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PARITY ANOMALIES IN GAUGE THEORIES IN 2+1 DIMENSIONS*

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ABSTRACT

We show that the introduction of massless fermions in an abelian gauge theory in 2+1 dimensions does not lead to any parity anomaly despite a non-commutativity of limits in the structure function of the odd part of the vacuum polarisation tensor. However, a parity anomaly does exist in non-abelian theories due to a conflict between gauge invariance under large gauge transformations and the parity symmetry.

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Recently, there has been renewed interest in 2+1 dimensional field theories[1] especially regarding the parity anomalies that occur when massless fermions are introduced in these theories.[2] However, there have been several contradictory claims in the literature regarding these anomalies in abelian and non-abelian gauge theories. In this letter, we would like to clarify the picture and show that there are no so called "parity anomalies" in QED_3, despite a non-commutativity of limits, and that the anomaly does exist in QCD_3 due to a conflict between global gauge invariance and the parity symmetry.

We begin with the Lagrangian in QED_3 with one 2 component massive fermion

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi - m\bar{\psi}\psi \tag{1}$$

The vacuum polarisation tensor can be written as

$$\Pi_{\mu\nu}(p^2, m^2) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi_{\text{even}}(p^2, m^2) + im\epsilon_{\mu\nu\alpha}p^\alpha\Pi_{\text{odd}}(p^2, m^2) \tag{2}$$

The one loop contribution of the fermions to $\Pi_{\text{even}}(p^2, m^2)$ is linearly divergent and has to be regulated. Depending upon the regulator we choose, $\mu(p^2, m^2) = m\Pi_{\text{odd}}(p^2, m^2)$ takes different values. Using a gauge and parity invariant regularisation procedure like dimensional regularisation we get

$$\mu(p^2, m^2) = \frac{e^2 m}{4\pi}\int_0^1 dx \frac{1}{[-p^2x(1-x) + m^2]^{\frac{1}{2}}} \tag{3}$$

whereas regulating the integral by adding an explicitly parity violating Pauli-Villars regulator field with a mass Λ yields

$$\mu(p^2, m^2) = \frac{e^2 m}{4\pi}\int_0^1 dx \frac{1}{[-p^2x(1-x) + m^2]^{\frac{1}{2}}} + \frac{e^2}{4\pi}\frac{\Lambda}{|\Lambda|} \tag{4}$$

The extra contribution to μ due to the regulator field can, however, be cancelled by adding a local counter term to the Lagrangian. This freedom of adding counter terms implies that it is consistent to have a theory with any $\mu(0,m^2)$ including $\mu(0,m^2)=0$ even when m$\neq$0. But the structure function $\mu(p^2,m^2)$ is only shifted by a constant and hence physical consequences like screening for non-zero p still exist.

However, a parity anomaly exists only if a non-zero $\mu(p^2,0)$ is induced that cannot be cancelled by a local counter term. From Eq. (3),we see that

$$\mu(p^2,m^2)=\begin{cases}\frac{e^2m}{4p} & p\gg m\\ \\ \frac{e^2}{4\pi}\frac{m}{|m|} & p\ll m\end{cases} \tag{5}$$

Hence, for $m=0$, there is an ambiguity, since the results depend on the order in which m and p go to 0 -i.e.,

$$\mu(0,0)=\begin{cases}0 & m\to 0 \text{ before } p\to 0\\ \\ \frac{e^2}{4\pi}\text{sign}(m) & p\to 0 \text{ before } m\to 0\end{cases} \tag{6}$$

The freedom of adding counter terms obviously cannot cancel the p-dependent $\mu(p^2,m^2)$ for all p.

