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COOLING RINGS FOR TEV COLLIDERS*

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1. INTRODUCTION

We are now familiar[1] with the relation for the beam power for a quantum beamstrahlung limited collider:

$$N \approx 1.7 \times 10^{12} \left(\frac{\epsilon_n \beta^* \delta^3}{\sigma'_z}\right)^{1/2} \quad \text{(mks)} \tag{1a}$$

$$f \approx 3.6 \times 10^{-24} \, \frac{\mathcal{L}\sigma'_z}{\gamma\delta^3} \quad \text{(mks)} \tag{1b}$$

$$P_{beam} \approx .5 \times 10^{-24} \, \mathcal{L} \, (\epsilon_n \, \beta^* \, \sigma'_z)^{1/2} \, \delta^{-3/2} \quad \text{(mks)} \tag{1c}$$

Burt Richter and others have rather arbitrarily considered various desirable values for these constants. I will try:

$\beta^* = 1$ mm $\quad (10^{-3}$ m) $\quad$ (final focus strength)
$\sigma'_2 = 1\ \mu \quad (10^{-6}$ m) $\quad$ (bunch length at collision)
$\delta = .16 \quad$ (beamstrahlung fractional energy loss)
$\mathcal{L} = 10^{33}$ cm^{-2} sec$^{-1} \quad (10^{37}$ m^{-2} sec^{-1}) $\quad$ (luminosity)
$\epsilon'_n = 1.35 \times 10^{-8}$ m $\quad$ (normalized emittance)
$\gamma' = 3 \times 10^6 \quad$ (1.5 TeV) $\quad$ (collision energy)

With these values one obtains a

$N \approx 4 \times 10^8 \quad$ (particles per bunch)
$f \approx 3.0$ kHz $\quad$ (repetition frequency)
$P_{beam} \approx .3$ M Watts $\quad$ (power per beam)

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A value which yields a total wall plug power for both beams at 1% efficiency (c.f. SLAC eff. is less than 10^{-3}) of:

$$P_{wall} \approx 60 \text{ M Watts}$$

which is reasonable.

The question I want to address here is: can one obtain $\sim \epsilon_n = 10^{-8}$ in any plausible cooling ring. In order to answer this one must consider not only quantum fluctuations but also intra beam scattering, cooling rates and ring acceptance.

2. COOLING RATE

Cooling arises in a ring because the synchrotron energy loss occurs not only longitudinally, but also, if the beam has a finite angular divergence, transversely. The rf cavities make up the longitudinal component but leave the loss of transverse component.

The rate of cooling of transverse momentum is proportional to the rate of loss of energy (mostly longitudinal and made up by the rf). Thus the time τ_q to lower the transverse momentum by "e" is given by

$$2.718 \approx \text{"e"} = \int \frac{\Delta E}{E} = \frac{2\ e^2}{3\ m_0 c} \frac{\beta^4 \gamma^3}{\rho^2} \tau_q\ F_m$$

$$\approx 9 \times 10^{-7} \frac{\gamma^3}{\rho^2} \tau_q\ F_m \quad \text{mks}$$

where F_m is the fraction of the ring filled with magnets. Thus

$$\tau_{qx,y} \approx \frac{3 \times 10^6}{J_{x,y}} \frac{\rho^2}{F_m \gamma^3} \qquad \text{mks} \tag{2a}$$

J_x is the partition function[2] in the bending plane which is equal to 1 for a separated function lattice. In any case:

$$J_x + J_y + J_z = 4 \tag{2b}$$

J_y is hard to shift from 1. J_L is 2 in a separate function lattice and can, at best be lowered to say .5, at which point $J_x \approx 2.5$.

Equation (2) assumes no mixing between horizontal and vertical emittance. Or alternatively it implies that both are being cooled simultaneously, as for instance is true initially. As equilibrium is approached, however, the horizontal emittance is being blown up by fluctuations and intrabeam scattering, while the vertical is not. Under these conditions equation (2) is only true in the absence of mixing. If we introduce a mixing parameter ς which is =0 for no mixing and =1 for full mixing then J_x can be substituted by $J_x + \varsigma J_y$. However, this is true only when the vertical emittance is cold. Initially, we must set $\varsigma = 0$ whether there is or is not mixing.

Adding this term and substituting the field B for ρ:

$$\rho \approx 1.7 \times 10^{-3} \ \ \gamma/B \tag{3a}$$

$$\tau_q \approx \frac{8.3}{J_x + \varsigma J_y} \frac{1}{B^2 \ \gamma \ F_m} \quad \text{mks} \tag{3b}$$

For instance the SLAC cooling ring has $B \approx 2$ tesla, $\gamma \approx 2.4 \times 10^3$, $F_m \approx .36$, $J_x \approx 1$ and since we are considering initial cooling, $\varsigma = 0$. The equation then gives $\tau \approx 2.4 \times 10^{-3}$ sec. This may be compared with the published[3] value of 3×10^{-3} sec, which is near enough for our purposes.

