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INDUCED QUANTUM NUMBERS IN 2+1 DIMENSIONAL QED*

RICHARD BLANKENBECLER AND DANIEL BOYANOVSKY

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

ABSTRACT

In this paper the vacuum polarization effects of fermions interacting with Abelian gauge fields are studied in 2+1 dimensions. We find that for a gauge field configuration with magnetic flux $2\pi F$, there is induced charge Q, spin S and angular momentum J given by $Q = -\frac{1}{2}\frac{m}{|m|}F$, $S = \frac{F}{4}$, $J = -\frac{1}{4}\frac{m}{|m|}F^2$ (m = fermion mass). A simple argument is offered to explain the physics of these quantum numbers. Some subtleties associated with the induced spin and angular momentum are explained in detail. The induced spin is shown to be related to the 1+1 dimensional chiral anomaly.

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1. Introduction, Motivation and a Glimpse at the Physics

By now it is well established that gauge theories in (2+1) dimensions offer a variety of very interesting phenomena. One important aspect is the possibility of giving a "topological" mass to the gauge bosons without spoiling gauge invariance.[1,2] It was realized that in an interacting theory of fermions and gauge fields, the fermionic vacuum polarization effects in the presence of the gauge field give rise to unusual induced currents.[3,4] These currents have unexpected parity properties and the effective action for the gauge fields obtained from these induced currents contains a "topological" mass for the gauge fields.

This phenomenon occurs in theories with two component fermions and it has been shown that these peculiar induced currents are a result of the unusual spin properties of these theories.[5]

One of the interesting aspects is that magnetic fields (taken as background fields) induce charge in the ground state, and that electric fields induce currents perpendicular to it.

Recently it was realized that in the presence of an external magnetic field with finite flux, a gauge vortex, the fermionic vacuum has an induced angular momentum which is a function of the total flux.[6] This result is very puzzling because the equations of motion for the fermions are rotationally invariant; angular momentum must then be time independent, and since it is an adiabatic invariant, its eigenvalues cannot depend on the (time dependent) flux. This result has been recently challenged by Brown.[7] In this paper we try to explain using simple arguments (which are non-perturbative), the physics of these induced quantum numbers.

We discuss the origin of the vacuum charge and point out the physical effects of the external background field on the fermionic spectrum.

We show that in the presence of the gauge vortex the Dirac sea is distorted to accommodate more (or less) states relative to the free vacuum. It is this change in the number of negative energy states that is responsible for the induced quantum numbers. Now the "vacuum" of the theory is defined so that all these negative energy states are filled, and all the positive energy states are empty. Thus there are induced quantum numbers in this vacuum when the vortex is switched on.

In Section 2 we provide very simple "counting of states" arguments that explain both the physics and the values of the "vacuum" quantum numbers. We compute in a simple manner the charge, angular momentum and spin induced by the external vortex configuration. This section is devoted to a simple and intuitive understanding of these vacuum polarization effects.

Section 3 is a more technical discussion of some subtleties of angular momentum and spin. The physics of the time dependence of the induced angular momentum is discussed. Also the total matter-field angular momentum is discussed with particular attention on the surface effects and boundary currents. Two different derivations of the induced spin are worked out and some subtleties inherent to its definition are clarified. We show that the induced spin is completely determined by the chiral anomaly in the 1+1 Euclidean dimensions.

2. Counting Arguments

In this section we provide some simple arguments that lead to the understanding of the induced quantum numbers. To start with we will review the nature of the Dirac spectrum in the presence of localized gauge vortices. Although this problem has received much attention,[8–10] we will give a brief derivation of the main features that will be of importance for the discussion of the induced quantum numbers under discussion.[11]

Review:

We choose the Dirac algebra to be

$$\gamma^0 = \sigma_3 \quad \gamma^1 = i\sigma_1 \quad \gamma^2 = i\sigma_2 \tag{2.1}$$

in terms of the usual Pauli matrices. Then the Dirac equation with a static background gauge field in the $A_0 = 0$ gauge reads

$$H\psi = E\psi \qquad H = -i\vec{\alpha}\cdot\vec{\nabla} - \vec{\alpha}\cdot\vec{A} + \sigma_3 m \ . \tag{2.2}$$

We write the gauge vortex configuration in the symmetric gauge

$$A_i = -\epsilon_{ij}\,\frac{x_j}{r^2}\,F(r) \qquad \text{with} \quad F(0) = 0 \ , \quad F(\infty) = F \ , \tag{2.3}$$

and the magnetic field and total flux are

$$B(r) = \frac{1}{r}\,\frac{dF(r)}{dr} \qquad\qquad \int B d^2x = 2\pi F \ . \tag{2.4}$$

This choice of the symmetric gauge is in fact very general. Indeed, in the $A_0 = 0$ gauge, there is the freedom of time independent gauge transformations.

