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DILATON COUPLING AND BRST QUANTIZATION OF BOSONIC STRINGS*

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ABSTRACT

BRST quantization of the bosonic string on a flat world sheet in an arbitrary background field is discussed. It is shown that by demanding the nilpotence of the BRST charge we may obtain the equations of motion of all the massless fields in the theory, provided we couple the dilaton field to the divergence of the ghost number current in the σ-model.

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1. Introduction

Among the most remarkable of the miracles that have been discovered in the recent explorations of string theory is the connection between the vanishing of the renormalization group β functions for σ models and the equations of motion for the space time fields which describe the particle excitations of the string.[1,2]

In principle, these space time equations of motion follow from the string field theory Lagrangian. In practice they can be extracted from the tree level S matrix.[3] That they should also be given by requiring the σ model which describes string propagation to be conformally invariant is a wonderful, if poorly understood, bonus. Only by exploiting this connection have nontrivial solutions of string theory been found.[4]

The equation of motion for the dilaton field plays a crucial role in this picture. On the one hand it is necessary if one wants to derive the full set of equations from an action. On the other hand, to obtain it one must include couplings in the σ model Lagrangian which are functions of the intrinsic two dimensional metric.[1,5] These seem rather odd since the metric is not a dynamical variable of the two dimensional field theory.† The reason that the metric appears in the two dimensional Lagrangian is easy to understand. In order to write the equations of motion we must go off shell. If the equations are indeed equivalent to conformal invariance (metric independence in the conformal gauge) then the metric must reappear in the off shell formalism.

A similar phenomenon occurs in the new covariant, gauge invariant free string actions.[6] The off shell description of free strings requires an infinite number of auxiliary fields. While the on shell fields live in a space of functionals of $X^M(\sigma)$, the off shell fields require the reparametrization ghost Hilbert space, $\Phi(X^M, b, c)$, for their description.

† Of course, if we worked in a gauge other than the conformal gauge, all couplings would depend on the two dimensional metric.

Comparison of these two formalisms suggests that there is a formulation of string field theory in which the fundamental field depends on $X^M(\sigma)$ and a metric variable $\chi(\sigma)$.* On the other hand it suggests a treatment of the dilaton equation of motion in terms of a coupling to the ghost fields in the σ model Lagrangian. This is the subject we pursue in the present paper.

We show that by adding to the σ model a coupling between $X^M(\sigma)$ and ghosts, we can obtain the equations of motion for space time fields by imposing the fundamental stress tensor operator product relation of conformal field theory.[7] This was suggested in a more general context by Friedan.[8] It is equivalent to insisting that the BRST charge be nilpotent. Our calculations thus generalize those of Kato and Ogawa.[9]

Throughout the paper we will work on a flat two dimensional surface and to lowest order in σ model perturbation theory. However, before beginning the calculation we present a heuristic argument that shows that our results are equivalent to those of Ref. [1,5]. We are going to calculate the c number anomaly in the operator product of two stress tensors. Friedan has argued that this is related to the trace of the stress tensor in a background two metric.[7] Let us bosonize the ghost fields in terms of a scalar field ϕ. The axial ghost number current is

$$j_\alpha = \partial_\alpha \phi \tag{1.1}$$

In a general two metric this current has an anomaly

$$\partial_\alpha j^\alpha = \Box\, \phi = \tfrac{3}{8}\, \sqrt{g}\; R^{(2)} = \tfrac{3}{8}\, \Box\, \chi \tag{1.2}$$

where the metric in conformal coordinates is $e^{2\chi}\delta_{\mu\nu}$. This anomaly equation is

* This was first suggested in the work of Siegel and Zwiebach,[6] who bosonized the ghost fields. The bosonized ghost field is proportional to the log of the conformal factor of the two metric. This is just the ghost number anomaly equation. The possibility of such a formalism for string fields was discussed by T. Banks and M. Gell-Mann at the Santa Fe Workshop. See also Tseytlin, Ref. 10.

enforced by the ghost Lagrangian

$$\mathcal{L} = - \sqrt{g}\, \frac{g^{\alpha\beta}}{2\pi}\, (\partial_\alpha \phi \partial_\beta \phi) - \frac{3}{8\pi}\, \sqrt{g}\, R^{(2)} \phi \, . \tag{1.3}$$

The dilaton field $\Phi(X(\sigma))$ will be coupled to the current j^α as it is in string field theory[6]

$$\mathcal{L}_{\text{dil}} = \frac{2}{3\pi}\, \partial_\alpha \Phi\, \partial_\beta \phi\, \sqrt{g}\, g^{\alpha\beta} \, . \tag{1.4}$$

Note that this is a renormalizable coupling of the σ model to ϕ. Integrating out ϕ in the conformal gauge, we find the $\sqrt{g}\, R^{(2)} \Phi$ coupling of Ref. 1. We also find a correction term for the σ model kinetic energy

$$G_{MN}\, \frac{\partial X^M}{\partial \sigma^\alpha}\, \frac{\partial X^N}{\partial \sigma^\beta}\, g^{\alpha\beta} \sqrt{g} \to \left(G_{MN} + \frac{32}{9}\, \alpha' \partial_M \Phi \partial_N \Phi \right)\, \frac{\partial X^M}{\partial \sigma^\alpha}\, \frac{\partial X^N}{\partial \sigma^\beta}\, g^{\alpha\beta} \sqrt{g} \, . \tag{1.5}$$

This is a field redefinition for the space time fields, and affects the form of their equations of motion without changing the physical content. Note, however, that the field redefinition for G_{MN} is of two loop order in the σ model perturbation expansion. We will not encounter it in the explicit one loop computations that we present in the next section.

The wary reader may be a bit suspicious of our cavalier derivation of the equivalence of coupling the dilaton to the ghosts or the two metric. We will spend the rest of this paper giving a careful proof of this result through one loop order. In Chapter 2 we present a BRST invariant renormalizable Lagrangian which couples the nonlinear σ model variables X^M to the ghosts via the dilaton field. We argue that a computation of the operator product of two stress tensors in this theory is equivalent to a computation in the σ model with a modified stress tensor*

$$T_{\alpha\beta} \to T_{\alpha\beta} - \frac{1}{2\pi}\, [\partial_\alpha \partial_\beta \Phi(X(\sigma)) - \eta_{\alpha\beta} \partial^2 \Phi] \, . \tag{1.6}$$

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* A similar interpretation of the dilaton field coupling has recently been discussed by Lovelace.[11]

In Chapter 3 we compute the operator products of this modified stress tensor through one loop order in the background field expansion. We show that the usual conformal algebra is satisfied only if the space time fields obey the equations of motion derived in Ref. 1. We argue that this is equivalent to nilpotence of the BRST charge. This result, which is undoubtedly known to the authors of Ref. 7, has not (to our knowledge) been carefully derived in the literature.

Appendix A contains a detailed graph by graph account of the computations described in Chapter 3, while Appendix B gives the details of Chapter 2.