3. EQUILIBRIUM EMITTANCE FROM QUANTUM FLUCTUATIONS

The existence of an equilibrium emittance arises because of the existence, in a ring, of a dispersion η. Different momenta have different orbits and when a sudden charge of momentum occurs due to the radiation of a photon, the particle finds itself in a position away from its equilibrium. Before it can be re-accelerated by the cavity it starts oscillating about its new orbit and, as a result, gains transverse momentum. This effect, balanced against the cooling, yields[2] an equilibrium emittance ϵ_{qn} (the q is for quantum, the n is for normalized)

$$\epsilon_{qn} = \frac{C_q}{J_x + \varsigma J_y} \gamma^3 \left\langle \frac{H}{\rho} \right\rangle \tag{4a}$$

where

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \approx 3.8 \times 10^{-13} \text{ m} \tag{4b}$$

Since

$$\rho \approx 1.7 \times 10^{-3} \ \frac{\gamma}{B}$$

$$\epsilon_{qn} \approx 2.2 \times 10^{-10} \ \frac{1}{J_x + \varsigma J_y} \ \gamma^2 \ \langle HB \rangle \tag{4c}$$

The function H depends on lattice parameters round the ring:

$$H = \frac{1 + \beta'^{\,2}/4}{\beta} \ \eta^2 - \beta' \eta \eta' + \beta \eta'^{\,2} \tag{4d}$$

β and η are the lattice parameters in the bending plane, the hyphen indicating the differential with respect to length. Note that where $B = 0$ it does not matter what H is.

Obviously the average over the lattice of a function like H is rather complicated and depends on the lattice. For a given number of bending magnets n it can be minimized and, assuming $J_x = 1$, $\xi = 0$ one obtains[4]

$$\epsilon_q \approx 8.3 \times 10^{-15} \ \ \gamma^3 \left(\frac{2\pi}{n} \right)^3 \quad \text{(m)} \tag{5}$$

Unfortunately the lattice required to achieve this minimum involves a relatively large amount of length devoted to manipulating β, β', η etc between each magnet.

As a result it would tend to have a low fraction of magnets F_m. This is not only bad for the cooling rate (see equation (2)) but will be bad for intrabeam scattering also. I will therefore choose to consider a more conventional ring with F_m as large as possible and with a sufficiently small phase advance per cell that I can take the approximation that β_x and η are constants around the ring. I will however, following Steffen, introduce one novelty[5]:

I will assume that each bending magnet is really a wiggler whose average bending field $\bar{B}$ is finite but less than average absolute field B. I define α_1 = average B in a magnet/local absolute B's

Remembering the definition of F_m, the average radius (R) of the ring is given by:

$$R = \rho/(\alpha_1 \ F_m) \tag{6}$$

Given these assumptions then

$$H \approx \eta^2/\beta_x \tag{7a}$$

since

$$\eta \approx R/Q^2 = \beta_x^2/R \tag{7b}$$

Thus[6]

$$H \approx \frac{\beta_x^3}{R^2} = \frac{\beta_x^3}{\rho^2} \ \alpha_1^2 \ F_m^2 \tag{7c}$$

And using

$$\begin{aligned} \rho &\approx 1.7 \times 10^{-3} \ \frac{\gamma}{B} \\ H &\approx 3.5 \times 10^5 \ \frac{\beta_x^3 \ B^2 \ \alpha_1^2 \ F_m^2}{\gamma^2} \end{aligned} \tag{7d}$$

putting this into equation (4c):

$$\epsilon_{qn} \approx 7.7 \times 10^{-5} \ \frac{\beta_x^3 B^3 F_m^2 \alpha_1^2}{J_x + \varsigma J_y} \tag{8}$$

To see how good this approximation is I again consider the SLC cooling ring[3] for which $\bar{\beta}_x \approx .77$ m, $B = 2$ Tesla, $F_m = .36$, $\alpha_1 = 1$, $J_x = J_y = 1$ and $\varsigma = 1$ which gives $\epsilon_{qn} \approx 1.8 \times 10^{-5}$. The published value is 2×10^{-5}.

A slightly more familiar form of equation (8) may be obtained by noting again $Q = R/\beta_x$ then for $J_x = 1$ and $\varsigma = 0$:

$$\epsilon_{qn} \approx 3.8 \times 10^{-13} \ \frac{\gamma^3}{Q^3} \ \frac{1}{F_m \ \alpha_1} \tag{9}$$

Further, if I assume a 65° phase advance per half cell (SLC) then the bending angle θ per cell is

$$\theta = \frac{2\pi\ 65}{360\ Q}$$

and

$$\epsilon_{qn} \approx \frac{2 \times 10^{-13}}{J_x} \frac{\gamma^3 \theta^3}{F_m\ \alpha_1} \tag{10}$$

which may be compared with P. Wilson's equation[7]

$$\epsilon_{qn} \approx 4.8 \times 10^{-13} \frac{\gamma^3 \theta^3}{F_m}$$

Thus my number is more optimistic than his, but also agrees better with the SLC ring.