Using this freedom, any gauge configuration with definite winding number can be cast into the symmetric gauge by a small gauge transformation. Since $\vec{A}$ in this gauge is rotationally covariant, the Dirac Hamiltonian (Eq. (2.2)) is rotationally invariant and $[H, \widehat{J}] = 0$ where

$$\widehat{J} = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \tfrac{1}{2}\,\sigma_3 \tag{2.5}$$

is the canonical angular momentum. (In any other gauge the symmetry corresponds to a rotation and a gauge transformation.) The eigenstates of H can be written as[8,9]

$$\begin{gathered}\psi(r,\theta) = \frac{1}{\sqrt{r}}\, e^{iJ\theta} \begin{pmatrix} e^{-i\theta/2} & f(r) \\ e^{i\theta/2} & g(r) \end{pmatrix} \\ J = \pm\tfrac{1}{2}\,, \pm\tfrac{3}{2}\,, \ldots \end{gathered} \tag{2.6}$$

and $f(r)$, $g(r)$ solve the radial equations

$$\begin{aligned}\left[+\frac{\partial}{\partial r} + \frac{1}{r}\,(J - F(r))\right]\; g(r) &= (E-m)\, f(r) \\ \left[-\frac{\partial}{\partial r} + \frac{1}{r}\,(J - F(r))\right]\; f(r) &= (E+m)\, g(r)\end{aligned} \tag{2.7}$$

For $m = 0$, the H in (2.2) is equivalent to the 1+1 dimensional Dirac operator in Euclidean space (hermitian) and the Atiyah-Singer index theorem predicts the existence of "zero modes" of definite chirality.[12,13] In this case 1+1 dimensional chirality is given by σ_3. Hence for gauge configurations with net flux we expect "zero modes" at $E = \pm m$, i.e. threshold states. From Eq. (2.7) it is easy to find the solutions

$$E = m \;:\; \psi \propto \frac{1}{\sqrt{r}} \begin{pmatrix} e^{\int^r [J - F(r')]\,\frac{dr'}{r'}} \\ 0 \end{pmatrix} \tag{2.8a}$$

$$E = -m \ : \ \psi \propto \frac{1}{\sqrt{r}} \begin{pmatrix} 0 \\ e^{-\int^r [J-F(r')] \frac{dr'}{r'}} \end{pmatrix} . \tag{2.8b}$$

Using the behavior of $F(r)$ given in (2.3) we find

$$f(r) \propto \begin{cases} r^J \quad for\ r \to 0 & \text{for } J > 0\ , \quad F \geq 1 \\ r^{(J-F)}\ for\ r \to \infty & J \leq F - \frac{1}{2} \end{cases} \tag{2.9a}$$

$$g(r) \propto \begin{cases} r^{-J} \quad for\ r \to 0 & \text{for } J < 0\ , \quad F \leq -1 \\ r^{-(J-F)}\ for\ r \to \infty & |J| \leq |F| - \frac{1}{2} \end{cases} \tag{2.9b}$$

For simplicity we will assume that F is integer; the general situation will be discussed later. Regularity at the origin requires that $J > 0$ for $f(r)$ and $J < 0$ for $g(r)$. Normalizability requires that $J \leq F - \frac{1}{2}$, $F \geq 1$ for $f(r)$ and $|J| \leq -F - \frac{1}{2}$, $F < 0$ and $|F| \geq 1$ for $g(r)$. The case of equality in the above conditions for J requires comment. When J satisfies the inequalities, the states are bound and normalizable. However, when the equalities are satisfied, these are resonant states; their norm diverges but slower than that of a true continuum state. In a large "box" of radius R the norm of these resonant states is $\sim \ell n\, R$. They are perfectly acceptable "zero modes".