2. Coupling of the Dilaton

The conventional BRST invariant action for the bosonic closed string in arbitrary background space-time metric G_{ij} and antisymmetric tensor field B_{ij} is

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\ [G_{ij}(X)\partial_\alpha X^i\partial^\alpha X^j + \epsilon^{\alpha\beta}B_{ij}(X)\partial_\alpha X^i\partial_\beta X^j + 2b_{++}\partial_- c^+ + 2b_{--}\partial_+ c^-] \tag{2.1}$$

where

$$\sigma^\pm = (\sigma^0 \pm i\sigma^1)/\sqrt{2}$$

and

$$\partial_\pm = (\partial_0 \pm i\partial_1)/\sqrt{2}\ . \tag{2.2}$$

The usual string variables τ and σ are related to $\sigma^\pm$ by $\sigma^\pm = e^{2(\tau\pm i\sigma)}$. Here both the (σ^0,σ^1) and the (τ,σ) coordinates are taken to be Euclidean. b and c are the antighost and the ghost fields respectively, originating from fixing of the reparametrization gauge invariance. In writing Eq. (2.1) we have worked in conformal coordinates and used local conformal invariance to eliminate the conformal factor χ of the two metric. Of course, this is only valid in 26 dimensions and for on shell background fields. Nevertheless, if the action (2.1) is truly conformally invariant in flat two space (with vanishing central charge) then the neglect of χ is justified. The action (2.1) is invariant under BRST transformations

$$\begin{aligned} \delta X^i &= -\lambda c^\alpha \partial_\alpha X^i \\ \delta c^+ &= -\lambda c^+ \partial_+ c^+ \\ \delta b_{++} &= \lambda (G_{ij}\partial_+ X^i \partial_+ X^j + \partial_+ b_{++} c^+ + 2b_{++}\partial_+ c^+) \end{aligned} \tag{2.3}$$

where λ is an anti-commuting parameter. δc^- and δb_{--} are obtained by replacing the $+$'s by $-$'s in the above equations. The associated Noether current is

$$\begin{aligned} J^B_\pm &= c^\pm \left\{ T^x_{\pm\pm} + \tfrac{1}{2} T^g_{\pm\pm} \right\} \\ &= \frac{1}{2\pi\alpha'} \left[c^\pm G_{ij}\partial_\pm X^i \partial_\pm X^j + c^\pm b_{\pm\pm}\partial_\pm c^\pm \right] \ . \end{aligned} \tag{2.4}$$

where T^x and T^g are the contributions to the stress tensor from the matter and the ghost fields respectively. Note that the currents $J^B_\pm$ are separately conserved, i.e. $\partial_- J^B_+ = \partial_+ J^B_- = 0$ if $T^x_{+-} = T^g_{+-} = 0$.

A modified action which is also BRST invariant is

$$\begin{aligned} S = \frac{1}{4\pi\alpha'} \int d^2\sigma \Big[& G_{ij}(X)\partial_\alpha X^i \partial^\alpha X^j \\ & + B_{ij}(X)\epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j + 2b_{++}\partial_- c^+ + 2b_{--}\partial_+ c^- \\ & + \tfrac{8}{3}(\partial_+\Phi(X) b_{--} c^- + \partial_-\Phi(X) b_{++} c^+) \Big] \ . \end{aligned} \tag{2.5}$$

The new BRST transformations

$$\begin{aligned} \delta X^i &= -\lambda e^{\frac{4}{3}\Phi} c^\alpha \partial_\alpha X^i \\ \delta(e^{\frac{4}{3}\Phi} c^+) &= -\lambda (e^{\frac{4}{3}\Phi})^2 c^+ \partial_+ c^+ \\ \delta(e^{-\frac{4}{3}\Phi} b_{++}) &= \lambda (G_{ij}\partial_+ X^i \partial_+ X^j + \partial_+ b_{++} c^+ + 2b_{++}\partial_+ c^+ + \tfrac{4}{3}\partial_+\Phi b_{++} c^+) \end{aligned} \tag{2.6}$$

leave (2.4) invariant. The corresponding BRST current is

$$J^B_+ = \frac{1}{2\pi\alpha'} e^{\frac{4}{3}\Phi}[c^+ G_{ij}\partial_+ X^i \partial_+ X^j - c^+\partial_+ c^+ b_{++}] \ . \tag{2.7}$$

The reader will have noted that Eqs. (2.5-2.7) are obtained from Eqs. (2.1-2.4) by the substitution $c \to e^{\frac{4}{3}\Phi}c$, $b \to e^{-\frac{4}{3}\Phi}b$. Hence one may think that the theory described by Eq. (2.5) is identical to the one described by Eq. (2.1). This is only partially true. In the absence of any external sources, the transformation $c \to e^{\frac{4}{3}\Phi}c$, $b \to e^{-\frac{4}{3}\Phi}b$ is non-anomalous, and hence the action (2.1) may be transformed to (2.4) by this transformation. This transformation, however, is anomalous in the presence of external sources that couple to the ghost stress tensor or the BRST current.

This means that an anomaly shows up in correlation functions with these operators. As a result the BRST current J^B_+ picks up an anomalous contribution as we rotate the ghost fields. The rest of this chapter is devoted to showing that this anomalous contribution is given by

$$-\frac{1}{2\pi} c^+\partial_+\partial_+\Phi(X) \tag{2.8}$$

which is precisely equivalent to modifying the matter stress tensor as in (1.6). Another way of looking at it is that if we start from the total stress tensor $T^x_{++} + T^g_{++}$ in the theory described by (2.4) and make a rotation of c and b to get to the action (2.1), T^g_{++} acquires an anomalous contribution $-\frac{1}{2\pi}\partial_+\partial_+\Phi$ during this rotation.

As we shall see in the next section, the nilpotence of the BRST operator is equivalent to the absence of singular terms in the operator product $J^B_+(\sigma)J^B_+(\sigma')$. Since the anomaly in the transformation $c \to e^{-\frac{4}{3}\Phi}c$, $b \to e^{+\frac{4}{3}\Phi}b$ comes from the ghost loop, we must investigate the ghost loop contribution to $J^B_+(\sigma)\ J^B_+(\sigma')$. Graphs like Fig. 1(a) where the ghost loop does not contain an insertion of J^B_+ do not give any anomalous contribution, since the transformation $c \to e^{-\frac{4}{3}\Phi}c$,

$b \rightarrow e^{+\frac{4}{3}\Phi} b$ is free of anomalies in the absence of an external source coupled to J_+^B (or the stress tensor). Graphs like Fig. 1(b), where the ghost loop has two insertions of J_+^B in it, are also anomaly free since these graphs are finite without any regularization so long as $\sigma \neq \sigma'$. Only graphs of the form shown in Fig. 2, where a ghost loop has one and only one insertion of J_+^B in it, give anomalous contributions. The effect of these graphs may be summarized by integrating out the ghost field to get the effective contribution to J_+^B from the graphs shown in Fig. 3. Of these, graphs with more than two insertions of $\partial_+\Phi$ (e.g. Fig. 3(c)) may be shown to vanish, only the graphs of Fig. 3(a) and (b) remain. The detailed analysis of these graphs is given in Appendix B. Here we just quote the result.

The total contribution from Fig. 3 gives an effective contribution to J_+^B of the form

$$-\frac{1}{2\pi}\, c^+\partial_+\partial_+\Phi \tag{2.9}$$

as given in Eq. (2.8). In the rest of the analysis we may treat the ghost fields as free fields with the action given in (2.1) . Thus the question of conformal invariance is reduced to the computation of operator product expansions in a σ model with modified stress tensor (1.6).