There is an obvious condition when using the wiggler. The local change in η within the wiggler must be kept small compared with the average η in the ring.

For small α_1 the orbits within the wiggler will consist of alternating arcs on either side of an essentially straight axis. The maximum orbit deviation from the axis, a, is given by

$$a = \ell_p^2/8\rho$$

where ℓ_p is the length of one arc, i.e. the length of an individual pole of the wiggler.

The change in dispersion, η', for unit dp/p, will be equal to a, and η' should be held to some small fraction f_ω of the average η. Thus

$$\eta' = a = \frac{\ell_p^2}{8\rho} = f_2\eta = \frac{f_\omega \beta_x^2}{R} = \frac{f_\omega \beta_x^2 \alpha F_m}{\rho}$$

and thus

$$\ell_p = \beta_x (8\alpha F_m f_\omega)^{1/2} \tag{11}$$

ℓ_2, the length of one pole of the wiggler magnet, can be compared with the total length of wiggler ℓ_ω

$$\ell_\omega = F_m\ \phi\ \beta_x \tag{12}$$

where ϕ is the phase advance per 1/2 cell (i.e. per bending magnet).

The minimum number n_ω of wiggles per wiggler is thus

$$n_\omega \geq \sqrt{\frac{\phi^2 F_m}{8\ \alpha_1\ f_\omega}} \tag{13a}$$

or for $\phi \approx 1$ radian, $F_m = 1/2$, $f_\omega = 1/4$

$$n_\omega \geq \sqrt{\frac{1}{4\alpha_1}} \tag{13b}$$

4. INTRABEAM SCATTERING

In the above sections we have assumed that the beam current is small and scattering of particles within a bunch is negligible. If the current is raised then eventually this intrabeam scattering becomes significant and will eventually determine the equilibrium emittance independent of the quantum fluctuation limit of equation (8).

In principle, it may be argued, intrabeam scattering within a spherical phase space will not charge that phase space and should not lead to a blow up. In practice, however, in any plausible electron cooling ring the phase space is very far from spherical. For instance a longitudinal momentum spread of 10^{-3} at 3 GeV corresponds to a longitudinal Δp_e of $.3 \times 10^{-6}$. This must be compared with the transverse momentum spread Δp_t which, even at $\epsilon_n = 10^{-8}$ and $\beta = 1$ m, is 1.7×10^{-6}. Thus $\Delta p_t \gg \Delta p_e$ and scattering transfers transverse phase space into the longitudinal. The resulting fluctuations in the momentum would perhaps be harmless but for the dispersion. As for the quantum effect the fluctuations in momentum in the presence of dispersion cause orbit jumps and result in a blow up of the transverse emittance.

The rate of growth due to these effects has been given[8] by

$$\frac{1}{\tau_c} = C_c \frac{I\gamma^2}{\varsigma \epsilon_{cn}^2} \left\langle \frac{H^{1/2}}{\sigma_p \gamma^3 \beta_y^{1/2}} \right\rangle \tag{14}$$

where $C_c \approx 10^{-10}$ m^2/(Amp sec) and $\epsilon_{vert} = \varsigma \epsilon_{horiz}$. The H here is the same as that above (equation (4d)) but the average is of course different.

Equilibrium is reached if this growth rate equals the quantum cooling rate $1/\tau_q$ thus

$$\epsilon_{cn} = \left\{ \frac{C_c\ I\ \tau_q}{\sigma_p\ \gamma\ \varsigma^2} \left\langle \frac{H^{1/2}}{\beta_y^{1/2}} \right\rangle \right\}^{1/2} \tag{15}$$

since

$$I = \frac{e\ N\ c}{\sqrt{2\pi}\ \sigma_z} \approx 1.9 \times 10^{-11}\ \frac{N}{\sigma_z}$$

$$\tau_q \approx 8.3/(B^2\ \gamma\ F_m) \qquad \text{from (3)}$$

thus

$$\epsilon_{cn} \approx \frac{1.2 \times 10^{-10}}{\varsigma} \left(\frac{N}{\sigma_z\ B^2\ \gamma^2\ F_m\ \sigma_p} \left\langle \frac{H^{1/2}}{\beta_y^{1/2}} \right\rangle \right)^{1/2} \tag{16}$$

c.f. the SLAC cooling ring: $N = 5 \times 10^{10}$, $\sigma_z = 6 \times 10^{-3}, B = 2$, $\gamma = 2.4 \times 10^3$, $F_m = .36$, $\varsigma = 1$, $\sigma_p = 7.3 \times 10^{-4}$, $H = .017$, $\bar{\beta}_y = 1.7$, *giving* $\epsilon_{cn} \approx 1.3 \times 10^{-6}$ *or* 4% *of* ϵ_{qn}.

