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Cosmic Texture*

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ABSTRACT

The notions of phase transitions and causality, combined with the standard cosmological model, lead to the appearance of topological defects in the early universe. The most familiar types of defects are solitons, strings and domain walls. Another type - *textures* - can exist when the spatial universe is compact. When these appear the whole universe takes on a winding number, and the consequences are quite amusing.

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1. Prologue

Topological defects – the solitons, vortices and domain walls – have a vast literature concerning them.[1] Here we will look at another variety – textures. These are stable, non-localized solutions to the classical equations of motion on a spatial manifold with compact dimensions.

In this letter I will always be considering homogeneous spatial manifolds with the topology of a three sphere, the closed Robertson-Walker spaces. In the language of homotopy theory, textures may appear when there is a symmetry breaking $G \to H$ and $\pi_3(G/H) \neq 1$; that is, when there are topologically non-trivial mappings from the three sphere into the manifold of degenerate vacua.[2]

In the next section we will see that a field configuration with a non-zero π_3 winding number need not be a texture: because textures are not topologically stable. Nonetheless, there exists a texture with winding number one which is stabilized by curvature.

Next, I look at the effect of a texture on the cosmological solution to Einstein's equations. I reach the bizarre conclusion that spatially compact universes with global textures can be indistinguishable from open or flat universes without textures. Thus our universe, which appears to be open, but nearly flat, could very well be tightly bound up. Even when this effect does not occur, the presence of a global texture leads to a size to the universe which is different from the standard picture.

Last, the case of gauge textures is discussed. I consider two examples. In the first there is no residual gauge symmetry and the texture is non-physical in that it has no associated energy density. In the second example there is an unbroken subgroup of the gauge symmetry. The non-abelian magnetic field of this residual symmetry has a vacuum expectation value at all points in the universe.

2. Textures From Global Symmetries

The simplest model in which textures may appear in a Robertson-Walker universe is the breaking of a global symmetry $O(4) \to O(3)$. Then the possible textures are classified by $\pi_3(O(4)/O(3)) = Z$. We take a single scalar fourplet $\vec{\phi}$ with Lagrangian

$$L = \int [\partial^\mu \vec{\phi} \cdot \partial_\mu \vec{\phi} + \lambda(\vec{\phi} \cdot \vec{\phi} - v^2)^2] \sqrt{-g} d^4x.$$

To begin we must choose coordinates. For a closed Robertson-Walker universe the appropriate spatial coordinates are periodic, so we use angular variables ξ, θ, φ for the spacelike coordinates and t for the cosmic time parameter. The metric which is homogeneous and isotropic is

$$d\tau = dt^2 - a^2(t)[d\xi^2 + \sin^2\xi(d\theta^2 + \sin^2\theta d\varphi^2)] \qquad \begin{pmatrix} 0 \le \xi \le \pi \\ 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{pmatrix}.$$

Let us suppose that the universe cools and goes through a phase transition, and that $\vec{\phi}$ gets a vacuum expectation value (VEV). The true vacuum is then described by a constant vector of length v, which by choosing a gauge can be written

$$\vec{\phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}.$$

In contrast, a texture exists when this vector maintains its length but varies from point to point in space. The texture with winding number one is

$$\vec{\phi} = v \begin{pmatrix} \cos\varphi \sin\theta \sin\xi \\ \sin\varphi \sin\theta \sin\xi \\ \cos\theta \sin\xi \\ \cos\xi \end{pmatrix}, \qquad (2.1)$$

which can be pictured as a vector normal to a 3-sphere embedded in a 4 dimensional Euclidean space.

The first thing to notice is that the texture-solution is not topologically stable. We can smoothly deform $\vec{\phi}$, without ever taking it out of the minimum of the potential, making it constant everywhere on the three sphere except for a region that is essentially flat. The energy of this configuration is

$$E = \frac{1}{2}\int \partial^k\vec{\phi}\cdot\partial_k\vec{\phi}d^3x, \qquad k = x_1, x_2, x_3$$

where we have switched to a flat coordinate system, appropriate to the flatness of the region where the gradient terms are non-zero. A simple scaling $x \rightarrow \alpha x$ gives $E \rightarrow \alpha E$, which means that the configuration wants to shrink. In this model there is nothing to prevent its collapse.[3] ⋆