3. Operator Product Expansion

We now proceed to check the BRST (or conformal) invariance of the σ-model described in Sec.2. The Virasoro generators of the string are given by $\oint(\sigma^+)^n T_{++} d\sigma^+$ and $\oint(\sigma^-)^n T_{--} d\sigma^-$ respectively, where $\oint$ denotes integration along a contour around the origin at fixed τ. They generate two independent conformal algebras with central charge c if $T_{++}(= T^x_{++} + T^g_{++})$ (and T_{--}) satisfy the operator product expansion,

$$T_{++}(\sigma)T_{++}(\sigma') = -\frac{1}{2\pi}\left[\frac{c}{(\sigma^+ - \sigma'^+)^4} + \frac{2}{(\sigma^+ - \sigma'^+)^2}T_{++}\left(\frac{\sigma+\sigma'}{2}\right) + \text{finite terms}\right]. \tag{3.1}$$

We will show that we obtain the equations of motion of spacetime fields by demanding that the operator product of the stress tensor has the form (3.1) with vanishing central charge c. This is equivalent to nilpotence of the BRST charge of the BRST quantized two dimensional field theory. The fact that nilpotence of Q_{BRST} follows from the conformal algebra for $L_n^x + L_n^g$ is known to the authors of Ref. 7, but a careful derivation has not appeared in print. In radial quantization nilpotence of Q_{BRST} is equivalent to the requirement that the operator product $J_+^B(\sigma)J_+^B(\sigma')$ is free of singularities as $\sigma \to \sigma'$. Now

$$\begin{aligned} J_+^B(\sigma)J_+^B(\sigma') =& c^+(\sigma)c^+(\sigma')T_{++}^x(\sigma)T_{++}^x(\sigma') \\ &+ \tfrac{1}{2}c^+(\sigma)c^+(\sigma')T_{++}^g(\sigma')T_{++}^x(\sigma) \\ &+ \tfrac{1}{2}c^+(\sigma)T_{++}^g(\sigma)c^+(\sigma')T_{++}^x(\sigma') \\ &+ \tfrac{1}{4}c^+(\sigma)T_{++}^g(\sigma)c^+(\sigma')T_{++}^g(\sigma') \end{aligned} \tag{3.2}$$

If T_{++}^x (T_{++}^g) satisfies the operator product expansion given in Eq.(3.1) with c replaced by c^x (c^g), the singular part of the first term in (3.2) may be evaluated exactly. Since b and c are free fields, the rest of the terms in (3.2) may also be evaluated exactly. The final result is,

$$J_+^B(\sigma)J_+^B(\sigma') = c^+(\sigma)c^+(\sigma')\frac{1}{(\sigma^+ - \sigma'^+)^4}[-\frac{c^x}{2\pi} - \frac{c^g}{2\pi}] \tag{3.3}$$

where $c^g = \frac{13}{2\pi}$. This vanishes if $c = (c^g + c^x)$ vanishes. In particular, it vanishes for D=26 in the free field case. Since the ghosts remain free in the presence of background fields, Q_{BRST} will remain nilpotent as long as the conformal algebra (with no central charge) is preserved. Thus the absence of an anomaly in the conformal algebra may also be interpreted as the criterion for the nilpotence of the BRST charge.

In actual computation of the operator product (3.1) we get extra singular terms on the right hand side of this equation of the form

$$\frac{1}{(\sigma^+ - \sigma'^+)^2} A_{ij}(X)\partial_+ X^i \partial_+ X^j + \frac{\sigma^- - \sigma'^-}{(\sigma^+ - \sigma'^+)^3} C_{ij}(X)\partial_- X^i \partial_+ X^j \tag{3.4}$$

All the operators are evaluated at $(\sigma + \sigma')/2$ in order to maintain the symmetry $\sigma \leftrightarrow \sigma'$. The first term in (3.4) may be removed by redefining the stress tensor* by adding to it an operator proportional to $A_{ij}\partial_+ X^i \partial_+ X^j$. We recalculate the operator product expansion with the new stress tensor. This affects the coefficient c in (3.1). If we now demand that the anomalous terms in the operator product expansion, as well as the coefficient c, vanish, we get three sets of constraints on the background fields

$$C_{ij} + C_{ji} = 0 \tag{3.5}$$

$$C_{ij} - C_{ji} = 0 \tag{3.6}$$

$$c^x = -\frac{13}{2\pi} \tag{3.7}$$

Eqs. (3.5-3.6) are equivalent to the vanishing of the β-functions of the σ model. This follows from the conservation law of the stress tensor.

$$\frac{\partial}{\partial \sigma^-} \langle T_{++}(\sigma) T_{++}(\sigma') \rangle = -\frac{\partial}{\partial \sigma^+} \langle T_{+-}(\sigma) T_{++}(\sigma') \rangle \tag{3.8}$$

and the fact that T_{+-} vanishes when the β-function vanishes. Later we shall

* Normally, a conserved current like the stress tensor is not expected to receive any renormalization counter terms. However, in the presence of background antisymmetric tensor field, dimensional regularization breaks the energy momentum conservation laws, which must be compensated by adding explicit counterterms to the stress tensor.

derive the relation between C_{ij} and the β-functions of the σ model through one loop.

The operator product expansion is most conveniently carried out using background field expansion.[12] If ξ^i denotes normal coordinates in the internal manifold, the background field expansion of various relevant quantities are as follows:

$$\partial_\alpha X^i = \partial_\alpha X^i_B + D_\alpha \xi^i + \left\{ \tfrac{1}{3} R^i_{k_1 k_2 j}(X_B) \xi^{k_1} \xi^{k_2} + O(\xi^3) \right\} \partial_\alpha X^j_B \tag{3.9}$$

$$G_{ij}(X) = G_{ij}(X_B) - \tfrac{1}{3} R_{ik_1 jk_2}(X_B) \xi^{k_1} \xi^{k_2} + O(\xi^3) \tag{3.10}$$

$$\begin{aligned} B_{ij}(X) = {} & B_{ij}(X_B) + D_k B_{ij}(X_B) \xi^k + \tfrac{1}{2} D_{k_1} D_{k_2} B_{ij}(X_B) \xi^{k_1} \xi^{k_2} \\ & - \tfrac{1}{6} \left\{ R^\ell_{k_1 i k_2}(X_B) B_{\ell j}(X_B) \xi^{k_1} \xi^{k_2} - (i \leftrightarrow j) \right\} + O(\xi^3) \end{aligned} \tag{3.11}$$

$$\Phi(X) = \Phi(X_B) + D_i \Phi(X_B) \xi^i + \tfrac{1}{2} D_i D_j \Phi(X_B) \xi^i \xi^j + O(\xi^3) \tag{3.12}$$

where

$$D_\alpha \xi^i = \partial_\alpha \xi^i + \Gamma^i_{jk}(X_B) \xi^j \partial_\alpha X^k_B$$

and R and Γ are respectively the Riemann curvature and Christoffel symbol constructed out of the metric G_{ij}. X_B is the background field which satisfies the classical equations of motions of the σ model. In the rest of the paper we drop the subscript B from X_B. The terms given in (3.9-3.12) are sufficient to calculate (in T_{++} and $\mathcal{L}$) all terms of order ξ^2, all terms of order ξ^3 with at least one derivative on ξ^i and all terms of order ξ^4 with both derivatives on ξ. In terms involving Φ, we keep terms linear in ξ with at least one derivative acting on ξ, and terms quadratic in ξ with both derivatives acting on ξ. As we shall see, these are the only terms required to calculate the coefficient c to two loop order, and the other terms in (3.5) and (3.6) to one loop order.