Defining the normalized longitudinal emittance:

$$\epsilon_{zn} = \gamma\ \sigma_p\ \sigma_z \tag{17}$$

$$\epsilon_{cn} \approx \frac{1.2 \times 10^{-10}}{\varsigma} \left(\frac{N}{\epsilon_{zn}\ B^2\ \gamma\ F_m} \left\langle \frac{H^{1/2}}{\beta_y^{1/2}} \right\rangle \right)^{1/2} \tag{18}$$

Finally I can substitute for H from equation (7d):

$$\epsilon_{cn} \approx 2.9 \times 10^{-9} \, \frac{1}{\varsigma\gamma} \left(\frac{N \, \alpha_1}{\epsilon_{zn} \, B}\right)^{1/2} \left(\frac{\beta_x^3}{\beta_y}\right)^{1/4} \tag{19}$$

Equation (19) would be correct if the blow up due to quantum fluctuations were negligible. If the two are comparable one obtains:[8]

$$\epsilon_n \approx \frac{1}{2}\left[\epsilon_{qn} + \left(\epsilon_{qn}^2 + \epsilon_{cn}^2\right)^{1/2}\right] \tag{20}$$

In the following I will consider rings in which:

$$\epsilon_{qn} = \epsilon_{cn} \tag{21a}$$

and thus from equation (20):

$$\epsilon_n \approx 1.2 \; \epsilon_{qn} \tag{21b}$$

5. LONGITUDINAL EMITTANCE

Synchrotron radiation not only cools in the transverse directions but also in the longitudinal. High momentum particles radiate more than low momentum ones and thus the momentum spread tends to reduce. Balanced against this the quantum fluctuations of the process itself tends to increase the momentum spread. An equilibrium is reached given[2] by:

$$\frac{\Delta p}{p} = \sigma_p \approx \left(\frac{2}{J_z}\right) \; 1.1 \times 10^{-5} \; (\gamma \; B)^{1/2} \quad \text{mks} \tag{22}$$

J_z is the longitudinal partition function which for normal separate function machines has the value 2. As noted above (equation (2b))

$$J_2 + J_x + J_y = 4$$

as in general $J_y = 1$, thus

$$J_z \approx 3 - J_x \tag{23}$$

The total normalized longitudinal emittance ϵ_{zn} is

$$\epsilon_{zn} = \gamma \; \sigma_p \; \sigma_z \leq \gamma' \; \sigma_p' \; \sigma_z' \tag{24}$$

and is related then to the minimum momentum spread σ_p' and bunch length σ_z' at the final collider energy γ'.

The bunch length σ_z is of course determined by the strength and frequency of the r.f.:

$$\sigma_z = \frac{R\,\alpha}{Q_s}\,\sigma_p \tag{25a}$$

where Q_s is the synchrotron tune

$$Q_s = \left(\frac{e\,\alpha\,U\,h}{2\pi\,E}\right)^{1/2}$$

$$\alpha \approx 1/Q_x^2$$

thus

$$\sigma_z \approx \frac{R}{Q_x}\left(\frac{2\pi\,\gamma\,(mc^2)}{U\,h}\right)^{1/2}\sigma_p \tag{25b}$$

where (mc^2) is the electron mass in electron volts, U is the voltage energy gain per revolution, h is the harmonic number (number of r.f. cycles per revolution), σ_p is beam momentum spread $\partial p/p$, and Q_x is the horizontal tune.

For our purposes however we can regard (Uh) as a free parameter and simply select ϵ_{zn} from bunch length and energy spread at full collider energy (equation (24)).

6. RINGS WITH QUANTUM AND INTRABEAM EMITTANCE MATCHED

Recall equation (4c)

$$\epsilon_{qn} \approx 2.2\times 10^{-10}\,\frac{1}{J_x+\varsigma J_y}\,\gamma^2\,\langle HB\rangle$$

and equation (18)

$$\epsilon_{cn} \approx \frac{1.2\times 10^{-10}}{\varsigma}\left(\frac{N}{\epsilon_{zn}\,B^2\gamma\,F_m}\langle\frac{H^{1/2}}{\beta_y^{1/2}}\rangle\right)^{1/2}$$

Setting $\epsilon_{qn} = \epsilon_{cn} = \frac{\epsilon_n}{1.2}$ and assuming that H is uniform about the ring, we can eliminate H and obtain:

$$\gamma = 3.57\times 10^{-8}\,\frac{N^{1/2}(J_x+\varsigma J_y)^{1/4}}{\epsilon_{zn}^{1/2}\varsigma\,F_m^{1/2}\,\epsilon_n^{3/4}\,B^{5/4}\,\beta_y^{1/4}} \tag{26}$$