Notice that there are F threshold states, *one in every partial wave.* It is easy to see that there are no bound states with $|E| < |m|$. These threshold bound states deplete both the positive and negative energy continuum -"borrowing" states from both parts of the spectrum. This causes an asymmetry, η in the Dirac spectrum[14]

$$\eta = \int_0^\infty [\rho_v(E) - \rho_v(-E)]\ dE\ , \tag{2.10}$$

where $\rho_v(E)$ is the density of states in the presence of the vortex. Clearly η is odd under charge conjugation. This quantity has to be properly regularized and has been studied by several authors and is given by[15]

$$\eta = \frac{m}{|m|}\, F \tag{2.11}$$

where m is the fermion mass in Eq. (2.2).

The physics is basically the same as in one-dimensional systems,[16] Eq. (2.11) and completeness allow us to understand the change in the spectrum. The threshold states deplete $|F|/2$ states from the positive continuum ($E > |m|$) and $|F|/2$ states from the negative ($E < -|m|$) continuum. Each of these states is an eigenstate of angular momentum J. Hence these threshold states deplete one-half of a state each partial wave J (up to $|J| = |F| - \frac{1}{2}$, see Eq. (2.9)) from both the positive and negative continuum.

A) <u>Vacuum charge:</u>

The "vacuum" with or without the vortex is constructed by filling up all the negative energy states and leaving all the positive energy states empty. The *induced* vacuum charge is defined as

$$\langle Q \rangle_v = \int_{-\infty}^{0} \left[\rho_v(E) - \rho_0(E)\right] dE \tag{2.12}$$

where $\rho_v(E)$ ($\rho_0(E)$) is the density of states in the presence (absence) of the vortex. Clearly equation (2.12) is nothing but the difference in the number of negative energy states between the two situations.

Several cases arise for different signs of F and m.

a) $m > 0,\ F > 0$: the threshold states are given by (2.8a) and (2.9a) with energy $E = m$ (i.e. $E > 0$). These states are empty. Since these states *deplete*

$F/2$ states from the negative continuum (i.e. there are $F/2$ states less with the vortex configuration compared to the free situation) the *induced* charge in Eq. (2.12) becomes

$$\langle Q \rangle_v = -\frac{F}{2} \, . \tag{2.13}$$

b) $m > 0, \; F < 0$: the threshold states are given by Eqs. (2.8b) and (2.9b) with energy $E = -m$ $(E < 0)$. These states are filled. These states *deplete* the $E < -m$ continuum by $|F|/2$ states, but now the threshold states must be accounted for in the charge. Now

$$\langle Q \rangle_v = -\frac{|F|}{2} + |F| = \frac{|F|}{2} = -\frac{F}{2} \, . \tag{2.14}$$

The first term on the right-hand side of Eq. (2.14) $(-|F|/2)$ is the *deficit* of states in the $E < -m$ continuum. The second term $(|F|)$ is the number of (occupied) threshold states.

c) $m < 0$: In this case the energy of the threshold states changes sign. For $F > 0$ they have energy $E = -|m|$ and they are occupied and have to be counted for the charge. For $F < 0$ they have energy $E = |m|$, they are empty and do not contribute to the charge. Then for $m < 0$

$$\langle Q \rangle_v = \frac{F}{2} \, . \tag{2.15}$$

Hence in the general case

$$\langle Q \rangle_v = -\frac{1}{2} \frac{m}{|m|} F = -\frac{1}{2} \frac{m}{|m|} \int \frac{B}{2\pi} \, d^2x \, . \tag{2.16}$$

The result (2.16) has been found by many authors using different techniques. Our analysis offers a simple alternative and will allow us to understand other quantum numbers carried by the "vacuum" in the presence of vortex configurations.

B) Spin:

We will now use the above analysis to compute the *induced spin* in the "vacuum". It has been noticed that spin is peculiar in two space dimensions for two component fermions. Indeed the spin is a pseudoscalar and fermions have only *one spin projection* along the missing z-direction:[2,5]

$$S = \frac{1}{2}\,\frac{m}{|m|}\,\frac{E}{|E|}\;. \tag{2.17}$$

The $E/|E|$ in Eq. (2.17) distinguishes the positive and negative energy states. It has been shown that it is this property of spin that is the one responsible for the parity anomalous induced currents in these theories. Of course spin is not a good quantum number and it is only defined in the rest frame.