To resolve this ambiguity, let us consider the physical mass of the photon and how it is measured. The bare propagator may be obtained from the Lagrangian as

$$(\Delta_{\mu\nu})^{\text{bare}}(p^2)=\frac{-i}{p^2}(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}) \tag{7}$$

The full inverse propagator is given by

$$(\Delta^{-1}_{\mu\nu})^{\text{full}}(p^2,m^2)=(\Delta^{-1}_{\mu\nu})^{\text{bare}}(p^2)+\Pi_{\mu\nu}(p^2,m^2) \tag{8}$$

where $\Pi_{\mu\nu}(p^2,m^2)$ is given in Eq. (2) and yields

$$(\Delta_{\mu\nu})^{\text{full}}(p^2,m^2)=\frac{-i}{(p^2-\mu_R^2)(1+\Pi_{\text{even}}(p^2,m^2))}(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}-i\mu_R\epsilon_{\mu\nu\alpha}\frac{p^\alpha}{p^2}) \tag{9}$$

with

$$\mu_R \equiv \mu_R(p^2, m^2) = \frac{\mu(p^2, m^2)}{(1 + \Pi_{\text{even}}(p^2, m^2))} \tag{10}$$

However, to compute $\mu_R(p^2, m^2)$ to $O(e^2)$, we can set $\Pi_{\text{even}}(p^2, m^2) = 0$, so that $\mu_R(p^2, m^2) = \mu(p^2, m^2)$. We note that $\Pi_{\text{even}}(p^2, m^2)$ is infrared divergent when p and m go to zero, so that a naive conclusion may be that $\mu_R(0,0) = 0$, irrespective of the value of $\mu(0,0)$. However, since $\Pi_{\text{odd}}(p^2, m^2)$ and $\Pi_{\text{even}}(p^2, m^2)$ are calculated in perturbation theory to first order in e^2, we cannot allow the $O(e^2)$ contribution to $\Pi_{\text{even}}(p^2, m^2)$ to overwhelm the zeroth order value of one. Hence, for consistency,we shall use $\mu_R(p^2, m^2) = \mu(p^2, m^2)$= physical mass of the photon.

Now, let us consider how the mass of the photon is measured physically. One way is through scattering of massless fermions at finite momentum, which means that we have to take the $m \to 0$ before $p \to 0$. Equivalently,we can measure the mass of the photon by measuring the force between static charges. The interaction potential between two static charges at a distance $\mathbf{r}$ apart is given by

$$V_{\text{int}}(\mathbf{r}) = \int d^2\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{r}} \Delta_{00}(p_0 = 0, \mathbf{p}, m^2) \tag{11}$$

where

$$\Delta_{00}(p_0 = 0, \mathbf{p}, m^2) = \frac{i}{\mathbf{p}^2 + \mu^2(p^2, m^2)} \tag{12}$$

(when $\Pi_{\text{even}}(p^2, m^2)$is neglected). Using a step function ansatz for $\mu(p^2, m^2)$ with $\mu(p^2, m^2) = 0$ for $p > m$ and $\mu(p^2, m^2) = \frac{e^2}{4\pi}\text{sign}(m) \equiv \mu$ for $p < m$, we get

$$V_{\text{int}}(\mathbf{r}) = \int_0^m d^2\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{r}} \frac{i}{\mathbf{p}^2 + \mu^2} + \int_m^\infty d^2\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{r}} \frac{i}{\mathbf{p}^2} \tag{13}$$

Hence, the force between two static charges, $\mathbf{F}(\mathbf{r})$, is given by

$$\mathbf{F}(\mathbf{r}) = -\frac{\partial V}{\partial \mathbf{r}} = \int_0^m dpd\theta p^2 \cos\theta e^{i\mathbf{p}\cdot\mathbf{r}} \frac{\mu^2}{\mathbf{p}^2 + \mu^2} + \int_0^\infty dpd\theta \cos\theta e^{i\mathbf{p}\cdot\mathbf{r}} \tag{14}$$

Since $\mu^2/(\mathrm{p}^2+\mu^2)$is bounded by 1,the first integral in Eq. (14) is bounded by m. The second integral is precisely the force due to the exchange of a massless photon in three dimensions and is proportional to $1/r$. Hence, when $r^{-1} \gg m$, i.e., $r \ll m^{-1}$, the force between two static charges may be described by a massless photon. As $m \to 0$, as long as we are interested in the physically relevant distances, $r \ll m^{-1} \to \infty$, the photon stays massless. We claim therefore,that there is no parity anomaly in QED_3, since there is no induced mass for the photon, when the fermions are massless. However there could be a dynamical breakdown of the parity symmetry,analogous to chiral symmetry breakdown,[3] which is currently under investigation.[4]