For instance, if for a linear collider we chose the parameters listed in section 1:

$$\begin{aligned}
&\epsilon_n' = 1.35\times 10^{-8}\\
&\left.\begin{aligned}\sigma_2' &= 10^{-6}\\ \gamma' &= 3\times 10^{6}\\ \sigma_p' &= .33\%\end{aligned}\right\}\quad \epsilon_{zn}' = \epsilon_{zn} = 10^{-2}\ \mathrm{m}\\
&\delta = .16\\
&N = 4\times 10^{8}
\end{aligned}$$

In order to operate at $\epsilon_n' = 1.3\times 10^{-8}$ we need on equilibrium emittance ϵ_n

The ring radius R is given by

$$R \approx 1.7 \times 10^{-3} \ \frac{\gamma}{B} \ \frac{1}{F_m \alpha_1} \tag{29}$$

For our example $\gamma = 4.8 \times 10^3, \ B = 2, \ F_m = .5$ and $\alpha = .06$ thus

$$R = 130 \text{ m}$$

Now

$$Q_x = \frac{R}{\beta_x} \approx 390$$

$$Q_y = \frac{R}{\beta_y} \approx 100$$

$$\frac{Q_x}{Q_y} \approx 4$$

Now we can look at what σ_p and σ_z are. For σ_p I will use equation (22), which is for quantum fluctuations only. It will at least give the right order of magnitude

$$\sigma_p \approx \left(\frac{2}{3 - J_x}\right) \ 1.1 \times 10^{-5} \ (\gamma \ B)^{1/2}$$

which for $J_x = 1, \ \gamma = 4.8 \times 10^3$ and $B = 2$ gives

$$\sigma_p \approx 1.0 \times 10^{-3}$$

σ_z is then given by

$$\sigma_z = \epsilon_{zn}/(\gamma \ \sigma_p)$$

which for $\epsilon_{zn} = 10^{-2}, \ \gamma = 4.8 \times 10^3$ gives

$$\sigma_z = 2 \times 10^{-3} \text{ m}$$

Finally we calculate the cooling rate given by equation (2)

$$\tau \approx \frac{8.3}{J_x} \ \frac{1}{B^2 \ \gamma \ F_m}$$

which for $\gamma = 4.8 \times 10^3, \ B = 2, \ F_m = .3, \ J_x = 1$:

$$\tau = .9 \times 10^{-3} \text{ sec}$$

We note that the diameter is not so unreasonable, it is less than PEP. The cooling rate is relatively fast and most parameters are not unreasonable. But the tunes are very high. Will such a ring have any acceptance?

some what lower than this. I take

$$\epsilon_n = 3/4\ \epsilon' \approx 1 \times 10^{-8} \text{ m rad}$$

choosing

$$\varsigma = 1 \qquad i.e. \text{ full } x, y \text{ mixing}$$

$$J_\gamma = J_y = 1 \qquad \text{normal partition functions}$$

$$F_m = .5 \qquad 50\% \text{ full of bending magnets}$$

$$B = 2 \qquad (20 \text{ Kg})$$

and

$$B_y = 1.4 \text{ m}$$

then

$$\gamma = 4.8 \times 10^3$$

$$E_e = 2.4 \text{ GeV}$$

The β_x we can now obtain from equation (8) turned around, and with equation (21)

$$\alpha_1^{2/3}\ \beta_x \approx 22\ (J_x + \varsigma J_y)^{1/3}\ \frac{\epsilon^{1/3}}{B\ F_m^{2/3}} \tag{27}$$

In our example $\epsilon_n = 10^{-8},\ B = 2,\ F_m = .5,\ J_x = J_y = \varsigma = 1$:

$$\alpha_1^{2/3}\ \beta_x = 4.8 \times 10^{-2}$$

Now if α_1 the wiggler parameter were equal to 1 this implies a β_x of 5 cm which is not very reasonable at $E = 2.4$ GeV. So what is a reasonable β_x? At SLC $\bar{\beta}_x \approx .77$ m, at a γ of 2.4×10^3 and quadrupole apertures, a, of 2.5 cm. Scaling gives

$$\beta_x \propto (a\ \gamma)^{1/2}$$

Or normalizing to the SLC cooling ring

$$\beta_x \geq .1\ (a\ \gamma)^{1/2} \qquad (\text{mks}) \tag{28}$$

If we take the aperture, a, to be 2.5 mm (note that the beam will be only tens of microns in diameter) then for 2.4 GeV

$$\beta_x(\text{reasonable}) \approx .34 \text{ m}$$

and thus $\alpha_1 \approx .06$.

