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A STRING PRIMER*

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Abstract

This is an elementary introduction to the classical and quantum mechanics of a single bosonic string, and to some aspects of its supersymmetric and heterotic extensions.

1. Introduction

This last year we have witnessed a dramatic revival of interest in string theories. It followed the remarkable observation of Green and Schwarz,[1] that gauge and gravitational anomalies miraculously cancel in a theory of open and closed superstrings with SO(32) as the gauge group. Here, then, is an aesthetically pleasing theory with a rich enough spectrum to potentially encompass all known phenomenology (including gravitational interactions),[2] and for which no one has yet been able to demonstrate an inconsistency. We might, in particular, for the first time have a consistent theory of quantum gravity. This should by itself suffice as motivation to further pursue this mathematically rich and largely unexplored direction. It's nevertheless, fair to say that not only the phenomenological merits,[3] but the very predictive power of string theories[4] remains at present questionable.

In these two lectures I will discuss the most elementary and old-fashioned aspects of the classical and quantum dynamics of a single string. Rather than a review, I would call it a preview of the several thorough reviews that already exits.[5–8] My hope is to demystify some of the most common buzz words of string theory, in preparation for the more technical talks during the last week of this School. In Section 2 I discuss the classical dynamics of a bosonic string. In Section 3 I discuss its quantization, and the emergence of a unique critical dimension of space-time. Finally in Section 4 I briefly consider its supersymmetric extensions, including

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the heterotic strings,[9] and the emergence of two unique gauge groups. This is the only material in these notes that is less than ten years old.

2. Classical Relativistic String[10]

The classical trajectory followed by a free relativistic particle from (space-time) point A to point B is obtained by extremizing an action-functional over all possible time-like curves $x^\mu(\tau)$ between A and B (see Fig. 1). The simplest choice for an action is the geometric length of the trajectory†

$$S = -m \int d\tau \, \left(\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} \right)^{1/2}$$

which is invariant under arbitrary reparametrizations $x^\mu(\tau) \to x^\mu(\tilde{\tau}(\tau))$. Particularly convenient is the unit-speed parametrization where $(dx^\mu/d\tau)(dx_\mu/d\tau) = 1$, and the equation of motion takes the simple form

$$\frac{d^2 x^\mu}{d\tau^2} = 0$$

showing that the classical trajectory is a straight line. Such a parametrization is always possible, provided the trajectory is time-like, that is $(dx^\mu/d\tau)(dx_\mu/d\tau) > 0$ everywhere.

By complete analogy we can derive the classical mechanics of a freely moving relativistic string by extremizing the invariant area of its trajectory,[11] which is now a two-dimensional "world-surface" $x^\mu(\sigma,\tau)$. In what follows we shall also use ς^0 and ς^1 in place of τ and σ respectively. The world surface is locally approximated by a Minkowski plane spanned by a time-like vector $\dot{x}^\mu \equiv \partial x^\mu/\partial\tau$ and a space-like vector $x'^\mu \equiv \partial x^\mu/\partial\sigma$ as shown in Fig. 2. The distance squared between two points on the surface with coordinates ς^α and $\varsigma^\alpha + d\varsigma^\alpha$ is

$$dx^\mu dx_\mu = (\partial_\alpha x^\mu \partial_\beta x_\mu) \; d\varsigma^\alpha \; d\varsigma^\beta \equiv g_{\alpha\beta}(\varsigma) \; d\varsigma^\alpha \; d\varsigma^\beta$$

which defines the "induced-metric"

$$g_{\alpha\beta} = \begin{pmatrix} \dot{x}^2 & \dot{x}\cdot x' \\ \dot{x}\cdot x' & x'^2 \end{pmatrix} \tag{2.1}$$

Using elementary Minkowski geometry we can calculate the area of the infinitesimal parallelogram of Fig. 2:

† Our convention for the space-time metric $\eta^{\mu\nu}$ is $(+1,-1,\ldots,-1)$.

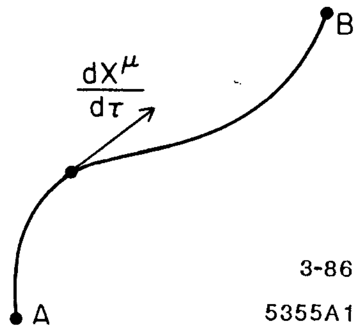

Fig. 1. Typical trajectory of a point-particle, over which the action should be minimized. The velocity $dx^\mu/d\tau$ must lie everywhere in the forward light-cone.

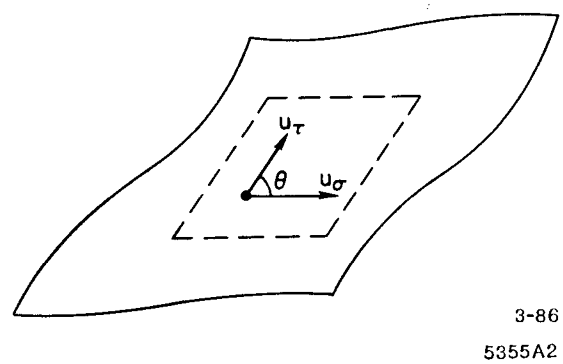

Fig. 2. A typical string trajectory. It can be locally approximated by a Minkowski plane spanned by $u_\tau = \dot{x}^\mu d\tau$ and $u_\sigma = x'^\mu d\sigma$.

$$\text{Area } (u_\sigma, u_\tau) = |u_\sigma||u_\tau| \sinh\theta = |u_\sigma||u_\tau| \sqrt{\cosh^2\theta - 1}$$

$$= [(u_\sigma \cdot u_\tau)^2 - u_\sigma^2 \; u_\tau^2] = (-\det g)^{1/2} \; d\sigma \; d\tau$$

The Nambu action[11] for the string is proportional to the total area of the world surface

$$S = -\frac{1}{4\pi\alpha} \int d\sigma \; d\tau \; (-\det g)^{1/2} \tag{2.2}$$

where α is called the Regge slope, has dimensions of area and we shall here set it equal to 1/2. That $\det g \leq 0$ is guaranteed by the fact that the surface is everywhere time-like.