Using (3.9-3.12) we get

$$\mathcal{L} = \frac{1}{4\pi\alpha'} \left[D_\alpha \xi^a D^\alpha \xi^a + 2\epsilon^{\alpha\beta} \partial_\beta X^k S_{ijk} \xi^i D_\alpha \xi^j \right.$$

$$+ R_{ijk\ell}\xi^j\xi^k\partial_\alpha X^i\partial^\alpha X^\ell - \epsilon^{\alpha\beta}D_kS_{ij\ell}\xi^j\xi^k\partial_\alpha X^i\partial_\beta X^l$$

$$+ \tfrac{2}{3}\, S_{ijk}\xi^i\epsilon^{\alpha\beta}D_\alpha\xi^jD_\beta\xi^k$$

$$+ \tfrac{4}{3}\, R_{ijk\ell}\partial^\alpha X^i\xi^j\xi^kD_\alpha\xi^\ell + \tfrac{4}{3}\, D_kS_{ij\ell}\epsilon^{\alpha\beta}\partial_\beta X^i\xi^j\xi^kD_\alpha\xi^\ell$$

$$+ \tfrac{1}{3}\, R_{ijk\ell}\, D_\alpha\xi^iD^\alpha\xi^\ell\xi^j\xi^k - \tfrac{1}{2}D_kS_{ij\ell}\xi^j\xi^k\epsilon^{\alpha\beta}D_\alpha\xi^iD_\beta\xi^\ell$$

$$+ \text{terms irrelevant for computation in this order}] \qquad (3.13)$$

$$T_{++} = \frac{1}{2\pi\alpha'}\, [2G_{ij}\partial_+X^iD_+\xi^j + D_+\xi^aD_+\xi^a$$

$$+ R_{ijk\ell}\xi^j\xi^k\partial_+X^i\partial_+X^\ell + \tfrac{4}{3}\, R_{ijk\ell}\partial_+X^i\xi^j\xi^kD_+\xi^\ell$$

$$+ \tfrac{1}{3}\, R_{ijk\ell}D_+\xi^iD_+\xi^\ell\xi^j\xi^k$$

$$- \alpha'\left\{2D_jD_k\Phi\partial_+X^jD_+\xi^k + D_k\Phi D_+D_+\xi^k + \tfrac{1}{2}\,(D_iD_j\Phi)D_+D_+(\xi^i\xi^j)\right\}$$

$$+ \text{terms irrelevant for computation at this order.}] \qquad (3.14)$$

where

$$S_{ijk} = -\tfrac{1}{2}\,(D_iB_{jk} + D_jB_{ki} + D_kB_{ij}) \qquad (3.15)$$

$$\xi^a = e^a_i(X)\xi^i \qquad (3.16)$$

$$D_\alpha\xi^a = \partial_\alpha\xi^a + \omega^{ab}_i\xi^b\partial_\alpha X^i\ . \qquad (3.17)$$

$e^a_i(X)$ is the vielbein field which transform the coordinate index i to the tangent space index a, and ω^{ab}_i is the spin connection constructed from the Christoffel symbol Γ. In (3.13) and (3.14) we may replace all the ξ^i by $E^i_a\xi^a$ and $D_\alpha\xi^i$ by $E^i_aD_\alpha\xi^a$, where E^i_a is the inverse of e^a_i.

We first evaluate the contribution involving the operator $\partial_+X^i\partial_-X^j$ in the operator product expansion $\langle T_{++}(\sigma)T_{++}(\sigma')\rangle$. The graphs contributing to this term are given in Fig. 4. The contribution from each individual graph is listed in Appendix A, here we only quote the sum of all the contributions:

$$\frac{1}{4\pi^2}\,\partial_- X^i \partial_+ X^j \,\frac{(\sigma^- - \sigma'^-)}{(\sigma^+ - \sigma'^+)^3}\,[R_{ij} + S_{imn}S_j^{mn} - 2D_iD_j\Phi + D_kS_{ij}{}^k - 2D^k\Phi S_{ijk}]\;. \tag{3.18}$$

Vanishing of the symmetric and antisymmetric part of the above equation gives

$$R_{ij} + S_{imn}S_j^{mn} - 2D_iD_j\Phi = 0 \tag{3.19}$$

$$D_kS_{ij}{}^k - 2S_{ijk}D^k\Phi = 0\;. \tag{3.20}$$

We should emphasize that the graphs contributing to these terms are free of divergences and hence may be calculated without introducing any regularization.

Next we focus on the contribution to the terms proportional to $\partial_+X^i\partial_+X^j$ in the operator product expansion. These contributions are given in Fig. 5. Since we are interested in the anomalous terms in the operator product expansion, we must subtract out the contribution of the term proportional to T_{++} on the right hand side of (3.4). Such contributions to one loop order are given in Fig. 6. Although the total contribution to the anomalous terms is finite, the individual graphs contributing to the operator product expansion are divergent and hence we must regularize the integrals. We use dimensional regularization for our analysis. The $\epsilon^{\alpha\beta}$ tensor always appear quadratically in this computation and is reduced by the formula

$$\epsilon^{\alpha\beta}\epsilon^{\gamma\delta} = \delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma} \tag{3.21}$$

which is then continued to $2-2\epsilon$ dimensions. Again, the details of the calculation are given in Appendix A. The final result for the anomalous term in the operator product expansion is

$$-\frac{1}{2\pi^2}\,S_{ik\ell}S_j{}^{k\ell}\partial_+X^i\partial_+X^j\,\frac{1}{(\sigma^+ - \sigma'^+)^2}\;. \tag{3.22}$$

This contribution may be removed by adding to T_{++} a term

$$-\frac{1}{2\pi}\, S_{ik\ell}S_j{}^{k\ell}\partial_+X^i\partial_+X^j \ . \tag{3.23}$$

This term will contribute to the operator product expansion through the graphs shown in Fig. 7. We must also take into account the contribution from this new term to the right hand side of (3.1). The leftover contribution precisely cancels the anomaly given in Eq. (3.22).

We must point out at this stage that the anomalous term proportional to $\partial_+X^i\partial_+X^j$ vanishes for $S=0$. This reflects the fact that a conserved current like the stress tensor does not receive any finite or infinite renormalization if the regularization prescription obeys the conservation law. This is true for dimensional regularization in the absence of the antisymmetric tensor field, however the stress tensor ceases to be conserved in $2-2\epsilon$ dimensions in the presence of a background antisymmetric tensor field. This is the origin of the finite renormalization of the stress tensor. In fact, it can be verified that if we use Pauli-Villars regularization, which respects the conservation of the stress tensor in the presence of background antisymmetric tensor field, there is no anomalous contribution of the form (3.23) in the operator product expansion.

We may now proceed to calculate the contribution to the central charge to two loop order. The graphs contributing to this term are displayed in Fig. 8. Note that there is an explicit contribution from the term given in (3.23), as shown in Fig. 8(i). The sum of these graphs is given by

$$+\frac{1}{4\pi}\,\frac{6}{(\sigma^+-\sigma'^+)^4}\left[\frac{\mathrm{D}}{12\pi}+\frac{\alpha'}{2\pi}\,\left\{-D^2\Phi+(D\Phi)^2+\tfrac{1}{4}\,R+\tfrac{1}{12}\,S_{ijk}S^{ijk}\right\}\right]\ . \tag{3.24}$$

The $\mathrm{D}/12\pi$ term cancels against the corresponding contribution from the ghost fields for $\mathrm{D}=26$. Thus a consistent formulation of the string theory requires that the rest of the terms in (3.24) must vanish

$$-D^2\Phi + (D\Phi)^2 + \tfrac{1}{4}R + \tfrac{1}{12}\,S_{ijk}S^{ijk} = 0\ . \tag{3.25}$$

Linear combinations of Eqs. (3.19-3.20) and (3.25) give us the correct equations of motion for the massless states of the string.

From the conservation law of the stress tensor of Eq.(3.8) it follows that when the β-function vanishes the central charge of Eq.(3.25) is independent of the coordinates X. To show this let us consider Eq.(3.8) when the trace T_{+-} of the stress tensor vanishes. In this case we have $\partial_- \langle T_{++}(\sigma)T_{++}(\sigma')\rangle = 0$. From the operator product expansion (3.1) we see that the most singular term on the left hand side of Eq.(3.8) is $\frac{\partial_i c \partial_- X^i}{(\sigma^+ - \sigma'^+)^4}$. Hence $\partial_i c$ must vanish.