7. ACCEPTANCE

I know of no generally accepted scaling law or equation for the acceptance of a lattice. What follows is therefore not to be taken too seriously. I will assume that the acceptance is limited by non-linear effects coming from sextupoles inserted to correct chromaticity (i.e. changes of Q with momentum). As before I will assume a lattice with essentially constant values of β, ς etc, i.e. a lattice with a sufficiently small phase advance per cell that I can think of the focussing as being continuous.

I define k to be a focussing strength, ℓ_q the quadrupole lengths and G the quadrupole field gradients:

$$k = \ell_q \, G \tag{30}$$

$$\beta \propto 1/k^{1/2} \tag{31}$$

and note

$$2\,\frac{d\beta}{\beta} = -\frac{dk}{k} = \frac{dp}{p} \tag{32}$$

In order to correct this variation of β with momentum we insert sextupoles around the ring. Again we assume that the phase advance is so small that the sextupole effect is essentially continuous and corresponds to a variation Δk of the focussing strength k with the average radial position ΔR of the beam:

$$\Delta k = S\; dr = S\;\eta\;\frac{dp}{p} \tag{33}$$

where S is the sextupole strength. Adding this term to equation (32) we obtain

$$\begin{aligned} 2\,\frac{d\beta}{\beta} &= -\frac{dk}{k} + \frac{\Delta k}{k} \\ &= \frac{dp}{p} - \frac{S}{k}\eta\,\frac{dp}{p} \end{aligned} \tag{34}$$

So, for no charge in tune β $(d\beta = 0)$ we require

$$S = \frac{k}{\eta} \tag{35}$$

If the sextupole strength is provided by sextupoles of length ℓ_s at every quad length ℓ_q then we note that

$$S = \ell_s\,\frac{dG_x}{dr} \tag{36}$$

and since in a sextupole

$$B = \frac{B_p}{(a/2)^2}\cdot r^2$$

$$G_s = \frac{B_p}{(a/2)^2}\cdot 2r$$

$$\frac{dG_s}{dr} = 2\cdot\frac{B_p}{(a/2)^2}$$

so

$$S = \ell_s \, 2 \, \frac{B_p}{(a/2)^2} \tag{37}$$

and since

$$k = \ell_q \, \frac{B_p}{a/2} \tag{38}$$

and

$$S = \frac{k}{\eta}$$

we find

$$\frac{\ell_s}{\ell_q} = \frac{a}{4\eta} \tag{39}$$

For our example $a = 2.5$ mm, $\eta = 9 \times 10^{-4}$ so $\ell_s/\ell_q = .7$ which means a lot of sextupole!

Now for a small enough emittance the effect of the sextupole strength is only seen as a charge in quadrupole strength. As the emittance rises however the more extreme orbits will see the nonlinear effects of the sextupoles. The relative magnitude of these non linear effects can be assessed by looking at the charge of focussing strength $\Delta k'$ arising from the maximum amplitude of oscillation σ.

$$\Delta k' = -S \, \hat{\sigma} \tag{40}$$

My assumption will be that non linear effects will become serious when this shift in focussing strength becomes a significant fraction f_S of the normal focussing strength k

$$f_S = \frac{\Delta k'}{k} = \frac{S\hat{\sigma}}{k} \tag{41}$$

now

$$\hat{\sigma} = \sqrt{\frac{\hat{\epsilon}_n \hat{\beta}}{\gamma}} \qquad \text{and} \qquad S = k/\eta$$

so

$$\hat{\epsilon}_n = \gamma \, \frac{f_S^2 \eta^2}{\hat{\beta}} \tag{42}$$

Now in order to reduce intrabeam scattering it is desirable to have $\beta_y > \beta_x$ and for the same reason one likes strong mixing so that $\epsilon_y \approx \epsilon_x$. Under these circumstances $\hat{\sigma}$ will be in the vertical direction y:

$$\hat{\epsilon}_{ny} = \frac{\gamma \, f_S^2 \, \eta^2}{\beta_y} \approx \frac{\gamma \, f_S^2 \, \beta_x^4}{R^2 \, \beta_y} \tag{43a}$$

$$\hat{\epsilon}_{ny} = \gamma \, f_S^2 \, R \, \frac{Q_y}{Q_x^4} \tag{43b}$$

the fraction, or fudge factor, f_S we can obtain from the SLC example

$$\hat{\epsilon}_n(SLC) = 10^{-2} = 2.4 \times 10^3 \; f_S^2 \; 5.6 \; \frac{3.25}{7.25^4}$$
$$f_S = 2.5 \times 10^{-2} \tag{44}$$

i.e. our scaling law implies that when the non linear focussing is more than 2.5% of the linear focussing the orbits become unstable. A not unreasonable conclusion.