However Eq. (2.17) indicates that in the free theory, the spin in the vacuum is infinite since all the $E < 0$ states are filled and have spin $\left(-\frac{1}{2}\,\frac{m}{|m|}\right)$.

In the presence of the vortex configuration these spins tend to be aligned parallel to the (localized) magnetic field. The deficit (or excess) of states in the positive and negative energy part of the spectrum arises from a process that involves a spin-flip transition. This spin-flip changes the sign of the energy of the state, thereby producing an asymmetry in the spectrum.

The vortex removes (or adds) states from (to) the Dirac sea ($E \leq -|m|$) the total change in the number of $E < 0$ states is given by the *induced* charge in Eq. (2.16). Every one of these states has spin (Eq. (2.17))

$$S = -\frac{1}{2}\,\frac{m}{|m|}\;.$$

Therefore the total change in the induced spin in the presence of the vortex is

$$\langle S \rangle_v = \left(-\frac{1}{2}\frac{m}{|m|}\right)\left(-\frac{1}{2}\frac{m}{|m|}F\right) = \frac{F}{4} \tag{2.18a}$$

or

$$\langle S \rangle_v = \langle Q \rangle_v \left(-\frac{1}{2}\frac{m}{|m|}\right) . \tag{2.18b}$$

The induced spin given by (2.18) is a pseudoscalar but is *odd* under charge conjugation. This last property may be surprising since under charge conjugation an electron at rest with spin up becomes a positron at rest with spin up. However the expression (2.18) is the spin *induced* in the "vacuum". In the presence of the vortex the vacuum is clearly *not* an eigenstate of charge conjugation because there is a net vacuum charge. Again, it is the asymmetry between the positive and negative parts of the spectrum that is responsible for this phenomena.

Indeed charge conjugation in the free theory implies that the spectrum *is* symmetric. Changing the sign of the flux (F) leads to the opposite spin polarization. The asymmetry changes sign (see Eq. (2.11)) and so does the *induced* spin.

If the reader feels uneasy about these arguments on the physics of these polarization effects, we offer in Section 3 two alternative derivations of the above result along with a more thorough formal treatment of the subtleties involved, and the relationship of the induced spin to the chiral anomalies in 1+1 dimensions.

C) Angular Momentum:

In the symmetric gauge (Eq. (2.3)), the time independence of the angular momentum $\hat{J}$ defined in Eq. (2.5) is a consequence of the rotational invariance of the Dirac Hamiltonian (2.2), i.e. $[H, \hat{J}] = 0$. Notice that because of this property, $\hat{J}$ must be time independent even when $F(r)$ in (2.3) depends on time.

If we imagine an adiabatic process in which a vortex is switched on, the total angular momentum of the system must remain the same, since H is rotationally invariant at all times.

In a recent paper Paranjape[6] has found that there is a non-trivial *induced* angular momentum contrary to the naive expectation, hence we feel it is of interest to understand the physical origin as well as the value of this induced angular momentum. We will again make use of the counting arguments developed earlier in this section.

The principal ingredients are the F (integer) threshold states given by eqs. (2.8) and (2.9). Recall that there is one threshold state in every partial wave up to $|J| = |J_F|$ ($|J_F| = |F| - \frac{1}{2}$ see Eq. (2.19)).

There is a depletion of $|F|/2$ states in the $E < -|m|$ continuum. This is achieved by depleting $\frac{1}{2}$ of a state in every partial wave up to $|J_F|$. This is true due to the conservation of J. As $F(r)$ is adiabatically switched on, the energy levels move and produce the asymmetry in the spectrum. For each state, J remains constant and as these states leave the $E < 0$ region of the spectrum they carry angular momentum. Repeating the arguments elaborated upon for the vacuum charge (cases a and b above) we find

a) $m > 0,\ F > 0$: the threshold states ((2.8a), (2.9a)) have energy $E = m$ and they are empty. There is a deficit of $\frac{1}{2}$ state per J up to J_F and $J > 0$ then

$$\langle J \rangle_v = -\frac{1}{2} \sum_{J=\frac{1}{2}}^{F-\frac{1}{2}} J = -\frac{F^2}{4} \ . \tag{2.19}$$

b) $m > 0$, $F < 0$: the threshold states (92.8b) and (2.9b)) have energy $E = -m$ and $J < 0$. There is a deficit of $\frac{1}{2}$ state for $J < 0$ $(J \geq J_F)$ from $E < -m$, but the threshold states contribute also

$$\langle J \rangle_v = -\frac{1}{2} \sum_{|J|=\frac{1}{2}}^{|F|-\frac{1}{2}} (-|J|) + \sum_{|J|=\frac{1}{2}}^{|F|-\frac{1}{2}} (-|J|) = -\frac{F^2}{4} \,. \tag{2.20}$$