However, the configuration (2.1) will not collapse because it is stabilized by curvature. Suppose that

$$\vec{\phi} = v\begin{pmatrix} \cos\varphi\sin\theta\sin[\xi/\alpha] \\ \sin\varphi\sin\theta\sin[\xi/\alpha] \\ \cos\theta\sin[\xi/\alpha] \\ \cos[\xi/\alpha] \end{pmatrix} \quad \text{for} \quad 0 < \xi \leq \pi\alpha$$

$$\vec{\phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix} \quad \pi\alpha < \xi \leq \pi. \tag{2.2}$$

This has winding number one and smoothly interpolates between (2.1) and total

⋆ What happens to this topological configuration? The probability for tunneling to undo the wind in a texture gets larger as the volume that contains it gets smaller. The texture will shrink until it vanishes by quantum tunnelling.

collapse as α ranges from 1 to 0. The energy is

$$E = 4\pi v^2 a \int_0^{\pi\alpha} \left[\frac{1}{\alpha^2} + 2\frac{\sin^2[\xi/\alpha]}{\sin^2 \xi}\right] \sin^2 \xi d\xi = 2\pi v^2 a \left[\frac{\pi}{2\alpha} - \frac{1}{4\alpha^2} \sin 2\pi\alpha + \pi\alpha\right].$$

There are two interesting limits to this formula. First, as $\alpha \to 0$, $E \propto \alpha$, which recovers the argument just made for topological instability. The second limit is $\alpha \to 1$, where $E \propto [1 + (1-\alpha)^2 + ...]$. Since this energy decreases as $\alpha \to 1$ and as $\alpha \to 0$ there must be an energy barrier between the two extremes, and there will be a stable minimum at $\alpha = 1$.

What about textures with higher winding numbers? These will not be stable by the following argument. An n-texture spread out over the three-sphere is really a superposition of n individual 1-textures, each confined to a different patch of the universe. The preceeding remarks showed that if these patches are small enough then the individual 1-textures are unstable to collapse. Since the patch size goes down as n goes up, there will be some maximum value of n for which the n-texture is stable. I beleive but will not attempt to prove that this maximum value is 1.

3. Global Textures in Cosmology

The configuration (2.1) has a unique stress-energy tensor:

$$(T)^{\mu}_{\nu} = \frac{v^2}{2a^2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (3.1)$$

This may appear familiar. But unlike the stress energy tensor for massless radiation, which is traceless and has $\rho = 3p$, in this case $\rho = -3p$; the pressure is negative; and $T^{\mu}_{\mu} = 2\rho$.

Einstein's field equations and the Cosmological Principle yield

$$\left[\frac{\dot{a}}{a}\right]^2 - \frac{8\pi G\rho}{3} = -\frac{k}{a^2} \qquad k = \begin{pmatrix} 1 & \Rightarrow & \text{Closed} \\ 0 & \Rightarrow & \text{Flat} \\ -1 & \Rightarrow & \text{Open} \end{pmatrix}, \tag{3.2}$$

where G is Newton's constant and k is the trichotomic constant determining the global property of the spatial universe: For our purposes we must have k=1. ρ is the total energy density, consisting of radiation or matter, in or out of thermal equilibrium, and, as we have seen, possibly a component due to the winding of $\langle\varphi\rangle$ around the universe. Note that the cosmological constant is assumed to be zero here and in the following.

Early in the Hot Bang curvature effects were negligible, every degree of freedom was relativistic, and the energy density redshifted as $\sim a^{-4}$. Usually we say that near a phase transition some fraction of this energy density will freeze out as matter and begin to redshift as $\sim a^{-3}$, eventually coming to dominate. Now let us suppose that a texture freezes out at some temperature as well. Thus, before the phase transition ρ redshifts as a^{-4} and a^{-3}, and afterward it also has a part that redshifts as a^{-2}. Einstein's equation becomes

$$\left[\frac{\dot{a}}{a}\right]^2 - \frac{8\pi G}{3}[\rho_r + \rho_m + \rho_w] = -\frac{1}{a^2},$$

and it is clear that the universe may become texture dominated.