The Nambu action is invariant under non-singular reparametrizations of the surface

$$\varsigma \to \tilde{\varsigma}(\varsigma) \quad ; \quad g_{\alpha\beta} \to \left(\frac{\partial \varsigma^\gamma}{\partial \tilde{\varsigma}^\alpha}\right) \left(\frac{\partial \varsigma^\delta}{\partial \tilde{\varsigma}^\beta}\right) g_{\gamma\delta}$$

We are thus allowed to impose two gauge conditions, or functional constraints; a particularly convenient choice is to make the tangent vectors $\dot{x}^\mu$ and x'^μ everywhere orthonormal up to a scale factor:

$$\dot{x}^\mu \; x'_\mu = 0 \tag{2.3a}$$

$$\dot{x}^2 + x'^2 = 0 \tag{2.3b}$$

so that the induced metric becomes conformally flat

$$g_{\alpha\beta} = \dot{x}^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv e^{\phi} \cdot n_{\alpha\beta}$$

We shall refer to Eqs. (2.3) as the (Lorentz) covariant gauge. In terms of $\varsigma^{\pm} = (\varsigma^0 \pm \varsigma^1)$ we can rewrite them as:

$$(\partial_+ x)^2 = (\partial_- x)^2 = 0 \tag{2.4}$$

which makes it clear that they leave a residual invariance under reparametrizations of the form $\varsigma^+ \to \tilde{\varsigma}^+(\varsigma^+)$ and $\varsigma^- \to \tilde{\varsigma}^-(\varsigma^-)$. In Euclidean parameters ($\varsigma^0 \to i\varsigma^0$, $\varsigma^+ \to z$, $\varsigma^- \to \bar{z}$) this residual symmetry is the conformal groups of all menomorphic (and antimenorphic) transformations.

Now using Eqs. (2.4), we can rewrite the Nambu action

$$S = -\frac{1}{4\pi} \int d\varsigma^+ \, d\varsigma^- (\partial_+ x^{\mu})(\partial_- x_{\mu}) \tag{2.5}$$

so that the string coordinates x^{μ} satisfy the simple wave equation

$$\partial_+ \partial_- x^{\mu} = \ddot{x}^{\mu} - x''^{\mu} = 0 \tag{2.6}$$

In what follows we will restrict ourselves to closed strings, meaning that the x^{μ} are periodic in σ with a period that we set equal to π. For open strings the requirement of a stationary action would have lead to the edge condition $x'^{\mu}|_{\substack{\text{string}\\ \text{endpoints}}} = 0$.

Now the most general solution to the wave Eq. (2.6) on a periodic $0 < \sigma \leq \pi$ strip is

$$\begin{aligned} x^{\mu} = q^{\mu} + P^{\mu} \cdot \tau + \sum_{n \neq 0} \frac{i}{2n} \, a_n^{\mu} \, e^{-2in(\tau - \sigma)} \\ + \sum_{n \neq 0} \frac{i}{2n} \, \tilde{a}_n^{\mu} \, e^{-2in(\tau + \sigma)} \end{aligned} \tag{2.7}$$

where q^{μ} is the initial center-of-mass position of the string, $P^{\mu} = \frac{1}{\pi} \int_0^{\pi} d\sigma \, \dot{x}^{\mu}$ is its conserved total momentum (as can be easily seen by using the Noether theorem for an infinitesimal translation), and the amplitudes of the left- and right-moving modes obey the reality conditions:

$$a_n^{\mu *} = a_{-n}^{\mu} \quad ; \quad \tilde{a}_n^{\mu *} = \tilde{a}_{-n}^{\mu}$$

It still remains to make sure that the gauge conditions (2.4), which have been used to arrive at the wave-equations of motion, are satisfied. After some straightforward algebra, they can be written in the following form:

$$L_N \equiv -\frac{1}{2} \sum_{n=-\infty}^{\infty} a^\mu_{N-n}\, a_{n\mu} = 0 \tag{2.8a}$$

$$\tilde{L}_N \equiv -\frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{a}^\mu_{N-n}\, \tilde{a}_{n\mu} = 0 \tag{2.8b}$$

where $a^\mu_0 \equiv \tilde{a}^\mu_0 \equiv \frac{1}{2}P^\mu$.

