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Regularities of Fermion Masses and Mixing Angles Near Infra-red Fixed Points*

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ABSTRACT

The relations $\left(\frac{m_d}{m_s}\right)^{1/2} - \left(\frac{m_u}{m_c}\right)^{1/2} \simeq 0(\lambda)$ and $\left(\frac{m_s}{m_b}\right)^{1/2} - \left(\frac{m_c}{m_t}\right)^{1/2} = 0(\lambda^2)$ are given by means of renormalization group equations in the standard SU(3) × SU(2) × U(1) model with three generations. Thus the key ingredient of Fritzsch mass matrix is obtained which yields the mixing angles of quark sectors in good agreement with recent experiments.

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The standard SU(3) $\times$ SU(2) $\times$ U(1) model [1] is remarkably successful in describing the observed strong, weak and electromagnetic interactions, phenomenologically. However, an unsatisfactory feature of the present model is the large number of parameters which are put in by hand. In the three generation version, there are three gauge couplings, nine quark and lepton masses, three mixing angles, one phase factor, the W boson mass and the Higgs mass. One of the most puzzling aspects of particle physics at the present time is observed generation structure of quarks and leptons. The problem in particle physics is that of explaining the quark masses and their mixings. They will serve as important clues in search for the more fundamental theory from which the standard model can be obtained as the low-energy effective theory. For example, recently buttressed string theory seems to provide the grand unified theory of everything, the resulting ten-dimensional theory is $N = 1$ supersymmetric theory [2], from which the SU(3) $\times$ SU(2) $\times$ U(1) model can be derived as the low energy effective theory.

Now we assume that the unified theory of interactions is a non-Abelian gauge theory based on a simple group G which breaks to SU(3) $\times$ SU(2) $\times$ U(1) at a mass scale M_x. If the renormalization group equations describing the evolution of the various couplings in the theory passes stable infra-red fixed points, then resulting value towards these fixed points will be determined by low energy SU(3) $\times$ SU(2) $\times$ U(1) gauge group [3]. In statistical physics, we know that there are several fixed points in the scalar theories with several couplings. The fermion masses in terms of a fixed point were studied either for ordinary SU(3) $\times$ SU(2) $\times$ U(1) model or recently for supersymmetric grand unified theories [4]. Their main interest predicts the top quark mass or the masses of the fourth generation.

In this paper, we would like to discuss the renormalization group constraints in the regularities of fermion masses and mixing angles. Because the rate of convergence to infra-red fixed point values is different for the different couplings, for example, the large Yukawa couplings converge very fast to their fixed point values, the small Yukawa couplings slowly converge to fixed points. So a small perturbation is introduced near the fixed points, when we consider the entire quark mass matrices. We found, that the success of the Fritzsch mass matrix [5], which is reproducing the observed pattern of the Kobayashi-Maskawa mixing [6], can be explained by standing the renormalization group equations of Yukawa and gauge couplings near the infra-red fixed points.

In standard SU(3) $\times$ SU(2) $\times$ U(1) model with three generations, quark and lepton sectors are

$$\begin{pmatrix} U_i \\ D_i \end{pmatrix}_L, \quad U_{iR}, \quad D_{iR}$$

$$\begin{pmatrix} E_i \\ N_i \end{pmatrix}_L, \quad E_R, \quad N_R$$

where $i = 1,2,3$ is color index, $U = u, c, t(?)$; $D = d, s, b$, $E = e, \mu, \tau$; $N = \nu_e, \nu_\mu, \nu_\tau$.

There is one Higgs doublet with hypercharge $Y = 1$,

$$\phi = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix} = \begin{pmatrix} \phi^+(x) \\ \frac{1}{\sqrt{2}}(v + H(x) + i\chi(x)) \end{pmatrix}$$

where $< H(x) >=< \chi(x) >= 0$ and

$$< \phi^0(x) >= v = 1/[\sqrt{2}\ G_F]^{1/2} = 246 \text{ GeV}$$

Only observable Higgs particle is neutral scalar boson H. The self interaction of the Higgs field and all the interaction of Higgs boson H with gauge bosons and fermions are determined by the masses of these particles, which is proportional to the mass of these particles. The Yukawa couplings

$$\mathcal{L}_Y = \phi^0\Big[\overline{D}_L M_D D_R + \overline{U}_R M_U^+ U_L + \overline{E}_L M_L E_R\Big] \\ +\phi^{0+}\Big[\overline{U}_L M_D D_R - \overline{U}_R M_U^+ D_L + \overline{N}_L M_L L_R\Big] + \text{h.c.} \tag{1}$$