This equation can be rearranged as

$$\left[\frac{\dot{a}}{a}\right]^2 - \frac{8\pi G}{3}[\rho_r + \rho_m] = \frac{\gamma - 1}{a^2},$$

where I have made the definition $\gamma \equiv \frac{\pi v^2 G}{3}$. The scale factor can be renormalized by $[a/\sqrt{|\gamma - 1|}] \equiv \tilde{a}$, to obtain

$$\left[\frac{\dot{\tilde{a}}}{\tilde{a}}\right]^2 - \frac{8\pi G}{3}[\rho_r + \rho_m] = -\frac{\tilde{k}}{\tilde{a}^2} \qquad \tilde{k} = \begin{pmatrix} 1 & \text{if } \gamma < 1 \\ 0 & \text{if } \gamma = 1 \\ -1 & \text{if } \gamma > 1 \end{pmatrix}.$$

Thus we come to the strange conclusion: If $\tilde{k} = 1$ then the universe is really smaller than it would appear. If $\tilde{k} = 0$ or -1 the spatially closed universe mimics the flat or open universes!

Since the universe appears to be matter dominated,or possibly just now becoming curvaturedominated, we should have

$$\frac{|\gamma - 1|}{a^2} \lesssim \frac{8\pi G}{3}\rho_m$$

at the present epoch. If $|\gamma - 1| \ll 1$ then the physical radius of the universe could be much less than previously thought. This could occur only if the symmetry breaking scale is near the Planck mass, admittedly a far-fetched proposition, but perhaps not-too-distantly-fetched in the speculative light of recent superunified models.

4. Gauge Textures

The global symmetry of the previous example may be promoted to a gauge symmetry. For the purposes of illustration, the model is gauged in two different ways.

4.1 SU(2)

First, we can define a spinor

$$\phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$$

and write the Lagrangian, without changing its content, as

$$L = \int [\partial^\mu \phi^\dagger \partial_\mu \phi + \lambda(\phi^\dagger \phi - v^2)^2]\sqrt{-g}d^4x.$$

This form suggests the gauging of $SU(2)$, which we do here, by making the replacement

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu - iA_\mu.$$

Now, when ϕ takes on a VEV that winds around the 3-sphere, we find that the energy is minimized when A_μ also gets a VEV. Let

$$\phi = g(\xi, \theta, \varphi) \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad g \in SU(2).$$

Then for

$$\langle A_\mu \rangle = g^{-1}\partial_\mu g$$

we find that

$$\mathcal{D}_\mu \phi = 0.$$

Furthermore, since $\langle A_\mu \rangle$ is pure guage, $F_{\mu\nu} = 0$ and, surprisingly, the texture has zero energy: it is *non – physical*.

An alternative way to see this is to suppose we start with a situation where $\langle \phi \rangle$ is wound up as before and $\langle A_\mu \rangle$=0. We can have $\langle A_\mu \rangle \to g\partial_\mu g$ and the texture will vanish, but since g has a winding number this can only happen by an instanton-like event. Since this instanton occurs all over the 3-sphere I call it a cosmological instanton. Thus a texture may decay through a cosmological instanton. This effect is quite similar to proton decay through weak instantons in the Skyrme model, but since the texture is formed in a phase transition it is most likely that A_μ will immediately take on a VEV such that $\mathcal{D}_\mu\phi = 0$, and a *physical* texture does not form at all.

4.2 The Cosmic Magnetic Background

In the second way we will gauge the entire $O(4)$ symmetry. In this case there will be an unbroken O(3) gauge symmetry. We make the following definitions

$$T^1 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T^2 \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad T^3 \equiv \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U^1 \equiv \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad U^2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad U^3 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which have the properties

$$[T^i, T^j] = \epsilon^{ijk} T^k \quad [U^i, T^j] = \epsilon^{ijk} U^k \quad [U^i, U^j] = \epsilon^{ijk} T^k. \tag{4.1}$$

Also,

$$\hat{\varphi} \equiv \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \\ 0 \end{pmatrix} \qquad \hat{\theta} \equiv \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$$

$$\hat{\xi} \equiv \begin{pmatrix} \cos\varphi \sin\theta \cos\xi \\ \sin\varphi \sin\theta \cos\xi \\ \cos\theta \cos\xi \\ -\sin\xi \end{pmatrix} \qquad \hat{r} \equiv \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