Conditions (2.8) play a central role in the theory of strings. To better understand their meaning, note that L_N is the generator of the infinitismal menomorphic transformation*

$$z \to z + \epsilon_N \cdot z^{N+1} \tag{2.9}$$

which leaves the free-field action (2.5) invariant. This can be checked directly by use of the Noether theorem. A less direct, but useful for what follows method, is to show that the L_N's obey the infinite dimensional algebra of the two-dimensional conformal group. Indeed, the canonical Poisson brackets for the free field theory (2.5) read

$$\left\{\mathcal{P}^\mu(\sigma,\tau),\ x^\nu(\sigma',\tau')\right\}\Big|_{\tau=\tau'} = -n^{\mu\nu}\cdot\delta(\sigma-\sigma') \tag{2.10}$$

where $\mathcal{P}^\mu \equiv -\frac{\delta S}{\delta \dot{x}^\mu} = \frac{1}{\pi}\dot{x}^\mu$ is the momentum density. These can be equivalently rewritten in terms of the normal-mode oscillators:

$$\{a^\mu_n,\ a^\nu_m\} = \{\tilde{a}^\mu_n,\ \tilde{a}^\nu_m\} = -in^{\mu\nu}\cdot n\cdot\delta_{m,-n} \tag{2.11a}$$

$$\{P^\mu,\ q^\nu\} = -n^{\mu\nu} \tag{2.11b}$$

Thus, for any two functionals A and B of the string coordinates, we have

$$\{B,A\} \equiv \frac{\partial A}{\partial q^\mu}\frac{\partial B}{\partial P_\mu} - \frac{\partial A}{\partial P^\mu}\frac{\partial B}{\partial q_\mu} + i\sum_{n\neq 0} n\cdot\frac{\partial A}{\partial a^\mu_n}\frac{\partial B}{\partial a_{-n\mu}}$$

$$+ i\sum_{n\neq 0} n\cdot\frac{\partial A}{\partial \tilde{a}^\mu_n}\frac{\partial B}{\partial \tilde{a}_{-n\mu}}$$

One can then easily deduce that

$$\{L_N, L_M\} = i(N-M)L_{N+M} \tag{2.12}$$

with a similar expression for the L_N. That this is indeed the algebra of the conformal

* We jump from the Minkowski to the Euclidean notation freely, as this should cause no confusion.

group follows by commuting two infinitesimal transformations of type (2.9)

$$\left(\left(z+\epsilon_M z^{M+1}\right)+\epsilon_N\left(z+\epsilon_M z^{M+1}\right)^{N+1}\right)$$

$$-\left(\left(z+\epsilon_N z^{N+1}\right)+\epsilon_M\left(z+\epsilon_N z^{N+1}\right)^{M+1}\right)$$

$$=\epsilon_N\epsilon_M\cdot(N-M)z^{N+M+1}+0(\epsilon^3)$$

We can now understand the meaning of conditions (2.8). The string coordinates can be made to obey the conformally invariant wave equation in two dimensions; since however this conformal invariance is part of the original reparametrization invariance (*i.e.* gauge symmetry) of the string, it can have no physical meaning. Thus, its generators L_N and $\tilde{L}_N$ must be set equal to zero. Note in particular that $L_0=\tilde{L}_0=0$ leads to

$$-\frac{1}{8}P^2=\sum_{n>0} a_n\cdot a_n^*=\sum_{n>0}\tilde{a}_n\cdot\tilde{a}_n^* \tag{2.13}$$

which expresses the effective mass of the string in terms of the amplitudes of oscillation modes.

It is of course possible to solve the conditions (2.8) explicityly, or, put differently, to completely fix the residual invariance under conformal parametrizations, by setting the parameter τ proportional to some string coordinate

$$n^\mu\, x_\mu=(n\cdot P)\tau \tag{2.14}$$

It is convenient to choose a light-like vector

$$n^\mu\equiv\frac{1}{\sqrt{2}}\,(1,-1,0,\ldots,0)$$

and the notation $A^\pm=\frac{1}{\sqrt{2}}(A^0\pm A')$ and $\vec{A}=(A^2,A^3,\ldots,A^{D-1})=\{A^i\}$ for any D-vector A^μ. Then the only independent degrees of freedom of the string are q^-, P^+ and the "transverse" oscillators a_n^i and $\tilde{a}_n^i$ (including $a_0^i=\tilde{a}_0^i=\frac{1}{2}P^i$) since

$$q^+=a_n^+=\tilde{a}_n^+=0\qquad\text{for } n\neq 0 \tag{2.15a}$$

and the conditions (2.8) yield

$$a_N^-=\frac{1}{P^+}\sum_{n=-\infty}^{\infty}\vec{a}_{N-n}\cdot\vec{a}_n \tag{2.15b}$$

$$\tilde{\alpha}_N^- = \frac{1}{P^+} \sum_{n=-\infty}^{\infty} \vec{\tilde{\alpha}}_{N-n} \cdot \vec{\tilde{\alpha}}_n \tag{2.15c}$$

Note that by setting $N = 0$, we find that only transverse oscillations contribute to the effective mass. Equations (2.15) together with (2.7) give a complete description of the classical dynamics of a relativistic string.

3. Quantization: The Critical Dimension

As is well known, the symmetries of a classical system are not automatically preserved upon quantization. In canonical quantization operator ordering problems may forbid the implementation of the algebra of symmetry-group-generators. In the functional integral approach, the Noether current of the symmetry is not necessarily conserved due to ultraviolet divergences that require the introduction of a regulator. We then say that the corresponding symmetry is anomalous. As we will now see, the global Loventz invariance and the local reparametrization invariance of a string can be both preserved at the quantum level, without introducing extra degrees of freedom, only at a special number of space-time dimensions, namely $D = 26$.