where $M_U,\ M_D,\ M_L$ are non-hermitean 3×3 matrices. In this notation, the one-loop renormalization group equations for the Yukawa couplings and the gauge couplings take the following form [7],

$$c\,\frac{dM_U}{dt} = \frac{3}{2}\,\left(M_U M_U^+ - M_D M_D^+\right) M_U + (T - g_u)M_U \tag{2.1}$$

$$c\,\frac{dM_D}{dt} = \frac{3}{2}\,(M_D M_D^+ - M_U M_U^+)M_D + (T - g_d)M_D \tag{2.2}$$

$$c\,\frac{dM_L}{dt} = \frac{3}{2}\,M_L M_L^+ M_L + (T - g_\ell)M_L \tag{2.3}$$

and

$$\frac{d\alpha_i^2}{dt} = \frac{b_i}{(4\pi)^2}\,\alpha_i^4 \tag{3}$$

where $c = 32\pi^2,\quad t = \ell n\left(\frac{\mu^2}{M_X^2}\right),\quad b_3 = 3,\quad b_2 = -1,\quad b_1 = -11$

$$T = Tr\Big[3\left(M_D M_D^+ + M_U M_U^+\right) + M_L M_L^+\Big] \tag{4}$$

$$g_u = 32\pi\alpha_3(t) + 9\pi\alpha_2(t) + \frac{17}{3}\,\pi\alpha_1(t) \tag{5.1}$$

$$g_d = 32\pi\alpha_3(t) + 9\pi\alpha_2(t) + \frac{5}{3}\,\pi\,\alpha_1(t) \tag{5.2}$$

$$g_\ell = 9\pi\alpha_2(t) + 15\pi\alpha_1(t) \tag{5.3}$$

$\alpha_i \equiv g_i^2/4\pi$, *i.e.*, $g_3(t)$, $g_2(t)$ and $g_1(t)$ are SU(3), SU(2) and U(1) gauge couplings, respectively. From Eqs. (2.1) - (2.3) we see that the gauge contributions increase the fermion masses but the Yukawa contributions tend to decrease their masses. So the critical points are determined by zero of the right of Eqs. (2.1) - (2.3),

$$\frac{3}{2}\ \left(M_U M_U^+ - M_D M_D^+\right) M_U + (T - g_u) M_U = 0 \tag{6.1}$$

$$\frac{3}{2}\ \left(M_D M_D^+ - M_U M_U^+\right) M_D + (T - g_d) M_D = 0 \tag{6.2}$$

$$\frac{3}{2}\ M_L M_L^+ M_L + (T - g_\ell) M_L = 0 \tag{6.3}$$

This is a system of coupled non-linear equations. The fixed point of lepton sectors and that of quark sectors cannot be realized simultaneously. Here we only consider the fixed points of quark sectors which depend on the initial values of the gauge and Yukawa coupling of the $\mu = M_X$. The solutions of Eqs. (6.1) and (6.2) are

$$(a)\ \ M_U = 0, \qquad M_D = 0$$

$$(b)\ \ M_U = 0\ \Rightarrow\ M_D = 0$$

$$(c)\ \ M_D = 0\ \Rightarrow\ M_U = 0$$

$$(d)\ \ M_U \neq 0, \qquad M_D \neq 0$$

For the case of (d), from Eqs. (6.1) and (6.2), we have

$$Tr[\mathcal{M}_U + \mathcal{M}_D] = \frac{1}{6}\ (g_u + g_d) \tag{7.1}$$

$$\mathcal{M}_U - \mathcal{M}_D = \frac{4\pi\alpha_1(t)}{3}\ I \tag{7.2}$$

where

$$\mathcal{M}_U = M_U M_U^+, \quad \mathcal{M}_D = M_D M_D^+$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ,$$

Obviously, $\mathcal{M}_U$ and $\mathcal{M}_D$ are hermitean matrices, which can be diagnosed by two unitary matrices U and V, $\quad UU^+ = 1, \quad VV^+ = 1$

$$U\mathcal{M}_U U^+ = \text{diag }(m_u^2,\ m_c^2,\ m_t^2) \tag{8.1}$$

$$V\mathcal{M}_D V^+ = \text{diag }(m_d^2,\ m_s^2,\ m_b^2) \tag{8.2}$$

The Kobayashi - Maskawa matrix is

$$M_{KM} = UV^+ \tag{9}$$

Equation (7.1) gives a sum rule of quark masses

$$\sum m_{\text{up}}^2 + \sum m^2{}_{\text{down}} = \left[\frac{16}{3}\ \pi g_3(t) + \frac{2}{3}\ \pi g_2(t) + \frac{11}{6}\ \pi g_1(t)\right](24b)^2\ \text{GeV}^2\ ,$$