Using Eq. (3.8) we can also explicitly express the central term c and the coefficients C_{ij} defined by Eq.(3.4) in terms of the β-function. The trace of the stress tensor has the form

$$T_{+-} = \beta_{ij}(X)\partial_- X^i \partial_+ X^j \tag{3.26}$$

It has a background field expansion

$$\beta_{ij}(X)\partial_- X^i \partial_+ \xi^j + D_k\beta_{ij}\xi^k\partial_- X^i\partial_+\xi^j + \text{ irrelevant terms} \tag{3.27}$$

By comparing the $\frac{(1}{\sigma^+ - \sigma'^+})^3$ terms on both sides of Eq. (3.8) we can express C_{ij} in terms of β_{ij}. The relevant graph for calculating $\langle T_{++}(\sigma)T_{+-}(\sigma')\rangle$ is shown in Fig. 9. It gives a contribution

$$-\frac{1}{2\pi}\beta_{ij}\partial_- X^i\partial_+ X^j\ \frac{1}{(\sigma^+ - \sigma'^+)^2}\ . \tag{3.28}$$

Substituting this in (3.8) and comparing the coefficient of

$\partial_- X^i \partial_+ X^j / (\sigma^+ - \sigma'^+)^3$ terms on both sides, we get

$$C_{ij} = -\frac{1}{\pi}\,\beta_{ij}\ . \tag{3.29}$$

To express the central charge c in terms of the β-function we calculate the coefficient of the $\partial_- X^j$ term in $\langle T_{+-}(\sigma) T_{++}(\sigma')\rangle$. The relevant graphs are given in Fig. 10 and contribute,

$$-\frac{\alpha'}{4\pi}\,\partial_- X^i \left[D^j \beta_{ij} - 2D^j \Phi \beta_{ij} - S_i{}^{jk}\beta_{jk}\right] \frac{1}{(\sigma^+ - \sigma'^+)^3}\ . \tag{3.30}$$

Substituting this in Eq. (3.8) and comparing the coefficient of the $\partial_- X^i \, \frac{1}{(\sigma^+ - \sigma'^+)^4}$ term on both sides, we get

$$\begin{aligned} \partial_i c &= 3\alpha' \left[D^j \beta_{ij} - 2D^j \Phi \beta_{ij} - S_i^{jk}\beta_{jk}\right] \\ &= -3\pi\alpha' [D^j C_{ij} - 2D^j \Phi C_{ij} - S_i^{jk} C_{jk}] \end{aligned} \tag{3.31}$$

This can also be derived by using Eqs. (3.18), (3.24) and the Bianchi identities. In other words we have derived the Bianchi identities, showing that to this order the space time equations of motion follow from a generally covariant action. We emphasize, however, that (3.29) and (3.31) are valid only in the lowest order in perturbation theory, and receive corrections in higher orders.

The central charge may also be calculated from the trace anomaly on a curved world sheet, as the coefficient of the $\sqrt{g}\,R^{(2)}$ term.[7,13] If the background metric is taken to be conformally flat, this is equivalent to calculating $\langle T_{+-} T_{+-}\rangle$, which may be related to $\langle T_{++} T_{++}\rangle$ by using conservation of the stress tensor. We must, however, mention at this point that if dimensional regularization is being used in the calculation, the trace must be taken only at the end of the calculation, after continuing back to two dimensions. Let us, for example, consider the contribution,

$$
\begin{aligned}
A_{\alpha\beta\gamma\delta} &\equiv \int e^{ip\cdot(\sigma-\sigma')} \left\langle T_{\alpha\beta}(\sigma)\, T_{\gamma\delta}(\sigma')\right\rangle \\
&\equiv f_1(p^2) p_\alpha p_\beta p_\gamma p_\delta + f_2(p^2)\{\delta_{\alpha\beta} p_\gamma p_\delta + \delta_{\gamma\delta} p_\alpha p_\beta\} \\
&\quad + f_3(p^2)\{\delta_{\alpha\gamma} p_\beta p_\delta + \delta_{\alpha\delta} p_\beta p_\gamma + \delta_{\beta\gamma} p_\alpha p_\delta + \delta_{\beta\delta} p_\alpha p_\gamma\} \\
&\quad + f_4(p^2)\delta_{\alpha\beta}\delta_{\gamma\delta} + f_5(p^2)(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})
\end{aligned}
\tag{3.32}
$$

ignoring the terms involving the ϵ tensor for this analysis. Conservation of the stress tensor gives,

$$
\begin{aligned}
p^2 f_1 + 2f_3 + f_2 &= 0 \\
p^2 f_2 + f_4 &= 0 \\
p^2 f_3 + f_5 &= 0 \ .
\end{aligned}
\tag{3.33}
$$

In two dimensions

$$
A_{\alpha\alpha\gamma\gamma} = (p^2)^3 f_1 \tag{3.34}
$$

using Eqs. (3.33). On the other hand, A_{++++} is given by $f_1 p_+^4$. This clearly shows that the coefficients of $A_{\alpha\alpha\gamma\gamma}$ and A_{++++} are related to each other. On the other hand, in 2-ϵ dimension

$$
A_{\alpha\alpha\gamma\gamma} = (p^2)^2 f_1 + 2\epsilon(f_2 + f_3) \ . \tag{3.35}
$$

Since f_2 and f_3 are ultraviolet divergent $\epsilon(f_2 + f_3)$ is finite and (3.35)is no longer related to A_{++++}. Hence in order to get the correct expression for the conformal anomaly, we must calculate $T_{\alpha\beta}$ in an arbitrary two dimensional gravitational field, and then take the trace in two dimensions, rather than starting from the dimensionally regularized expression for $T_\alpha{}^\alpha$.

4. Conclusion

We have seen that the equations of motion of the dilaton and other massless fields may be obtained by imposing conformal invariance (or equivalently nilpotence of the BRST charge) on a BRST invariant field theory on a flat world sheet. This reconciles the string field theory picture of the origin of the dilaton with the picture arising from σ models.

Our calculations have been done by direct evaluation of the operator product expansion. A Ward identity following from conservation of the two dimensional stress tensor enabled us to relate all two loop contributions to the central charge to the one loop contribution to the less singular terms in the operator product expansion. Thus our method may be more easily extended to higher orders.

We have also seen that the two dimensional Ward identity shows that the dilaton equation of motion follows from the other equations. From the space-time point of view, this result is a consequence of the Bianchi identities, and ultimately of the fact that the space-time equations are derivable from an action. Thus we may hope to use this two dimensional identity to prove that the equations following from $Q^2 = 0$ are derivable from an action. At present this is only known to the lowest order.

Our work also shows that the string equations of motion follow from $Q^2 = 0$ where Q is a background dependent BRST charge. This connection (first conjectured by Friedan[8]) is also evident in the interacting open string field theory constructed by Witten.[14]

Finally we would like to remind the reader of the opposite side of the connection between the string field theory and the σ-model approaches. We have shown that the dilaton can be coupled to the ghost as it is in the string field theory. On the other hand, there should be a formulation of the string field theory in which the ghosts are replaced by two dimensional metric. Hopefully this will aid us in finding a more geometrical formulation for string theory.

ACKNOWLEDGEMENTS

We wish to thank M. Peskin for useful discussions. One of us (A.S.) would also like to thank R. Akhoury and Y. Okada for helpful discussions.