There are two different ways of quantizing the classical string. We shall briefly describe them both, even though they are totally equivalent at the critical dimension, because they offer complementary views that are particularly useful given our present incomplete understanding of the theory of strings. A historical analog is that of spontaneously broken gauge theories, for which unitarity is manifest in one gauge, and renormalizability in another.

(A) Light-cone quantization

The starting point here is the completely fixed parametrization, defined by Eqs. (2.14) and (2.4). We then postualte canonical commutation relations, according to the rule

$$\{A, B\} \leftrightarrow i[A, B]$$

only for the independent variables q^-, P^+, $\vec{\alpha}_n$ and $\vec{\tilde{\alpha}}_n$. Explicitly

$$[\alpha_n^i,\ \alpha_m^j] = [\tilde{\alpha}_n^i,\ \tilde{\alpha}_m^j] = +\delta^{ij} \cdot n \cdot \delta_{m,-n} \tag{3.1a}$$

$$[P^+, q^-] = +i \tag{3.1b}$$

The Hilbert space can be constructed by starting with a ground state, annihilated by all positive-frequency oscillators, and characterized by some D-momentum p^μ:

$$a^i_n|0,p\rangle = \tilde{a}^i_n|0,p\rangle = 0 \qquad (n > 0)$$

$$P^\mu|0,p\rangle = p^\mu|0,p\rangle$$

Negative-frequency oscillators act to create excited states. Note that unlike conventional field theories, where the oscillators are labelled by the momentum in real space, here they are labelled by the frequency of the excitation in the direction along the string. This frequency is nevertheless related to the invariant mass of the corresponding string state, by means of Eqs. (2.15b) and (2.15c) which we can rewrite in the form

$$\begin{aligned} \frac{1}{8}M^2 &= \frac{1}{8}(2P^+P^- - \vec{P}\cdot\vec{P}) \\ &= \sum_{n>0} \vec{a}_{-n}\cdot\vec{a}_{+n} - \alpha_0 = \sum_{n>0} \vec{\tilde{a}}_{-n}\cdot\vec{\tilde{a}}_{+n} - \alpha_0 \end{aligned} \tag{3.2}$$

We have here normal-ordered the oscillators so that positive frequencies appear to the right of negative frequencies, and have introduced the constant α_0 expressing the effect of zero-point oscillations

$$\alpha_0 = (D-2)\sum_{n=0}^{\infty}\frac{1}{2}n$$

The sum is of course formally infinite, but we can calculate it by introducing a cutoff and looking for the cutoff-independent finite piece. This is justified because a change in the cutoff can be compensated by a rescaling of the parameters of the world surface, that should not affect α_0, which is the physically observable mass of the lowest string state. In the "heat-kernel" regularization we find

$$\sum_0^{\infty}\frac{1}{2}\,ne^{-\beta n} = -\frac{1}{2}\frac{d}{d\beta}\left(\frac{1}{1-e^{-\beta}}\right) = \frac{1}{2\beta^2} - \frac{1}{24} + O(\beta)$$

so that

$$\alpha_0 = -\frac{(D-2)}{24} \tag{3.3}$$

The reader can verify that the same result obtains in any other regularization, for instance:

$$\alpha_0 = \lim_{s\to -1} \sum_1^{\infty} \frac{1}{2} n^{-s}$$

This is of course completely analogous to the Casimir effect of quantum electrodynamics: the cutoff independent piece of the zero point energy that scales appropriately with cavity size, is unique and experimentally observable.

We now go back to the construction of the Hilbert space. First we note that in view of the commutation relations (3.1a), and the hermiticity property $a_n^{i+} = a_{-n}^i$, all states have positive definite norm; the price we pay, however, is the loss of manifest Lorentz invariance. Given that left- and right-moving modes must occur in equal numbers [because of Eq. (3.2)], the first excited state is $a_{-1}^i \, \tilde{a}_{-1}^j \, |0, p\rangle$, and has a mass $\frac{1}{8}M^2 = 1 - \frac{(D-2)}{24}$. But the only spin-2 particle (or 2-index symmetric field) with only transverse degrees of freedom is the graviton, which is massless. Thus, if our string theory is to be consistent with Lorentz invariance, we must have $0 = 1 - \frac{(D-2)}{24} \Rightarrow D = 26$!

It is of course also possible to arrive at the critical dimension by a more direct argument, namely by explicitly calculating the algebra of Lorentz generators

$$M^{\mu\nu} = \frac{1}{2\pi} \int_0^{\pi} d\sigma \, (x^\mu \, \dot{x}^\nu - x^\nu \dot{x}^\mu)$$

The ordering of operators in $M^{\mu\nu}$ is completely fixed by demanding hermiticity; the only ambiguity is the constant α_0 in the equation for $a_0^- = \frac{1}{2}P^-$, which we can leave undetermined. A lengthy but straightforward calculation then shows[10] that the commutator $[M^{i-}, M^{j-}]$ has anomalous pieces that vanish if and only if $D = 26$ and $\alpha_0 = 1$.

In conclusion, the Hilbert space of the quantum string in the light-cone gauge is manifestly positive-definite, but Lorentz invariance is only restored at $D = 26$. Note also that the lowest lying state has a negative mass, $\left(\frac{1}{8}M^2 = -1\right)$ *i.e.* it is a tachyon. This is of course a serious problem which will find its resolution in the supersymmetric versions of the bosonic string.