For the case $m < 0$ the threshold states have the opposite sign of the energy, a similar analysis gives the general result

$$\langle J \rangle_v = -\frac{m}{|m|} \frac{F^2}{4} \,. \tag{2.21}$$

This expression disagrees by an overall factor from that of Ref. 6. In fact, the above result is a consequence of the asymmetry in the spectrum and the time independence of $\widehat{J}$. There is a deficit (or excess) of states in the Dirac sea; these states carry angular momentum and there is then a net angular momentum in the "vacuum" given by (2.21).

Note that if F is time dependent, then the induced angular momentum $\langle J \rangle_v$ is also time dependent (even though it would seem to be formally time independent). As discussed before, the reason behind this phenomenon is that as F varies adiabatically in time, states with definite angular momentum are being lost from (or gained by) the Dirac sea. These states are eigenstates of $\widehat{J}$; hence there is a deficit (or excess) of angular momentum in the "vacuum" with the vortex present.

The following question arises: is there an anomaly in angular momentum; i.e. is angular momentum conserved?

The canonical angular momentum is the volume integral of the $\mu = 0$ component of

$$\mathcal{M}^{\mu ij}(x) = \epsilon^{ij}\overline{\psi}(x)\gamma^{\mu}\widehat{J}\ \psi(x)\ , \tag{2.22}$$

where $\psi(x)$ is the field operator and $\widehat{J}$ is given by Eq. (2.2). Conservation of angular momentum is then given by

$$\partial_{\mu}\mathcal{M}^{\mu ij} = 0\ . \tag{2.23}$$

Hence even when $\langle J\rangle_v$ is time dependent the angular momentum will be conserved if there is a flow of angular momentum current out of a large circle at spatial infinity.

At this stage we recall that the induced vacuum currents are

$$\langle J^{\mu}\rangle_v \propto \epsilon^{\mu\nu\rho}F_{\nu\rho} + \cdots \tag{2.24}$$

which are obviously conserved. For time varying background fields, the induced charge varies in time and there is a current flow at infinity.

An analogous phenomenon occurs for the angular momentum. As F varies, there is a charge accumulation and a flow of current at infinity. The charge accumulation gives rise to the induced angular momentum and current flow at infinity carries the compensating angular momentum. Angular momentum (global) is therefore conserved.

In a two-dimensional (spatial) world, the situation is obscured by the fact that a magnetic field may have a net flux; i.e. there is *no return* flux. Alternatively one can think that the return flux is at spatial infinity.

We postpone until the next section the discussion of the return flux and the subtleties regarding the canonical versus the Belinfante form of the total matter-field angular momentum. In the next section we will analyze the gauge invariance of the result (2.21) and reconcile this with the non-invariance of $\widehat{J}$ as given in Eq. (2.5).

3. Some Formalities

This section is devoted to clarifying certain subtle points and filling in certain details of the arguments used in the preceding section.

A) Total Angular Momentum:

The rotational invariance of the matter-field Lagrangian (QED in 2+1 dimensions) ensures the conservation of the total canonical (Noether) angular momentum

$$\begin{aligned} M_C = &\int d^2x\, \psi^+(x) \left[(\vec{r} \times (\vec{p} - \vec{A})) + \tfrac{1}{2}\, \sigma_3 \right] \psi(x) + \int d^2x\, \vec{r} \times (\vec{E} \times B) \\ &+ \int d^2x\, \partial_i (E^i \epsilon^{k\ell} r_k A_\ell) \ . \end{aligned} \tag{3.1}$$

The last term (a pure surface term) is the integral of a gauge non-invariant quantity. However, since it depends on the fields at spatial infinity, it is invariant under gauge transformations that do not change the flux. Hence M_C is gauge invariant.