APPENDIX A

In this appendix we present some details of the calculation that led to the results in Chapter 3. We start by writing the ξ propagator in position and momentum spaces respectively

$$\Delta(\sigma - \sigma') = -\frac{\alpha'}{2}\, \ell n(\sigma - \sigma')^2 \tag{A.1}$$

$$\widetilde{\Delta}(p) = 2\pi\alpha'\, \frac{1}{p^2}\ . \tag{A.2}$$

The tree graphs may be evaluated in coordinate space directly. For graphs involving one or more loops, it is convenient to evaluate them in momentum space (using dimensional regularization, whenever necessary) and then convert them back to position space, remembering that the Fourier transform of $1/p^2$ is $\ell n(\sigma - \sigma')^2$, and each power of p_α is equivalent to $i\partial_\alpha$ in the position space.

With these rules, we may proceed to evaluate the graphs in Fig. 4-10. In these graphs the $\times$ denotes the vertices originating from T_{++}, $\otimes$ denotes vertices originating from the terms in T_{++} involving the dilaton field Φ, ⊠ denotes vertices originating from the term given in Eq. (3.23), ○ denotes vertices originating from T_{+-}, and all the other vertices originate from the Lagrangian $\mathcal{L}$. The double lines denote the background field. If the background field carries any derivative of X^i we display it explicitly on the graph. In Fig. 8 we have refrained from displaying the background fields explicitly.

In calculating the set of graphs given in Figs. 4-10, we may omit all terms that involve the spin connection ω explicitly, either from a vertex involving T_{++}, or in the Lagrangian. A separate calculation verifies that these graphs cancel among themselves. Also in Figs. 4, 5, 7, and 8 we have not bothered to explicitly show the graphs which are related to others through the exchange $\sigma \leftrightarrow \sigma'$. The evaluation of most of the graphs is now straightforward. Hence we will concentrate only on a few graphs whose evaluations are not so straightforward.

First consider Fig. 4(f). This contribution is given by

$$-\left(\frac{1}{2\pi^2\alpha'}\right)\partial_+X^i(\sigma)\left\{\frac{\partial}{\partial\sigma^+}\,\frac{\partial^2}{\partial(\sigma'^+)^2}\,\Delta(\sigma-\sigma')\right\}\;D_i\Phi(X(\sigma'))+(\sigma\leftrightarrow\sigma')\;. \tag{A.3}$$

We now expand $X^i(\sigma)$ as

$$X^i(\sigma)=X^i\left(\frac{\sigma+\sigma'}{2}\right)+\frac{\sigma^\alpha-\sigma'^\alpha}{2}\,\partial_\alpha X^i\left(\frac{\sigma-\sigma'}{2}\right)+O[(\sigma-\sigma')^2] \tag{A.4}$$

and make a similar expansion for $X^i(\sigma')$. When substituted into (A.3), the lowest order term is cancelled by the symmetry $\sigma\leftrightarrow\sigma'$. The leading contribution involving the $\partial_+X^i\partial_-X^i$ and $\partial_+\partial_-X^i$ operators is given by

$$\left[-\frac{1}{2\pi^2}\,\partial_+X^i\partial_-X^jD_iD_j\Phi+\frac{1}{4\pi^2}\,\partial^2X^iD_i\Phi\right]\frac{\sigma^--\sigma'^-}{(\sigma^+-\sigma'^+)^3}\;. \tag{A.5}$$

Using the equations of motion for the X^i fields, the term in the square bracket is reduced to

$$-\frac{1}{2\pi^2}\,\partial_+X^i\partial_-X^j[D_iD_j\Phi+S_{ijk}D^k\Phi]\;. \tag{A.6}$$

The evaluation of the rest of the graphs in Fig. 4 is straightforward and the sum of these contributions give us the result quoted in Eq. (3.18).

A similar manipulation is needed for the graphs of Fig. 5(j). In this case we get operators of the form $\partial_+X^i\partial_+X^j$ and $\partial_+^2X^i$. The $\partial_+^2X^i$ term precisely generates the $\partial_+^2\Phi$ term in T_{++} on the right hand side of Eq. (3.1).

The graph shown in Fig. 5(g) suffers from an infrared divergence. But so does the graph shown in Fig. 6(b) and when we compare the two sides of Eq. (3.1) their divergence gets cancelled. There are also several ultraviolet divergent graphs, which cancel from both sides of Eq. (3.1) when we take into account the graphs of Fig. 6 on the right hand of Eq. (3.1). Contributions from the rest of the graphs in Figs. 5-10 are fairly straightforward, and give us the answers quoted in the text.

APPENDIX B

In this appendix we analyze the contributions from Fig. 3. The ghost propagator in momentum space is given by

$$\langle c^+ b_{++} \rangle \equiv S(p) = -2i\pi\alpha' \, \frac{p^+}{p^2} \qquad (B.1)$$

and satisfies the identity

$$S(p)q^- S(p-q) = -2i\pi\alpha'(S(p-q) - S(p)) \ . \qquad (B.2)$$

With this identity it is easy to show that graphs with three or more insertions of $\partial_\mu \Phi$ in Fig. 3 vanish. The proof is essentially the same as that of the decoupling of a longitudinal photon from a fermion loop in QED. Consider a particular $\partial_+ \Phi$ insertion carrying momentum p_1. The term involving the propagators adjacent to this line then takes the form $p_1^- S(q)\, S(q - p_1)$, where q is some internal loop momentum. This may be reduced by Eq. (B.2). If we now sum over all insertions of this $\partial_+ \Phi$ vertex in the graph, and reduce each graph using Eq. (B.2), there will be pairwise cancellation between various terms. If the internal c line coming out of the J_+^B vertex has no ∂_+ operator acting on it, then we have complete cancellation after a shift of momentum (which is allowed for finite graphs). If the internal c line has a ∂_+ operator acting on it at the J_+^B vertex, then the cancellation is incomplete, and we get a factor of p_1^- times a graphs with one less $\partial_+ \Phi$ insertion, but otherwise having the same structure as Fig. 3.

But we may now repeat the analysis starting with another $\partial_+ \Phi$ vertex. Since we have already used up the ∂_+ acting on the internal c line at the J_+^B vertex to get a p_1^- factor, the pairwise cancellation after summing over all insertions and using (B.2), will now be complete, and there is no leftover contribution from these graphs.

This manipulation cannot be carried out for the graphs shown in Fig. 3(a) and can be carried out only once for the graph of Fig. 3(b), which reduces it to a graph of the form of Fig. 3(a). These contributions may be evaluated directly. Figure 3(a) gives

$$-\frac{1}{3\pi}\, c^+ \left[\tfrac{1}{2}\,\partial_+\partial_+\Phi\, e^{\frac{4}{3}\Phi} + \partial_+ \left(e^{\frac{4}{3}\Phi}\partial_+\Phi\right)\right] \tag{B.3}$$

whereas Fig. 3(b) gives

$$\frac{4}{9\pi}\, c^+\partial_+\Phi\, \partial_+\Phi\, e^{\frac{4}{3}\Phi}\ . \tag{B.4}$$

The sum of these contributions is

$$-\frac{1}{2\pi}\, c^+ e^{\frac{4}{3}\Phi}\partial_+\partial_+\Phi\ . \tag{B.5}$$

We may now make a field redefinition $c \to e^{-\frac{4}{3}\Phi}c$, $b \to e^{\frac{4}{3}\Phi}b$. This reduces the ghost Lagrangian to the free form given in (2.1) and (B.5) to the form (2.8).

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FIGURE CAPTIONS

1. Non-anomalous contributions to $J_+^B(\sigma)\, J_+^B(\sigma')$. Solid lines denote ghosts, dotted lines mesons and double lines background fields. $\times$ denotes a J_+^B vertex.