(B) Covariant quantization

In order to keep Lorentz invariance manifest, we can treat all oscillators as independent variables on equal footing, by postulating the canonical commutation relations

$$[a_n^\mu, a_m^\nu] = [\tilde{a}_n^\mu, \, \tilde{a}_m^\nu] = -n^{\mu\nu} \cdot n \cdot \delta_{m,-n} \qquad (3.4a)$$

$$[P^\mu, q^\nu] = +i\eta^{\mu\nu} \tag{3.4b}$$

We must then impose the vanishing of the conformal generators

$$L_N = -\frac{1}{2} \sum_{n=-\infty}^{\infty} a_n^\mu \, a_{N-n\ \mu} \qquad (N \neq 0) \tag{3.5a}$$

$$L_0 = -\frac{1}{8} P^2 - \sum_{n>0} a_{-n}^\mu \, a_{+n\ \mu} - \alpha_0 \tag{3.5b}$$

(and similarly for $\tilde{L}$'s which we don't mention explicitly throughout most of this paper). Operator ordering ambiguities enter only in the definition of L_0, which we have written as a normal-ordered piece that annihilates the vacuum, plus an arbitrary constant. If one adopts the symmetric definition $L_0 = -\frac{1}{2}\sum_{-\infty}^{\infty} a_{-n}^\mu \, a_{+n\ \mu}$, then $\alpha_0 = \frac{D-2}{24}$ as before since the zero-point oscillations of each time-like mode a_n^0, exactly cancel those of a corresponding space-like mode, say a_n^1.

Keeping for the time being α_0 arbitrary, we can calculate the quantum algebra of the conformal generators. We may expect a difference, with respect to the classical algebra (2.12), only in the commutator $[L_N,\ L_{-N}]$, since L_0 may not appear with the correct operator ordering. Evaluating this commutator explicitly, using Eqs. (3.4), we find

$$\begin{aligned}[L_N,\ L_{-N}] &= -N \sum_{n=-\infty}^{\infty} : a_{-n}^\mu \, a_{+n\ \mu} : + \frac{D}{2} \sum_{0}^{N} n(N-n) \\ &= 2N L_0 + \frac{D}{12}(N^3 - N) + 2N\alpha_0\end{aligned}$$

where double dots indicate normal ordering; thus the full algebra takes the form:

$$[L_N,\ L_M] = (N-M)\ L_{N+M} + \delta_{N-M}\left(\frac{D}{12}N^3 + \left(2\alpha_0 - \frac{D}{12}\right)N\right) \tag{3.6}$$

The presence of the C-number anomaly may lead us to think that the conformal reparametrization invariance of the string is broken at the quantum level in any number of dimensions. This is too naive, but for the moment let us simply point out that the anomaly certainly forbids our setting all $L_N = 0$ as operator equations. We will therefore demand the weaker conditions, that the positive and zero-frequency generators annihilate all physical states

$$L_N \ |\text{phys}\rangle = 0 \qquad (N \geq 0) \tag{3.7}$$

which suffice to ensure that the expectation values of all generators in any physical state vanish. Equations (3.7) are called the Virasoro[12] gauge conditions; in particular, $L_0 \ |\text{phys}\rangle = 0$ is the mass-shell condition for the corresponding physical state.

The Hilbert space is of course constructed by applying negative-frequency oscillators on the vacuum, $|0, p\rangle$, which is a physical state of mass $\frac{1}{8}M^2 = -\alpha_0$. Due to the indefinite metric in the commutators (3.4a), however, some of these states will have negative norm. We must then make sure that no such negative-norm states exist in the physical subspace defined by the Virasoro gauge conditions.

To simplify matters notice that states of the form

$$(a_{n_1}^{\mu_1} \dots a_{n_k}^{\mu_k}) \ (\tilde{a}_{m_1}^{\nu_1} \dots \tilde{a}_{m_k}^{\nu_k}) \ |0, p\rangle$$

form an orthocomplete basis of the unconstrained Hilbert space. They can be written as the product of one "left-moving" state constructed out of the oscillators a_n^μ, an one "right-moving" state constructed out of the $\tilde{a}_n^\mu$'s. Furthermore, the Virasoro gauge conditions (3.7) apply only to the left-moving piece, while the corresponding conditions $\tilde{L}_N \ |\text{phys}\rangle = 0$ apply to the right-moving piece. It is then easy to convince oneself that the physical Hilbert space also has an orthocomplete basis constructed out of products of one left- and one right-moving state, and it will therefore suffice to show the absence of negative norms in each sector separately. This simple observation, that the left and right sectors of a closed string separate naturally, is the basis for the beautiful construction of the heterotic string.[9] Note also that one of the sectors alone, forms the complete Hilbert space of an open string.

Now, the first excited left-moving state is

$$|\epsilon\rangle \equiv \epsilon^\mu \ a_{-1\ \mu} \ |0, p\rangle$$

and has a norm $\langle\epsilon|\epsilon\rangle = -\epsilon^\mu\epsilon_\mu$ and mass $\frac{1}{8}M^2 = 1 - \alpha_0$. The constraint $L_1 \ |\epsilon\rangle = 0$, on the other hand, gives the transversality condition $\epsilon^\mu P_\mu = 0$, which shows that ϵ cannot be time-like and hence $\langle\epsilon|\epsilon\rangle \geq 0$ as it should. Furthermore, if $\alpha_0 = 1$, the momentum P^μ is light-like, and there exists a longitudinal state ($\epsilon^\mu = P^\mu$) with zero norm.