The "improved" Belinfante angular momentum is given by[17,18]

$$M_B = M_C - \int d^2x\, \partial_i (E^i \epsilon^{k\ell} r_k A_\ell) \ . \tag{3.2}$$

When the fields fall-off fast enough at infinity M_B and M_C are equivalent. However it has been noticed[18] that for a vortex configuration, the surface terms in (3.1) and (3.2) are non-vanishing.

It is the canonical (Noether) angular momentum M_C that is time independent.[18] Let us now compute $\langle M_C \rangle_v$ when $\vec{A}$ is given by the vortex configuration (2.3).

In the presence of a magnetic field there is a charge induced in the vacuum given by Eqs. (2.15) and (2.16); the induced charge density is

$$\langle\rho(x)\rangle_v = -\frac{1}{2}\,\frac{m}{|m|}\,\frac{B(x)}{2\pi} + \cdots \tag{3.3}$$

where the dots stand for higher derivative terms of the form $(\nabla^{2n}B/m^2)$. We will only study the situation for large fermion mass and neglect the non-local terms in (3.3) since in any case they will integrate to zero in the final expressions.

The induced charge generates an electric field given by

$$\vec{\nabla}\cdot\vec{E}(x) = \langle\rho(x)\rangle_v \quad \text{or} \quad \vec{E}(x) = \int d^2y\,\frac{(\vec{x}-\vec{y})}{2\pi|\vec{x}-\vec{y}|^2}\,\langle\rho(y)\rangle_v\ . \tag{3.4}$$

Using Eqs. (2.3), (2.4) and (3.3), (3.4) and after some simple algebra, we find

$$\langle M_B\rangle = \int d^2x\,\left\langle\psi^+(x)\widehat{J}\psi(x)\right\rangle_v + \frac{1}{2}\,\frac{m}{|m|}\,F^2\ , \tag{3.5}$$

where $\widehat{J}$ is the canonical angular momentum given by Eq. (2.5).

The expectation value of the surface term in (3.1) and (3.2) is given by

$$-\frac{1}{2}\,\frac{m}{|m|}\,F^2\ , \tag{3.6}$$

and hence

$$\langle M_C\rangle_v = \int d^2x\,\left\langle\Psi^+(x)\widehat{J}\psi(x)\right\rangle_v\ . \tag{3.7}$$

Therefore, in the *symmetric gauge*, the canonical, gauge invariant, (M_C) Noether angular momentum for matter plus field is just the expectation value of the canonical, gauge dependent, angular momentum for the matter fields.[19]

This is, in fact, just as one expects; in the symmetric gauge the Dirac Hamiltonian is rotationally invariant and the eigenvalues of $\widehat{J}$ remain constant even when the vortex configuration varies in time.

When the gauge field is switched off, M_C in (3.1) coincides with the matter field canonical angular momentum. Since both $\widehat{J}$ is constant in time *and* M_C in (3.1) is time independent, Eq. (3.7) must hold.

Therefore when F varies in time the time dependence of $\langle J \rangle_v$ in Eq. (2.21) is not due to angular momentum being transferred to the electromagnetic fields. As was explained before, it is a consequence of the second quantization procedure of defining the vacuum as the filled Dirac sea in the presence of the vortex.

The physical interpretation of the surface term becomes clear when we incorporate the effects of the return flux.

In two space dimensions there is no analog of the 3-D Maxwells equation $\vec{\nabla} \cdot \vec{B} = 0$ which forces all the flux lines to close. In fact in two dimensions there can be vortices with no return flux and hence a net flux passes through the plane.

This situation can be envisaged by building an infinitely long solenoid along the z-axis in three dimensions and slicing a plane perpendicular to this axis at $z = 0$. This corresponds to a two dimensional vortex with no return flux. Since the solenoid is infinitely long along z, the return flux is spread out at infinity. However, for a finite solenoid of length 2L along this axis, the magnetic field at $z = 0$ (in the plane) is given by (see Eq. (2.4))

$$B_S(r) = \left[\frac{F'(r)}{r} - \frac{L\, F(\infty)}{(L^2 + r^2)^{3/2}} \right] \; . \tag{3.8}$$

As $L \to \infty$ the second term vanishes. However the integral of this term over the plane is finite and equal to $2\pi F$ independent of L. Indeed

$$\int B_S(r)\, d^2x = 0 \ . \tag{3.9}$$

The second term in (3.8) is the return flux. As $L \to \infty$ this return flux is pushed to infinity, its density goes to zero but its integral is constant (independent of L) and cancels the flux in the solenoid.