We may also study the theory in the large N limit, where N is the number of fermion families. To leading order in the $1/N$ expansion ($N \to \infty$, with $\alpha = e^2 N$ fixed), we find that

$$\mu(p^2, m^2, N) = \begin{cases} \frac{\alpha m}{4p} & p \gg m \\ \frac{\alpha}{4\pi}\frac{m}{|m|} & p \ll m \end{cases} \tag{15}$$

and

$$\Pi_{\text{even}}(p^2, m^2, N) = \begin{cases} \frac{\alpha}{16p} & p \gg m \\ \frac{\alpha}{12\pi m} & p \ll m \end{cases} \tag{16}$$

with all further corrections suppressed by factors of $1/N$ which vanish when $N \to \infty$. Hence, in this limit $\mu_R(0,0,N) = 0$ irrespective of the value of $\mu(0,0,N)$ and hence the ambiguity at $p = 0, m = 0$ is physically irrelevant.

This analysis can be easily extended to non-abelian theories. The Lagrangian is given by

$$L = \frac{1}{2}Tr(F^{\mu\nu}F_{\mu\nu}) + \bar{\psi} i\not{D}\psi - m\bar{\psi}\psi \tag{17}$$

Just as in QED_3, the one loop contribution of the fermions to the vacuum polarisation tensor can be computed and the results are given in Eqs. (3) and

(4) except for an extra factor of 1/2 on the right hand side of the equations now, coming from $TrT_aT_b = -1/2\delta_{ab}$. However, unlike in QED_3, there exists a topological Ward identity in QCD_3 - i.e.

$$4\pi(\frac{\mu}{g^2})_{\rm ren} = 4\pi(\frac{\mu}{g^2})Z_m(\frac{Z}{Z_g})^2 = \text{integer} \tag{18}$$

where Z_m, Z_g and Z are defined to be the renormalisation constants for the mass, three gluon vertex and the wave function, respectively, at zero momentum, and μ is the bare mass for the photon. Hence, for a globally gauge invariant theory, the regularisation scheme must be consistent with this Ward identity. The calculation of $Z_m(\frac{Z}{Z_g})^2$ was done in Ref.(5)[5] for the case of pure gauge fields. The fermions do not contribute to the ghost-ghost-gluon vertex $\tilde{Z}_g$,and the ghost wave function $\tilde{Z}$ at zero momentum and using the infinitesimal Ward identity $\tilde{Z}_g/\tilde{Z} = Z_g/Z$ we find that for SU(N) theories,

$$4\pi(\frac{\mu}{g^2})_{\rm ren} = 4\pi(\frac{\mu}{g^2}) + N + 4\pi(\frac{\mu(0,m^2)}{g^2}) \tag{19}$$

The first term on the right hand side of Eq. (19), $4\pi(\mu/g^2)$ is an integer (as a consequence of gauge invariance of the bare Lagrangian). We therefore must have, by Eqs. (18) and (19) that $4\pi\mu(0,m^2)/g^2 = integer$. Hence, it is clear that dimensional regularisation with $\mu(0,m^2) = (g^2/8\pi)(m/|m|)$ is not globally gauge invariant, whereas the Pauli-Villars procedure with $\mu(0,m^2) = (g^2/8\pi)(m/|m|+\Lambda/|\Lambda|)$ is invariant under the large gauge transformations which lead to the topological Ward identity (18). It is perhaps, not surprising that dimensional regularisation, though an infinitesimally gauge invariant procedure, fails to obey the topological Ward identity, since large gauge transformations with integral winding number cannot be defined in $3-\epsilon$ dimensions. If we have an even number of fermions, of course, either scheme is fully gauge invariant and $\mu(0,m^2)$ differs in the two cases only by a finite counter term (which is itself consistent with Eq. (18)). For an odd number of fermions, however, we claim that Pauli-Villars is the globally gauge invariant procedure - i.e. if we regulate the theory using dimensional regularisation, we need to add a $1/2$ integer counter term to make the effective theory gauge invariant under large gauge transformations.