At the next mass level ($\frac{1}{8}M^2 = 2 - \alpha_0$), the most general state is:

$$|\epsilon,\theta\rangle = \frac{1}{\sqrt{2}}\left(\epsilon^\mu\, a_{-2\ \mu} + \theta^{\mu\nu}\, a_{-1\ \mu}\, a_{-1\ \nu}\right)|0,p\rangle$$

with norm

$$\langle\epsilon|\theta\rangle = -\epsilon^\mu\,\epsilon_\mu + \theta^{\mu\nu}\,\theta_{\mu\nu} \tag{3.8}$$

and we now have two non-trivial Virasovo conditions:

$$L_2\,|\epsilon,\theta\rangle = 0 \Longrightarrow \epsilon\cdot p - \theta^\mu_\mu = 0$$

$$L_1\,|\epsilon,\theta\rangle = 0 \Longrightarrow \epsilon^\mu + \frac{1}{2}\,p_\nu\,\theta^{\mu\nu} = 0$$

where $\theta^{\mu\nu}$ is a symmetric tensor. Going to the rest frame, where

$$p^\mu = \left(\sqrt{8(2-\alpha_0)},0,\ldots,0\right)$$

we can solve for ϵ^μ and θ^{00} and then express the norm of the state [Eq. (3.8)] in terms of the remaining *independent* variables:

$$\langle\epsilon|\theta\rangle = \sum_{i\neq j}(\theta^{ij})^2 + (2(2-\alpha_0)-2)\sum_i(\theta^{0i})^2 + \left\{\sum_i(\theta^{ii})^2 - \frac{(2(2-\alpha_0)-1)}{(4(2-\alpha_0)+1)^2}\cdot\left(\sum_i\theta^{ii}\right)^2\right\} \tag{3.9}$$

where i and j run from 1 to $D-1$. The first term on the right-hand side is manifestly positive; the second is non-negative provided

$$\alpha_0 \leq 1 \tag{3.10a}$$

Finally one can easily show that the third takes its minimum value when $\theta^{11} = \theta^{22} = \cdots = \theta^{D-1\ D-1}$; this minimum value is non-negative if

$$D \leq \frac{(4(2-\alpha_0)+1)^2}{(2(2-\alpha_0)-1)} + 1 \tag{3.10b}$$

When $\alpha_0 = 1$ we must have $D \leq 26$; here then reappears the critical dimension which we have found in the light-cone gauge from a completely different route. At the marginal values $D = 26$, $\alpha_0 = 1$ (which, incidentally, are consistent with the symmetric definition of L_0 that yields $\alpha_0 = \frac{D-2}{24}$, as already discussed), one clearly has in addition to positive-norm states, several zero-norm states. Much like the longitudinal photons in quantum electrodynamics, these states can be shown to decouple, leaving a smaller space of physical excitations. We will not attempt to prove this here, nor shall we show that conditions (3.10a) and (3.10b) actually guarantee the absence of negative norm states at all higher mass levels as well.[13]

Let us simply summarize, by saying that in the covariant quantization $D = 26$ appears as the marginal dimension for which the physical Hilbert space becomes non-negative definite, and at which, furthermore, a large number of zero-norm states actually decouple from the theory. These results have been in fact understood in a different, very elegant way, proposed by Polyakov.[14] Starting from a first order formulation of the classical string action, he showed how covariant gauge fixing requires the introduction of Fadeev-Popov ghosts. When their contribution is taken into account, the c-number anomaly in the conformal algebra (3.6) cancels precisely in 26 dimensions, signaling the restoration of the string's reparametrization invariance at the quantum level. For lack of time, we will not go into the details of Polyakov's approach, even though they are at the core of several interesting recent developments.[15]

4. The Semi-, Fully- and Doubly-Super Strings

We will, instead, attempt a blitz-review of the $N = 1/2$, 1 and 2 supersymmetric extensions of the bosonic string, whose phenomenological merits are, incidentally, inversely proportional to their number N of (world-sheet) supersymmetries. Let us recall that in the covariant quantization, the bosonic string can be considered as a conformal two-dimensional field theory, with the constraints that the positive-frequency conformal generators annihilate physical states. By analogy, we can construct the Neveu-Schwarz-Ramond ($N = 1$) superstring[16,17] as a superconformal field theory

$$S = -\frac{1}{2\pi}\int d\varsigma^0\, d\varsigma^1\, \left(\partial^\alpha x^\mu\, \partial_\alpha x_\mu + \psi^\mu\gamma^0\gamma^\alpha\partial_\alpha\psi_\mu\right) \tag{4.1}$$

with the constraint that superconformal generators annihilate physical states. Here the ψ^μ are hermitean (Majorana) two-dimensional fermions, that move freely on the world surface, while the γ^α are the two dimensional Dirac matrices, which we take to be:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;;\qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \;;\qquad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Passing again to the coordinates $\varsigma^\pm = \varsigma^0 \pm \varsigma^1$, $\psi^\mu_\pm = (\frac{1\pm\gamma^5}{2})\psi^\mu$ we can rewrite the action (4.1) as follows