The vector potential that gives rise to $B_S(r)$ in Eq. (3.8) in the symmetric gauge *does not* have a long range tail (zero total flux), therefore the surface term in (3.1) vanishes.

Indeed with $B_S(r)$ the *total* induced angular momentum, vacuum charge and spin vanish. There is an accumulation of charge (and spin and $\langle J \rangle_v$) near the solenoid and an accumulation of the opposite Q, S, and J near the return flux. As $L \to \infty$ these quantum numbers near the return flux flow out to infinity. In fact the return flux plays the role of the antivortex.

The surface term in (3.1) takes into account the contribution of the return flux to the electromagnetic field angular momentum.

Therefore we believe this argument illuminates the fact that even if $\langle Q \rangle_v$ and $\langle J \rangle_v$ are time dependent (as F varies) both are conserved once the surface effects are taken into account.

B) Induced Spin:

The argument leading to expressions (2.18,a,b) for the induced spin may not be convincing to the reader. Unlike the charge and angular momentum, the states cannot be labeled by the spin, hence the derivation of (2.18) is at best heuristic. However it contains the correct physical interpretation. We now offer a more formal derivation of the induced spin.

Because there is only one spin polarization, the ground state spin is ill-defined. It is infinite even in the free theory.

The first question we face is what is a good definition of the *induced* spin? One would be tempted to propose

$$\underset{\sim}{S}(x) = \tfrac{1}{4}\sigma_{3\alpha\beta}\left[\psi_{\alpha}^{+}(x)\,,\,\psi_{\beta}(x)\right] \ . \tag{3.10}$$

This definition of spin is charge conjugation even. (The charge conjugate field operator is $\psi_c(A) = -i\gamma^1\gamma^0\overline{\psi}^T(-A)$). However this definition is ill-defined. Expanding the field operators in terms of positive energy (u_n) and negative energy (v_n) solutions of (2.2) we find

$$\left\langle \underset{\sim}{S}(x) \right\rangle = \tfrac{1}{4}\left\{ \sum_{E_n<0} v_n^{+}\sigma_3 v_n - \sum_{E_n>0} u_n^{+}\sigma_3 u_n \right\} \tag{3.11}$$

but because of (2.17) (recall $\sigma_3 = \gamma_0$)

$$\overline{v}_n v_n = -\overline{u}_n u_n \ . \tag{3.12}$$

Therefore $\left\langle \underset{\sim}{S} \right\rangle$ is infinite and not defined. In fact, the definition (3.10) amounts (when integrated in d^2x) to *adding* the number of positive and negative energy states.

From (3.12) we see that a more suitable definition of the *induced* spin is

$$S(x) = \frac{1}{4}\,\sigma_{3\alpha\beta}\,\left\{\psi_{\alpha}^{+}(x)\,,\,\psi_{\beta}(x)\right\} \tag{3.13}$$

which leads to

$$\langle S(x) \rangle = \frac{1}{4}\left\{ \sum_{E_n<0} \overline{v}_n v_n + \sum_{E_n>0} \overline{u}_n u_n \right\} \ . \tag{3.14}$$

Using the anticommutation relation for the fermi fields it achieves the form

$$\langle S(x)\rangle = \frac{1}{4}\,\mathrm{Tr}\left[\sigma_3 \delta^2(0)\right] . \tag{3.15}$$

From Eq. (3.12) we see that the definition (3.14) amounts to *subtracting* the number of positive energy states from the number of negative energy states, i.e. it is related to the spectral asymmetry of Eq. (2.11).

Expression (3.15) seems to indicate that this quantity ($= 0 \times \infty$) and hence is not well defined either, however from (3.14) we find that in the free case it is trivially zero by charge conjugation. Indeed, in the free case, charge conjugation yields $\overline{u}_E u_E = -\overline{v}_{-E} v_{-E}$.