2. Some anomalous contributions to $J_+^B(\sigma)\, J_+^B(\sigma')$.

3. Contributions to $(J_+^B)_{\text{effective}}$ from anomalous diagrams.

4. Contributions to $\partial_+ X^i \partial_- X^j$ in $T_{++}(\sigma)\, T_{++}(\sigma')$. $\times$ denotes a vertex originating from the Φ independent part of T_{++}, $\otimes$ denotes a vertex originating from the Φ dependent part of T_{++}.

5. Contributions to $\partial_+ X^i \partial_+ X^j$ in $T_{++}(\sigma)\, T_{++}(\sigma')$.

6. One loop contributions to T_{++} to be included on the right hand side of (3.1).

7. Contribution to $\partial_+ X^i \partial_+ X^j$ in $T_{++}(\sigma)\, T_{++}(\sigma')$ from (3.23). $\boxtimes$ denotes the vertex originating from the term (3.23) in T_{++}.

8. Contributions to the central charge to order α'.

9. Contributions to $\partial_+ X^i \partial_- X^j$ in $T_{+-}(\sigma)\; T_{++}(\sigma')$. $\bigcirc$ denotes a vertex originating from T_{+-}.

10. Contributions to $\partial_- X^i$ in $T_{+-}(\sigma)\; T_{++}\; (\sigma')$.

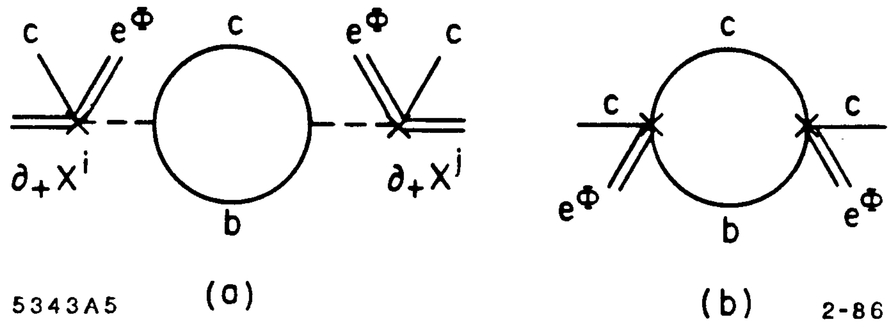


FIG. 1

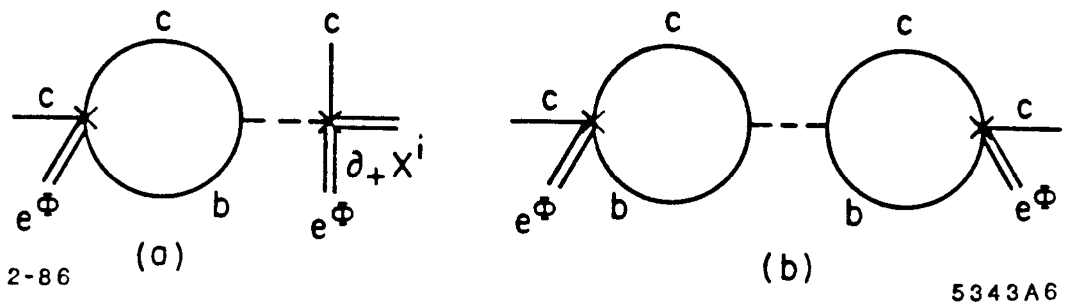


FIG. 2

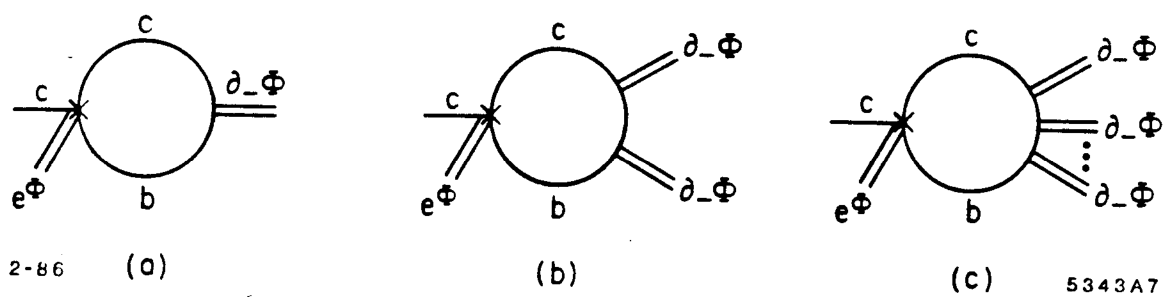


FIG. 3

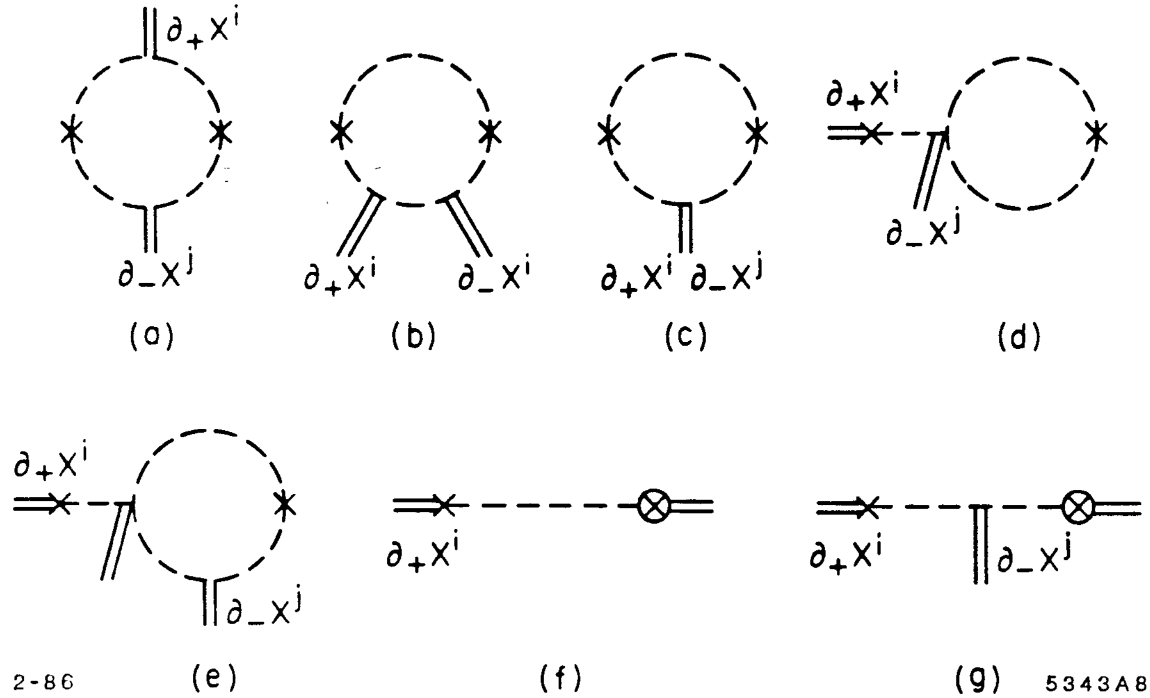


FIG. 4

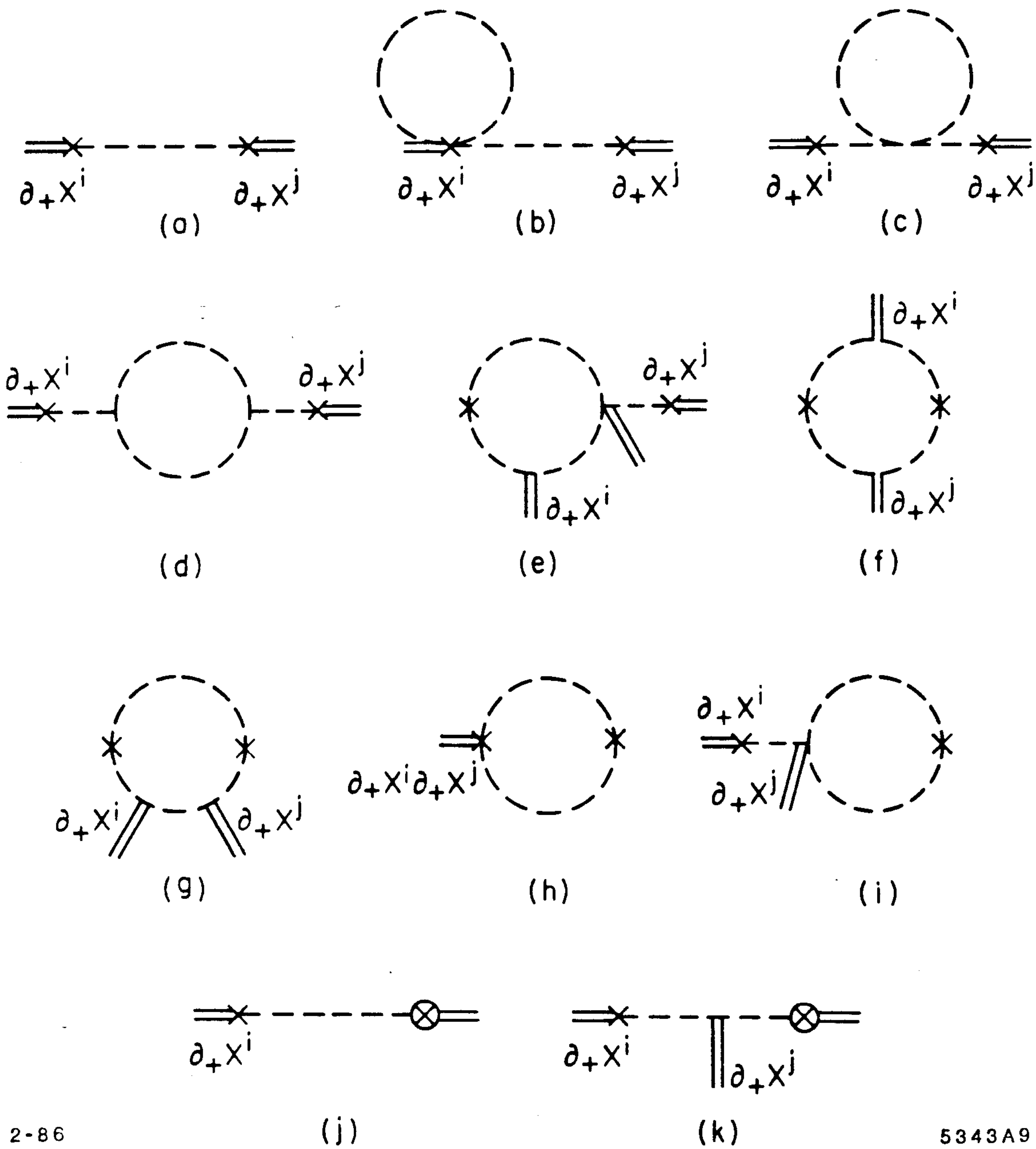
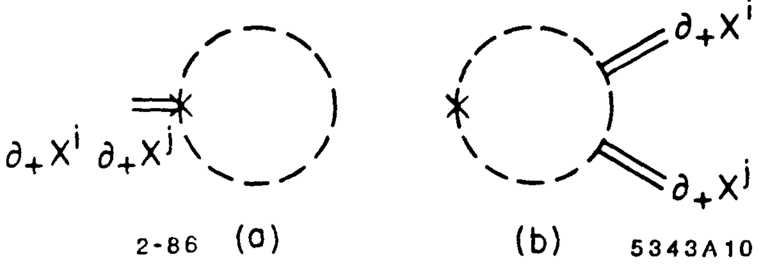


FIG. 5



$\partial_+ X^i$ $\partial_+ X^j$
$\partial_+ X^i$
$\partial_+ X^j$
2-86
(a)
(b)
5343A10

FIG. 6

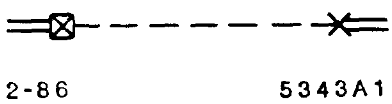


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FIG. 7

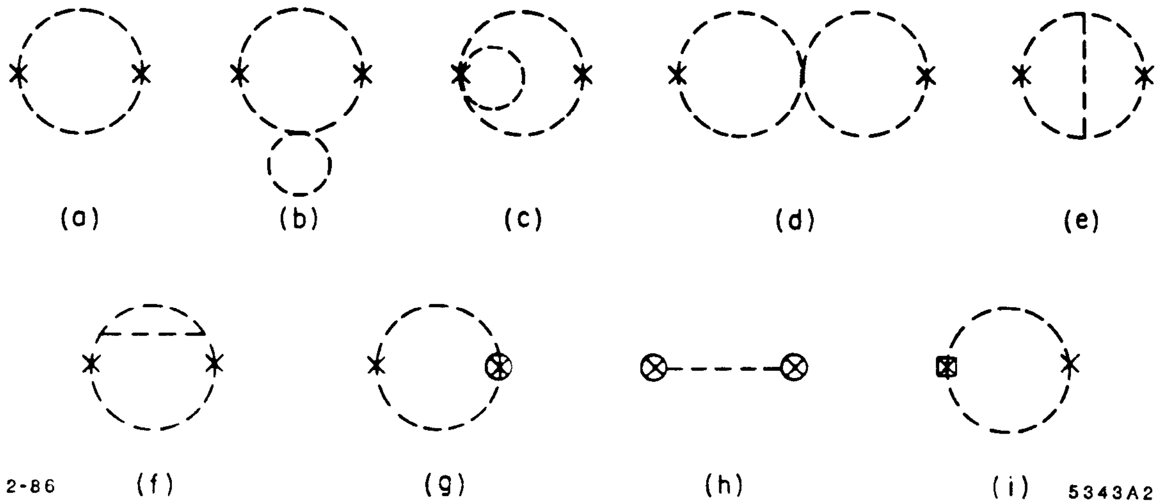

FIG. 8

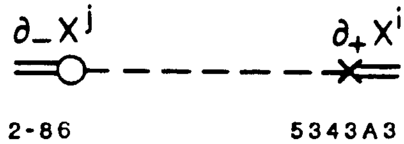

FIG. 9

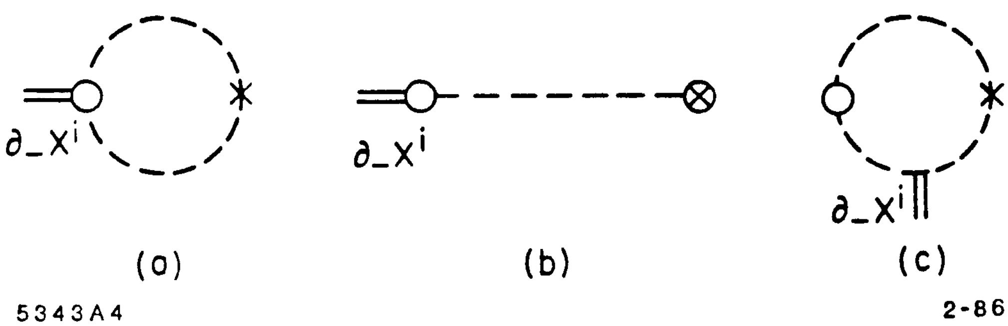

FIG. 10