$$S = -\frac{1}{4\pi}\int d\varsigma^+\, d\varsigma^- \left\{\partial_+ x^\mu\, \partial_- x_\mu + \frac{1}{2}\, \psi_-^\mu \partial_+ \psi_{-\mu} + \frac{1}{2}\, \psi_+^\mu \partial_- \psi_{+\mu}\right\} \qquad (4.2)$$

which makes it obvious that $\psi_\pm^\mu$ are the right- and left-moving components whose mode expansions are

$$\psi_-^\mu = \sum_n \frac{1}{2}\, \psi_n^\mu \cdot e^{-2in\varsigma^-}$$

$$\psi_+^\mu = \sum_n \frac{1}{2}\, \tilde{\psi}_n^\mu \cdot e^{-2in\varsigma^+}$$

with canonical anticommutation relations

$$\{\psi_n^\mu, \psi_m^\nu\} = \left\{\tilde{\psi}_n^\mu, \tilde{\psi}_m^\nu\right\} = -\eta^{\mu\nu}\, \delta_{n+m,0} \quad .$$

The superanalytic transformations that leave the action (4.2) invariant (up to total divergences) are:

$$\begin{aligned} \delta\varsigma^- &= f(\varsigma^-) \\ \delta\psi_-^\mu &= -\frac{1}{2}\,\frac{df}{d\varsigma^-}\cdot\psi_-^\mu \end{aligned} \qquad (4.3)$$

and

$$\begin{aligned} \delta x^\mu &= -\frac{1}{2}\,\epsilon(\varsigma^-)\,\psi_-^\mu \\ \delta\psi_-^\mu &= \epsilon(\varsigma^-)\,\partial_- x^\mu \end{aligned} \qquad (4.4)$$

where ϵ is an anticommuting Grassman function of the variable ς^- alone. There are of course corresponding antianalytic transformaions of the right-moving fields, which we will not write explicitly. Note incidentally that unlike the scalar fields x^μ, the fermionic fields ψ^μ transform non-trivially under the conformal transformations (4.3).

The superconformal generators are:

$$L_N = -\frac{1}{2}\sum :a_{N-n}^\mu\, a_{n\mu}: -\frac{1}{2} :\sum n\cdot\psi_{N-n}^\mu\, \psi_{n\mu}: -\alpha_0\cdot\delta_{N,0}$$

$$G_M = \frac{1}{2}\sum a_{M-m}^\mu\, \psi_{m\mu}$$

and the generalized Virasoro conditions become

$$L_N\ |\text{phys}\rangle = G_M\ |\text{phys}\rangle = 0 \qquad (N, M \geq 0) \qquad (4.5)$$

The subtraction constant α_0, which gives the mass of the string's unexcited ground state, depends on the boundary conditions we choose for the fermionic fields. It is consistent with their hermiticity, to choose them either all antiperiodic, as they wind once around the string, or all periodic. These two choices are called the Neveu-Schwarz[17] and Ramond[16] boundary conditions, respectively, and we will here consider them both. Note, incidentally, that other (twisted) boundary conditions are also permitted if one allows the global $SO(1, D-1)$ symmetry to be explicitly broken.[18]

For antiperiodic fermions we find:

$$\alpha_0^{NS} = \lim_{\beta \to 0} (D-2) \left(\sum_0^\infty \frac{1}{2}\, n e^{-\beta n} - \sum_0^\infty \frac{1}{2} \left(n + \frac{1}{2} \right) e^{-\beta(n+\frac{1}{2})} \right) = \frac{D-2}{16}$$

while for periodic fermions:

$$\alpha_0^R = \sum \frac{1}{2}\, n - \sum \frac{1}{2}\, n = 0$$

as expected, since the zero-point oscillations cancel in a supersymmetric theory between bosons and fermions. The critical dimension can be derived in precisely the same way as in the bosonic string, *i.e.* by demanding all negative norm states to disappear from the physical Hilbert space at the second massive level. One finds $D = 10$, so that the Neveu-Schwarz ground state is a tachyon with mass $\frac{1}{8}M^2 = -\frac{1}{2}$, while the first excited state $\epsilon_\mu \psi^\mu_{-1/2} \left|0, p\right\rangle_{NS}$ (with $p \cdot \epsilon = 0$) is a massless vector boson in the open string case, and a graviton, an antisymmetric tensor and a scalar, when combined with the corresponding right-mover, in the closed string case.

For Ramond boundary conditions, the ground state of the string must be degenerate, since the fermionic zero modes ψ_0^μ, which do not change the mass of a state, must act non-trivially on it. In other words, the Ramond ground state must be a representation of the Clifford algebra of the zero modes:

$$\left\{ \psi_0^\mu, \psi_0^\nu \right\} = -n^{\mu\nu}$$

i.e. a D-dimensional Majorana spinor, which we denote by $\left|\alpha, p\right\rangle_R$.