which were studied for the mass of top quark. Our interest is to discuss Eq. (7.2). From Eq. (7.2), (8.1) and (8.2), we have

$$\text{diag }(m_u^2,\ m_c^2,\ m_t^2) - \frac{4}{3}\ \pi\alpha_1(t)I = U\mathcal{M}_D U^+ \tag{10.1}$$

and

$$\text{diag }\left(m_d^2, m_s^2, m_b^2\right) - \frac{4}{3}\ \pi\alpha_1(t)I = V\mathcal{M}_U V^+ \tag{10.2}$$

which means the $\mathcal{M}_D(\mathcal{M}_U)$ also can be diagnosed by unitary matrix $U(V)$. So, in general, we have

$$M_U = \beta M_D \tag{11}$$

Thus, the charge 2/3 up and -1/3 down quark mass ratio in each generation should be the same.

$$\frac{m_{3/2}}{m_{-1/3}} = \text{generation independent} \tag{12}$$

In fact, the quark mass values have the following hierarchical structure [8],

$$\begin{pmatrix} m_u & m_c & m_t \\ m_d & m_s & m_b \end{pmatrix} = \begin{pmatrix} 5.1\ \text{MeV}, & 1.35\ \text{GeV}, & 30-50\ \text{GeV}(?) \\ 8.9\ \text{MeV}, & 175\ \text{MeV}, & 5.3\ \text{GeV} \end{pmatrix} \tag{13}$$

From Eq. (13), we see that the quark mass ratio $\frac{m_t}{m_b} = \frac{m_c}{m_s}$ is a better approximation than that of $\frac{m_u}{m_d} = \frac{m_c}{m_s}$. In my opinion, it is not difficult to understand. Because the structure of the fixed points in above cases have very rich properties, which not only depends on the initial values of the gauge and Yukawa couplings but also depends on the values of the matrix elements. The rate of convergence of heavy quark mass matrix element is more fast than that of light quark mass matrix element. The first generation masses are small. We need to give the correction of Eq. (11) so that it gives larger corrections for $\frac{m_u}{m_d} = \frac{m_c}{m_s}$ than for $\frac{m_c}{m_s} = \frac{m_t}{m_b}$. The corrections just reflect the fact the properties in the neighborhood of an infra-red fixed point may be very complex.

It is expected that there are non-perturbative effects near the fixed points, especially for the first generation. Because the physical fixed points can only be reached if the Yukawa couplings are sufficiently large. So, we assume that the right of Eqs. (2.1) - (2.3) tends to zero, but not equal to zero, i.e.,

$$\frac{3}{2}\,(M_U M_U^+ - M_D M_D^+) + (T - g_u) = \Delta/2 \tag{14.1}$$

$$\frac{3}{2}\,(M_D M_D^+ - M_U M_U^+) + (T - g_d) = -\Delta/2 \tag{14.2}$$

where Δ is very small.

It is noted that Eqs. (14.1) and (14.2) keep the sum rule of quark masses Eq. (7.1) invariant, but Eq. (7.2), therefore Eq. (11) change into the following,

$$M_D = \beta M_U + A \tag{15}$$

where $A = 1/6\ U^+\Delta$, which is also small and A^2 is neglected.

From Eq. (15), we have

$$UV^+VM_DV^+VU^+ = \beta\ UV^+VM_UV^+VU^+ + UV^+VAV^+VU^+$$

i.e.,

$$M_{KM}[\ \text{diag}\ (m_d, m_s, m_b)]M_{KM}^+ = \alpha\ \text{diag}\ (m_u, m_c, m_t) + \bar{A} \tag{16}$$

where $\bar{A} = UAU^+ = 1/6\ \Delta U^+$ is small also.

Now we can show that Eq. (16) indeed gives larger corrections for $\frac{m_u}{m_d} = \frac{m_c}{m_s}$ then for $\frac{m_c}{m_s} = \frac{m_t}{m_b}$.

An explicit representation of the Kobayash–Maskawa matrix is [9]

$$M_{KM} = \begin{pmatrix} c_1 & s_1c_3 & s_1s_3 \\ -s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2s_3e^{i\delta} \\ -s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix} \tag{17}$$

where $s_i = \sin\theta_i, \quad c_i = \cos\theta_i, \quad i = 1,\ 2,\ 3.$

Inserting Eq. (17) into Eq. (16), the diagonal elements of Eq. (16) give the following three relations,

$$\beta m_u = m_d c_1^2 + m_s s_1^2 c_3^2 + m_b s_1^2 s_3^2 + \bar{A}_{11} \tag{18.1}$$

