$ in the definition of ℓ_0 and in the argument of U_{inf}.
9. See the articles on experimental gravitation and laser interferometry in the book cited in Ref. 1 above.