It turns out that a consistent theory of *interacting* strings can be constructed by putting together both the Neveu-Schwarz and Ramond Hilbert spaces, and then projecting out the states of even two-dimensional fermion number

$$(-)^F \; \left|\text{phys}\right\rangle = + \; \left|\text{phys}\right\rangle \tag{4.6}$$

This is called the GSO[19] projection. In the Neveu-Schwarz sector it amounts to keeping only states with an odd* number of fermionic oscillators, for which $\frac{1}{8}M^2$ is

★ For reasons that can be explained only by introducing the Polyakov ghosts, the Neveu-Schwarz ground state of a superstring has *odd* world-sheet fermion number.[15]

an integer; in particular, the tachyon has been eliminated. In the Ramond sector, $(-)^F$ must anticommute with all zero modes ψ_0^μ, and is therefore proportional to the space-time chirality operator, so that (4.6) is a chiral projection.

Hence, in the end, the massless right-moving states of the superstring are a 10-dimensional vector $\psi^\mu_{-1/2}\,|0,p\rangle_{NS}$ and a Weyl-Majorana spinor $(1-(-)^F)\,|\alpha,p\rangle_R$; both have 8 physical degrees of freedom, so that the spectrum is space-time supersymmetric! For closed strings we should in fact combine the identical left and right sectors; what we obtain is the spectrum of $N=2$ ten-dimensional supergravity.[7] This emergence of space-time supersymmetry is one of the beautiful features of string theories. It is actually an open and very interesting problem, to find a formulation in which both Lorentz invariance and space-time supersymmetry are mainfest.[20,21]

Though mathematically beautiful, the closed superstring theory discussed above has little chance of describing the known low-energy phenomenology, since all its massless particles are in the supermultiplet of the graviton. It would be easier to imagine interesting compactifications, if the theory already contained chiral fermions and a gauge group in ten dimensions. This can be achieved by putting appropriate sources at the endpoints of open superstrings which, as shown by Green and Schwarz,[1] can be consistently done only for the gauge group $SO(32)$. An elegant alternative is the heterotic ($N=1/2$) superstring,[9] whose construction is based on the observation that the left- and right-moving modes of a closed string do not interact and can, therefore, be treated asymmetrically. Consider then a hybrid theory, whose left-moving sector contains the excitations $\partial_- x^\mu$ and ψ^μ_- of the ten-dimensional superstring, while its non-supersymmetric right moving sector contains in addition to the bosonic partners $\partial_+ x^\mu$, 32 extra *positive-metric* fermions χ^a_+ $(a=1,\ldots,32)$. The number 32 can be explained if one notes that 32 free fermions are equivalent, in two dimensions, to 16 free bosons, so that the right sector contains the excitations of the consistent 26-dimensional bosonic string.

Consistency of the interacting theory can, again be achieved by taking both Neveu-Schwarz and Ramond boundary conditions and then making a GSO projection onto *even* fermion number. This can in fact be done separately, for different groups of the χ-fermions. If N_p of those fermions are periodic, and all the rest antiperiodic, the subtraction constant α_0 is

$$\alpha_0=\frac{(D-2)}{24}-\frac{N_p}{24}+\frac{(32-N_p)}{48}=1-\frac{N_p}{16}$$

and hence the masses of right-moving states are given by $\frac{1}{8}M^2=-(1-\frac{N_p}{16})+$ integer. These can only match the masses of the left-movers of the superstring ($\frac{1}{8}M^2=$ integer) if $N_p=0$, 16 or 32, which leaves us with two options:

(a) Put all 32 fermions in the same group. The massless right-movers then come entirely from the antiperiodic sector. They are $\tilde{a}^\mu_{-1}\,|0,p\rangle_{NS}$ and

$\chi^a_{-1/2}\chi^b_{-1/2}\ |0,p\rangle_{NS}$. The latter transform in the adjoint of $SO(32)$, so that when combined with the superstring left-movers they yield the spectrum of an $SO(32)$ super-Yang-Mills, coupled to $N=1$ supergravity.

(b) Separate the fermions into two groups of 16, say χ^i and χ'^i. Apart from $\tilde{a}^\mu_{-1}\ |0,p\rangle_{NS}$, the massless right-movers and their transformation properties under the symmetry group $SO(16)\times SO(16)$ are:

$$\chi^i_{-1/2}\ \chi^j_{-1/2}\ |0,p\rangle_{NS-NS}\ :\ (\underline{120},\ 1)$$

$$\chi'^i_{-1/2}\ \chi'^j_{-1/2}\ |0,p\rangle_{NS-NS}\ :\ (1,\ \underline{120})$$

$$|\alpha,p\rangle_{R-NS}\ :\ (\underline{128},\ 1)$$

$$|\alpha',p\rangle_{NS-R}\ :\ (1,\ \underline{128})$$

where $|\alpha,p\rangle_{R-NS}$ is the ground state for periodic χ^i and antiperiodic χ'^j, and thus transforms as a (Weyl) spinor of $SO(16)$, and similarly for $|\alpha',p\rangle_{NS-R}$. Now $\underline{120}+\underline{128}$ can be combined to form the adjoint of the exceptional group E_8. The spectrum of this heterotic superstring is therefore that of $N=1$ supergravity coupled to $E_8\times E_8$ super-Yang-Mills in ten dimensions. This model is the starting point for several, phenomenologically viable, compactifications.[3]

We will here quit, by suggesting an instructive exercise. One can consider complex string coordinates x^μ, and 2-dimensional Dirac-fermionic partners ψ^μ. The action (4.1) then has an $N=2$ world-sheet supersymmetry which can be used to eliminate all negative-metric states.[22] The reader could constructively review the material we have discussed here, by deriving the generalized Virasoro conditions, the critical dimension and the spectrum of this theory.

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