$$\begin{aligned}\beta m_c = {} & m_d s_1^2 c_2^2 + m_s \left(c_1^2 c_2^2 s_3^2 + s_2^2 s_3^2 - 2 c_1 c_2 c_3 s_2 s_3 \cos\delta \right) \\ & + m_b \left(c_1^2 c_2^2 s_3^2 + s_2^2 c_3^2 + 2 c_1 c_2 c_3 s_2 s_3 \cos\delta \right) + \bar{A}_{22}\end{aligned} \tag{18.2}$$

$$\begin{aligned}\beta m_t = {} & m_d s_1^2 s_2^2 + m_s \left(c_1^2 s_2^2 c_3^2 + c_2^2 s_3^2 + 2 c_1 c_2 c_3 s_2 s_3 \cos\delta \right) \\ & + m_b \left(c_1^2 s_2^2 s_2^3 + c_2^2 c_3^2 - 2 c_1 c_2 c_3 s_2 s_3 \cos\delta \right) + \bar{A}_{33}\end{aligned} \tag{18.3}$$

Adding both sides of the above three equations, we get

$$\beta = \frac{m_d + m_s + m_b + Tr\bar{A}}{m_u + m_c + m_t} \tag{19}$$

For $m_b >> m_s >> m_d$ and $m_t >> m_c >> m_u$, considering only the first two generations, from Eq. (18.1) and (19), it is easy to show

$$\left|\frac{m_d}{m_s}\right|^{1/2} - \left|\frac{m_u}{m_c}\right|^{1/2} = \frac{1}{\left|\frac{m_d}{m_s}\right|^{1/2} + \left|\frac{m_u}{m_c}\right|^{1/2}} \cdot \frac{\bar{A}_{11}}{|m_s|} \tag{20}$$

Comparing with other terms in Eqs. (18.1) and (19), we know that $\frac{\bar{A}_{11}}{|m_s|} \sim 0(\lambda^2)$, where $\lambda = (M_{KM})_{us} \simeq 0.225$.

Now we consider the model with three generations. From Eq. (18.2) and (19), we also have the following approximate result,

$$\left|\frac{m_s}{m_b}\right|^{1/2} - \left|\frac{m_c}{m_t}\right|^{1/2} = \frac{1}{\left|\frac{m_s}{m_b}\right|^{1/2} + \left|\frac{m_c}{m_t}\right|^{1/2}} \cdot \frac{\bar{A}_{22}}{|m_b|} \tag{21}$$

An acceptable assumption is that the diagonal matrix elements of the matrix $\bar{A}$ have the same order of magnitude. Thus we have, approximately, $\frac{\bar{A}_{22}}{|m_b|} \sim 0(\lambda^3)$.

Then, from Eq. (20) and (21), we have

$$\frac{\left|\frac{m_s}{m_b}\right|^{1/2} - \left|\frac{m_c}{m_t}\right|^{1/2}}{\left|\frac{m_d}{m_s}\right|^{1/2} - \left|\frac{m_u}{m_c}\right|^{1/2}} \sim 0(\lambda) \tag{22}$$

Because we have (see Eq. (13))

$$\frac{\left|\frac{m_d}{m_s}\right|^{1/2} + \left|\frac{m_u}{m_c}\right|^{1/2}}{\left|\frac{m_s}{m_b}\right|^{1/2} + \left|\frac{m_c}{m_t}\right|^{1/2}} \sim 0(1)$$

and from Eq. (13), (20) and (21), we get the following results, finally,

$$\left(\frac{m_d}{m_s}\right)^{1/2} - \left(\frac{m_u}{m_c}\right)^{1/2} \simeq 0(\lambda) \tag{23}$$

$$\left(\frac{m_s}{m_b}\right)^{1/2} - \left(\frac{m_c}{m_t}\right)^{1/2} \simeq 0(\lambda^2) \tag{24}$$

Equations (23) and (24) are the very interesting results which were obtained from the discussion of the renormalization group equations. As the reference [10], recently, show that Eqs. (23) and (24) are the key ingredient that is needed for Fritzsch matrix to yield a set of Kobayashi-Maskawa angles in close agreement with the observed pattern.

So, our conclusion is that we can do regularities of fermion mass and mixing angles based on the renormalization group equations near the infra-red fixed points. It means that if the renormalization group equations describing the evolution of gauge couplings and Yukawa couplings possess stable neighborhood of the infra-red fixed points $\left(\frac{\Delta}{|m_s|} \sim 0(\lambda^2)\right)$ then in continuing from the scale of unified interactions to μ the couplings will be swept towards these neighborhoods of the fixed points irrespective of their initial values. It seems that the mass matrix of quark sectors and their mixing angles will be determined by the low energy gauge group SU(3) $\times$ SU(2) $\times$ U(1) mainly.

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