To look for a parity anomaly, we take the $m \to 0, p \to 0$ limit, being careful to take the physically motivated order of limits $m \to 0$ before $p \to 0$. We find that

$$\mu(p^2, 0) = 0 \tag{20}$$

in dimensional regularisation and

$$\mu(p^2, 0) = \frac{g^2}{8\pi} \frac{\Lambda}{|\Lambda|} \tag{21}$$

using a Pauli-Villars regulator field. Equations (20) and (21) show the incompatibility of global gauge invariance and parity. However, if we naively apply the topological Ward identity (19) to Eqs. (20) and (21), we would conclude that $\mu(0,0) = 0$ is the parity and gauge invariant answer. This apparent contradiction is resolved by noting that in the presence of massless fermions, Eq. (19) is modified by a non-perturbative anomaly.[6] For massless fermions, the effective action after integrating out the fermions is

$$L_{\text{eff}} = \frac{1}{2} Tr(F^{\mu\nu} F_{\mu\nu}) - i\ell n \det(\partial\!\!\!/ + gA\!\!\!/) \tag{22}$$

Under a large gauge transformation with winding number n

$$\det(\partial\!\!\!/ + gA\!\!\!/) \to (-1)^{nN} \det(\partial\!\!\!/ + gA\!\!\!/) \tag{23}$$

where N is the number of fermions and the determinant is regulated in a parity invariant way. Hence, for an odd number of fermions, the action is not gauge invariant unless this gauge non-invariance is compensated by

$$4\pi(\frac{\mu}{g^2}) = (\frac{N}{2})\text{integer} \tag{24}$$

It should be noted that the gauge non-invariance in Eq. (23) is not due to the two (or three) point functions, which are the only diagrams in need of ultra-violet regulation, since they have been explicitly shown to be zero - i.e., in Eq. (20) where we have used parity invariant dimensional regularisation. Thus, in order for the theory to be gauge invariant,we see from Eq. (24) that μ cannot be zero. Hence, since we require the theory to be gauge invariant, we have $\mu(p^2,0) = (g^2/8\pi)(\frac{\Lambda}{|\Lambda|})$ - i.e., the gluon has acquired a mass and the parity symmetry is broken.

Once again, let us look at the theory in the large N limit. For N odd, $\mu(p^2,m^2,N) = (\alpha/8\pi)(\Lambda/|\Lambda|)$ where $\alpha = g^2 N$ However,

$$\mu_R(p^2,m^2,N) = \frac{\mu(p^2,m^2,N)}{(1+\Pi_{\text{even}}(p^2,m^2,N))} \tag{25}$$

which yields

$$\mu_R(p^2,m^2,N) = \begin{cases} \frac{\alpha}{8\pi}\,\frac{\Lambda}{|\Lambda|}\,\frac{1}{(1+\frac{\alpha}{32p})} & p \gg m \\ \frac{\alpha}{8\pi}\,\frac{\Lambda}{|\Lambda|}\,\frac{1}{(1+\frac{\alpha}{24\pi m})} & p \ll m \end{cases} \tag{26}$$

which in turn shows that $\mu(0,0,N) = 0$, irrespective of the order of limits. But for $p \gg \alpha, \mu_R(p^2,0,N) \neq 0$. Hence, as long as N is odd and $\mu(p^2,m^2,N) \neq 0, \mu_R(p^2,0,N) \neq 0$ for all p even if $\mu_R(0,0,N) = 0$. Thus, even in the large N limit, parity is broken because of the topological Ward identity.

In conclusion, we would like to restate our results - there is no parity anomaly in QED_3,whereas in QCD_3 there is an anomaly due to the conflict between gauge invariance under large gauge transformations and the parity symmetry. The QED and QCD theories can be solved exactly in the large N limit for the mass of the photon and the gluon and though there are some differences from the perturbative case, the results as stated above do not change. However, there are several technical questions concerning the physical picture of regularisation ambiguities and the infra-red, ultra-violet connection that are still under investigation.

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