One more set of useful definitions is,

$$\begin{pmatrix} \mathcal{T}^{\varphi} \\ \mathcal{T}^{\theta} \\ \mathcal{T}^{\xi} \end{pmatrix} \equiv \begin{pmatrix} \hat{\varphi} \cdot \vec{T} \cos\xi - \hat{\theta} \cdot \vec{U} \sin\xi \\ \hat{\theta} \cdot \vec{T} \cos\xi + \hat{\varphi} \cdot \vec{U} \sin\xi \\ \hat{r} \cdot \vec{T} \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{U}^{\varphi} \\ \mathcal{U}^{\theta} \\ \mathcal{U}^{\xi} \end{pmatrix} \equiv \begin{pmatrix} -\hat{\theta} \cdot \vec{T} \sin\xi + \hat{\varphi} \cdot \vec{U} \cos\xi \\ \hat{\varphi} \cdot \vec{T} \sin\xi + \hat{\theta} \cdot \vec{U} \cos\xi \\ \hat{r} \cdot \vec{U} \end{pmatrix}.$$

Equipped with this machinery, the reader may verify that $\mathcal{T}^{i}$ all annihilate $\vec{\phi}$, and are therefore generators of the unbroken O(3) symmetry group; that $\mathcal{T}_i$ and $\mathcal{U}_i$ satisfy the algebra (4.1); and that the equation

$$\mathcal{D}_{\mu}\vec{\phi} = [\partial_{\mu} - A_{\mu}]\vec{\phi} = 0,$$

which gives the vacuum expectation of A_{μ}, is satisfied by

$$A_{\varphi} = \frac{1}{a}\mathcal{U}^{\varphi} \sin\xi \sin\theta \quad A_{\theta} = \frac{1}{a}\mathcal{U}^{\theta} \sin\xi \quad A_{\xi} = \frac{1}{a}\mathcal{U}^{\xi}.$$

One may now calculate the vacuum field strength in the coordinates, $(t, \varphi, \theta, \xi)$,

$$F_{\mu\nu} = \frac{v^2}{a^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{T}^{\xi} \sin^2\xi \sin\theta & \mathcal{T}^{\theta} \sin\xi \sin\theta \\ 0 & \mathcal{T}^{\xi} \sin^2\xi \sin\theta & 0 & -\mathcal{T}^{\varphi} \sin\xi \\ 0 & -\mathcal{T}^{\theta} \sin\xi \sin\theta & \mathcal{T}^{\varphi} \sin\xi & 0 \end{pmatrix},$$

which is a non-abelian magnetic field existing everywhere in the universe,

$$\vec{\mathcal{B}} = -\frac{v^2}{a^2}(\mathcal{T}^{\varphi} \sin\xi \sin\theta, \mathcal{T}^{\theta} \sin\xi, \mathcal{T}^{\xi}).$$

Such a VEV of course beaks Lorentz invariance, but a Lorentz transformation followed by a gauge rotation leaves the vacuum invariant. If we define $\mathcal{D}^{\mu}_{\mathcal{T}}$ as the derivative, covariant with respect to only the unbroken O(3) subgroup, then, since $\langle A^{\mu}_{\mathcal{T}} \rangle = 0$,

$$\vec{\mathcal{D}}_{\mathcal{T}} \cdot \vec{\mathcal{B}} = \vec{\partial} \cdot \vec{\mathcal{B}} = 0,$$

and

$$\vec{\mathcal{D}}_{\mathcal{T}} \times \vec{\mathcal{B}} = \vec{\partial} \times \vec{\mathcal{B}} = \frac{2v^2}{a^2}(\mathcal{U}^{\varphi} \sin\xi \sin\theta, \mathcal{U}^{\theta} \sin\xi, \mathcal{U}^{\xi}).$$

So at energies below the symmetry-breaking the effect of the texture is to give a non-abelian electric current to the vacuum, which is the source of $\vec{\mathcal{B}}$.

The stress energy tensor is

$$\mathrm{Tr}[F_{\mu\nu}F^{\lambda\nu} - \frac{1}{4}\delta^{\nu}_{\mu}F^2] = \frac{1}{4g^2a^4}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where g is the gauge coupling constant. This is to be contrasted with (3.1). No longer do we have the effect described in section 3. Gauging the texture results in an energy density which redshifts as a^{-4}, like radiation.

5. Epilogue

Though I have no prejudice about the possibility of incorporating these ideas into a truly realistic cosmological model, there are several interesting questions to be answered. The first, of course, is to what extent can the phenomena described here be ruled out by astrophysical observations? Others are, what happens to the cosmic magnetic background when confinement sets in? What could the role of fermions be in these models? Finally, how can the generalization of these effects inform model building with higher compactified dimensions?

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