Our scaling law thus predicts:

$$\hat{\epsilon}_{ny} \approx 6 \times 10^{-4} \; \gamma \; R \; \frac{Q_y}{Q_x^4} \tag{45}$$

For our example $\hat{\epsilon}_{ny} = 1.6 \times 10^{-6}$ which is very small, but still 160 times the equilibrium emittance.

8. CONCLUSIONS

I have summarized the assumptions in our example in Table I, and the calculated parameters in Table II, together with those for the SLC ring. As noted before there seems nothing impossible about such a ring although the magnet apertures of 2.5 mm, the tune of 390, and acceptance of 20 microns are certainly daunting.

Table I
Assumed Parameters of Example (A)
Including Variations Assumed in Later Examples

Collider Energy	E'	1.5 + 1.5	TeV
Collider Luminosity	$\mathcal{L}$	10^{33}	cm^{-2} sec^{-1}
Final Focus	β^*	1.0	mm
Final Bunch Length	σ_z'	1.0	μm
Final Mom. Spread	σ_p'	3.0×10^{-3} (D : 3.3×10^{-4})	
Beamstrahlung Mom. Loss	δ	.16 (B : .32)	
Horizontal-Vertical Mixing	ς	1	
Partition Function	J_x	1 (C : 2)	
Dipole Fraction of Circ.	F_m	.5	
Dipole Field	B	2 (E : 4)	Telsa
Tune Ratio	Q_x/Q_y	4 (G : 40)	
Magnet Apertures	a	2.5 (F : 10)	mm
Phase Advance/$\frac{1}{2}$ Cell	ϕ	65°	
$d\eta/\eta$ in Wiggle	f_w	.25	
sext./quad. Strength	f_s	2.5×10^{-2}	

Table II
Calculated Parameters for Example A
and Comparison with SLC Cooling Ring

		Ex – A	SLC	
ϵ_{qn}	Quantum ϵ_n Equilibrium	$.8 \times 10^{-8}$	2×10^{-5}	m
ϵ_{cn}	Coulomb ϵ_n Equilibrium	$.8 \times 10^{-8}$	1.3×10^{-6}	m
N	Particles/Bunch	4×10^{8}	5×10^{10}	
f	Pulse Repetition	3×10^{3}	120	Hz
P	Power/Beam	.3 MW	70 KW	
E	of Cooling Ring	2.4	1.2	GeV
R	Radius of Ring	130	5.6	m
α	Wiggler $B/\bar{B}$	.06	1	
ℓ	Wiggler	19	32	cm
ℓ	Pole	≤ 8	—	cm
η	Chromaticity	9×10^{-4}	1.7×10^{-2}	m
$\bar{\beta}_x$		.34	.77	m
$\bar{\beta}_y$		1.4	1.7	m
Q_x		390	7.25	
Q_y		100	3.25	
$\ell_{\text{sext}}/\ell_{\text{quad}}$		1.0		
$\hat{\epsilon}_n$	Acceptance	1.6×10^{-6}	1×10^{-2}	m
$\langle\hat{\sigma}\rangle$	Acceptance	20 μm	2.6 mm	
$\hat{\epsilon}_n/\epsilon_n$		160	500	
σ_p	dp/p in Ring	1×10^{-3}	$.73 \times 10^{-3}$	
σ_z	in Ring	2	5.9	mm
τ	Cooling Time Constant	.9	3	msec

In order to see how the ring depends on the assumptions, I have calculated a number of rings changing each assumption in turn (see Table III). What do I conclude:

1. Only a small gain is obtained (example C) by messing with the partition functions.
2. A very significant gain is made by using higher (presumably superconducting) bending fields. Example E using 4 Tesla magnets has a radius reduced from 130 to only 40 meters and the Q has dropped from 390 to 210. The physical acceptance has gone up a bit (20 μ to 27 μ) and the cooling rate has gone up too. Whether such advantages would compensate for the great complication of superconducting magnets I do not know, but this should be studied.
3. A reduction in the ring diameter is obtained (example G) by allowing β_y to be much larger than β_x. For $\beta_y/\beta_x = 40$ the diameter has dropped from 130 to 54 meters. But the acceptance has dropped and is now only 26 times the equilibrium value. This is not in principle unacceptable, the ring could be fed from another pre cooling ring, but we must remember that the acceptance law is not reliable and only lattice tracing would tell us how bad this example is.
4. As would be expected the ring gets bigger if the magnet apertures are increased (example F).
5. Far more serious, however, is the ring diameter increase if the momentum spread of the beam is reduced (example D). This is a serious question. I had assumed .3% $\Delta p/p$ at 1.5 TeV and no dilution. This implies 3% $\Delta p/p$ at 150 GeV if the final bunching were performed at this energy. The short bunches (1 μ) are desirable to suppress wake field effects but some have suggested that small momentum spread may also be required. If really true (and I personally doubt it) this would have serious consequences for the attainability of emittances of 10^{-8}.
6. If even lower beam power per luminosity is required. (For a 5 TeV machine, for instance), then we may attempt to obtain an even lower emittance (example H). This does look pretty bad. The sextupoles are 5 times as long as the quads and the acceptance is only 3 microns!
7. The power can be more easily reduced by allowing a higher beamstrahlung energy loss (example B) the resulting higher current in the cooling ring does make the ring larger and more expensive but to no where near the extent of a lower emittance.
8. Finally I give the parameters of a 10^{-7} m radian emittance case. With a radius of only 7 meters it would be a lovely ring to try and build. Note, however, that this would not be suitable for the SLC. The number of particles per bunch is far too low.