The reader will recognize in (3.15) the chiral anomaly in 1+1 dimensions (Euclidean) where σ_3 is γ_5. Therefore we regulate (3.15) in a gauge invariant way (gauge invariant under time independent gauge transformations) *a la* Fujikawa[20]

$$\langle S\rangle = \frac{1}{4} \lim_{M\to\infty} \mathrm{Tr}\left[\sigma_3 e^{-H^2/M^2} \delta^2(0)\right] , \tag{3.16}$$

where H is the Dirac Hamiltonian in (2.2). The standard result of the $\lim\limits_{M\to\infty}$ $\mathrm{Tr}(\ldots)$ is the two dimensional chiral anomaly and we find

$$\langle S\rangle = \frac{F}{4} \tag{3.17}$$

in agreement with (2.18).

Please notice that the definition of the induced spin and Eqs. (3.13) and (3.17) are *odd* under charge conjugation, but well defined. Indeed the reader can easily find that only a charge conjugation odd definition of the induced spin is well behaved.

We offer yet another method to compute $\langle S \rangle$ using point-splitting. From the definition of the fermion Green's function

$$iS_F(x', x, t' - t) = \langle 0 \,|\, T\psi(x', t')\overline{\psi}(x, t) \,|\, 0 \rangle \ ,$$

which is the usual time ordered product, we find

$$\langle 0 \,|\, \overline{\psi}(x, t)\psi(x, t) \,|\, \rangle = - \lim_{\epsilon_0 \to 0+} \lim_{\vec{\epsilon} \to 0} \langle 0 \,|\, T\psi(x, t)\overline{\psi}(\vec{x} + \vec{\epsilon}, t + \epsilon_0) \,|\, 0 \rangle \ e^{i \int_x^{x+\epsilon} A_\mu dx^\mu}, \tag{3.18}$$

where the exponential of the line integral ensures gauge invariance. After a tedious but straightforward calculation we find

$$\langle 0 \,|\, T\psi(x, t)\, \overline{\psi}(x, t + \epsilon_0) \,|\, 0 \rangle = \frac{m}{2\pi} \frac{1}{|\epsilon_0|} - \frac{1}{4\pi} \epsilon^{oij} \partial_i A_j \frac{\epsilon_0}{|\epsilon_0|} \ . \tag{3.19}$$

Hence

$$\langle 0 \,|\, \overline{\psi}(x, t)\psi(x, t) \,|\, 0 \rangle = \lim_{\epsilon_0 \to 0+} \left[-\frac{m}{2\pi} \frac{1}{|\epsilon_0|} + \frac{1}{4\pi} \epsilon^{ij} \partial_i A_j \frac{\epsilon_0}{|\epsilon_0|} \right] \ . \tag{3.20}$$

The first term on the right-hand side (divergent as $\epsilon_0 \to 0$) is identified as $\langle S \rangle$ in the free theory ($A_\mu = 0$). The second term corresponds then to the induced spin therefore the *induced* spin is (see Eq. (2.4))

$$\langle S \rangle_v = \int d^2x \, \frac{1}{2} \langle 0 \,|\, \overline{\psi}(x, t)\psi(x, t) \,|\, 0 \rangle_{\text{ind}} = \frac{F}{4} \ . \tag{3.21}$$

We hope that these alternative methods were helpful in that the reader may feel more comfortable with the counting arguments of the first section, which we believe illuminate the physics in simple terms.

So far we have only studied the integer F case. This allowed us to use simple counting arguments to understand the induced quantum numbers. However please notice that (3.17) and (3.21) were obtained for any arbitrary F. Also, the vacuum charge (computed by other methods) is given by Eq. (2.16) for arbitrary F. When F is not integer the counting argument is obscure but a detailed analysis[8,11] (or a field-theory calculation) will show that the expressions for $\langle Q \rangle_v$, $\langle S \rangle_v$ and $\langle J \rangle_v$ are indeed general. The continuum develops a singularity in the density of states at threshold involving the fractional part of F; this then contributes to the induced quantum numbers[11] in the same way as the zero modes.

4. Summary

We have studied the fermionic vacuum polarization effects in 2+1 dimensions. In the presence of a background vortex the Dirac sea contains more or less states (depending on the sign of the flux and fermion mass) relative to the ground state in the trivial theory.

These states carry charge, spin and angular momentum. The difference in the number of states in the Dirac sea with and without the external vortex field gives rise to the induced quantum numbers that can have general values.

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