I would like to thank Bob Siemann for starting me on this study, and albert Hoffman for his frequent help.

Table III

Calculated Parameters of Various Cooling Rings

	Example		A	B	C	D	E	F	G	H	I
				$\delta_{\text{beamstrahlung}} = .32$ (.16)	Partition f $J_x = 2$ (1)	Mom. Spread at Final Focus $\sigma'_p = 3.3 \times 10^{-4}$ (3.3×10^{-3})	Magnetic Field $B \equiv 4$ Telsa (2)	Magnet Aperture $a = 10$ mm (2.5 mm)	Ratio of Tunes $Q_x/Q_y = 40$ (4)	Emittance at Final Focus $\epsilon'_n = 1.35 \times 10^{-9}$ (1.35×10^{-9})	Emittance at Final Focus $\epsilon'_n = 1.35 \times 10^{-7}$ (1.35×10^{-8})
ϵ_n	Equilibrium	m	10^{-8}	10^{-8}	10^{-8}	10^{-8}	10^{-8}	10^{-8}	10^{-8}	10^{-9}	10^{-7}
N	Electrons/Bunch		4×10^8	1×10^9	4×10^8	4×10^8	4×10^8	4×10^8	4×10^8	1.3×10^8	13×10^8
f	Pulse Frequency	kHz	3	.37	3	3	3	3	3	3	3
P	Power/Beam	MW	.3	.1	.3	.3	.3	.3	.3	.09	.9
E	of Ring	GeV	2.4	3.8	2.6	6.6	1.1	2.1	1.4	6.7	.8
R	of Ring	m	130	300	130	800	50	290	54	2500	7
α_1	Wiggler $B/\bar{B}$		.06	.04	.07	.03	.04	.02	.09	.009	.4
ℓ_w	Length Wiggler	cm	19	24	20	32	13	36	15	32	12
ℓ_p	Length of Pole	cm	8	9	9	9	4.5	10	8	5	13
η	Chromaticity	m	9×10^{-4}	6×10^{-4}	10×10^{-4}	4×10^{-4}	11×10^{-4}	14×10^{-4}	13×10^{-4}	1.3×10^{-4}	60×10^{-4}
β_x		m	.34	.43	.36	.57	.23	.6	.27	.57	.2
β_y		m	1.4	1.7	1.4	2.3	.9	2.5	10.6	2.3	.8
Q_x	Horizontal Tune		390	690	350	1400	210	450	205	4400	34
Q_y	Vertical Tune		100	173	90	350	50	110	5	1100	8
$\ell_{\text{sext}}/\ell_{\text{quad}}$			.7	1.0	.6	1.6	.6	1.8	.5	4.8	.1
$\hat{\epsilon}_n$	Acceptance	m	1.6×10^{-6}	1×10^{-6}	2.2×10^{-6}	$.57 \times 10^{-6}$	1.7×10^{-6}	1.8×10^{-6}	$.3 \times 10^{-6}$	6×10^{-8}	4.4×10^{-5}
$\hat{\sigma}_y$	Acceptance	μ	20	15	25	10	27	34	30	3	150
$\hat{\epsilon}_n/\epsilon_n$			160	100	200	57	170	180	26	56	440
σ_p	dp/p in Ring		1×10^{-3}	1.3×10^{-3}	2.2×10^{-3}	1.7×10^{-3}	1×10^{-3}	1×10^{-3}	$.8 \times 10^{-3}$	1.8×10^{-3}	$.6 \times 10^{-3}$
σ_z	in Ring	mm	2	1.0	.8	.04	4.3	2.4	4.2	.4	9
τ	Cooling Time	msec	.9	.6	.8	.3	.5	1.0	1.5	.3	2.5

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6. Matt Sands has pointed out that this relation is more generally true, since

$$H = \alpha R/Q_x$$
$$\alpha = 1/Q_x^2$$
$$\text{thus} \qquad H \approx \beta_x^3/R^2$$

   see M. Sands, The Physics of Storage Rings, SLAC-PUB-121 (1979), p. 134.
7. See Ref. 1(c) who quotes: H. Wiedemann, 11th Int. Conf. on High Energy Accelerators (1980), p. 693.
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