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Skyrmions and Vector Mesons*

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ABSTRACT

We discuss the scattering of mesons of arbitrary spin and isospin from baryons in models in which the baryon is considered a soliton, or "skyrmion," in an effective Lagrangian of mesons. Model-independent linear relations between partial-wave amplitudes are derived for $\pi N \to \rho N$ and $\pi N \to \omega N$, and, where possible, are compared with experiment.

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The notable success of Skyrme model calculations has lent credence to the picture of the nucleon as a soliton, or "skyrmion," in the nonlinear sigma model of pions.[1,2] Of course, if this picture is at all sensible, we ought to expect that the more realistic the theory of mesons that we start with, the more accurate our predictions of baryonic properties will be.[3] In constructing a realistic theory, the most important modification to consider is the incorporation of additional low-lying mesons into the effective Lagrangian.

Work along these lines is just beginning. In the "ω-stabilized" Skyrme model of Adkins and Nappi,[4] one introduces a coupling $\omega_\mu B^\mu$ between the ω meson and the topological current of the theory. This coupling, which accounts for the decay $\omega \rightarrow 3\pi$, turns out to be sufficient to guarantee a stable soliton. Pleasingly, the static properties of the nucleon in this model constitute an improvement over the unadulterated Skyrme model. One can likewise construct stable solitons when the Lagrangian includes ρ-mesons, although the static properties of this model are as yet undetermined.[5]

In this paper we give a blueprint for calculating 2-body scattering amplitudes in models in which the skyrmion is coupled to an arbitrary number of different species of mesons. The processes we will focus on will be of the type

$$\phi B \longrightarrow \psi B'$$

where ϕ and ψ stand for generic mesons of arbitrary spin, isospin and parity,[♯1] and B and B' denote either a nucleon or a Δ. The treatment will be on a general level; we will not specify a Lagrangian. Nevertheless, as we shall see, the soliton picture

♯1 In fact, they need not be mesons at all, but might, for instance, represent a quark interacting with a skyrmion in a "hybrid" model of quarks and pions.

of baryons already implies nontrivial linear relations between these partial-wave scattering amplitudes, the experimental validity of which can be regarded as direct tests of the skyrmion approach to baryon physics. As a special case, we shall recover the results for $\pi N \to \pi N$ and $\pi N \to \pi\Delta$ familiar from Refs. 6-8. As in these papers, our results will be valid only to leading order in $1/N_c$, where N_c is the number of colors of the underlying strong interaction gauge group; the reader is referred to Sec. II of Ref. 6 for a discussion of this approximation.

Our fundamental assumption will be that the effective meson theory admits a soliton solution which is a singlet under the simultaneous action $\mathbf{I}+\mathbf{J}$ of isospin and angular momentum. Such is the case in the usual Skyrme model, where the skyrmion is a "hedgehog" configuration:

$$U_0 = \exp\big(iF(r)\hat{\mathbf{r}}\cdot\vec{\sigma}\big)\,. \tag{1}$$

This can be thought of as the pion field's having acquired a spatially varying vacuum expectation value

$$< \pi^a(x) >= \frac{f_\pi}{2}F(r)\hat{x}_a\,. \tag{2}$$

In a more general Lagrangian, there is no reason for the skyrmion to confine itself entirely to the pion field. For example, in the model of Adkins and Nappi,[4] the (non-propagating) time component of the ω likewise acquires a VEV:

$$< \omega_0(x) >= G(r)\,, \quad < \omega_i(x) >= 0. \tag{3}$$

Similarly, in the ρ-stabilized model of Ref. 5, the ρ field is characterized by

$$< \rho_0^a(x) >= 0\,, \quad < \rho_i^a(x) >= \epsilon_{ija}\hat{x}_j H(r)/r\,. \tag{4}$$

Note that Eqs. (3) and (4) also satisfy the fundamental assumption stated above.

Eqs. (1)-(4) typify what we shall refer to as a skyrmion in its canonical (*i.e.*, unrotated) orientation. Of course, by isospin invariance, one can construct a family of degenerate soliton solutions by rotating the canonical configuration through an angle $A \in SU(2)$, *viz* :

$$\begin{aligned} < \pi^a(x) > &\longrightarrow \mathcal{D}^{(1)}(A)_{ab} \frac{f_\pi}{2} F(r) \hat{x}_b \, ; \\ < \rho_i^a(x) > &\longrightarrow \mathcal{D}^{(1)}(A)_{ab} \epsilon_{ijb} \hat{x}_j H(r)/r \, . \end{aligned} \tag{5}$$

In fact, as shown by Adkins *et al.*,[2] and invoked below, the proper identification of the nucleon and the Δ requires a superposition of the soliton states corresponding to all such orientations, weighted by appropriately constructed wavefunctions $\chi(A)$.

Nevertheless, let us forget for the moment about the existence of these degenerate configurations, and focus exclusively on the soliton in its canonical orientation. We are thus (temporarily) interested in studying the two-point function $< \psi_j^b \phi_i^a >_0$ representing the simplified process

$$\phi U \longrightarrow \psi U,$$

where U stands for "unrotated skyrmion." The upper and lower indices on the meson fields denote isospin and spin, respectively; the mesons will be assumed to be in a representation $I_{\phi,\psi}$ of isospin and $S_{\phi,\psi}$ of spin. The nought on the propagator will remind us that the skyrmion is in its canonical orientation.

The key to our results is the observation that the vectorial sum $\mathbf{K} = \mathbf{S}_\phi + \mathbf{L} + \mathbf{I}_\phi$ of the meson's angular momentum and isospin will be conserved in such a process. (This will no longer be true when we consider scattering, not from unrotated skyrmions, but from nucleons and/or Δ's.) This conservation law is a direct consequence of our fundamental assumption. Consequently, the natural thing to do is to expand the meson field in eigenstates of $\mathbf{K}^2$ and K_z, which we can imagine doing as follows: First, ϕ and ψ are expanded in partial waves $|L, M>$ and $|L', M'>$, respectively. Orbital angular momentum is then added to isospin to form states $|\widetilde{K}, \widetilde{K}_z>$ and $|\widetilde{K}', \widetilde{K}'_z>$, where $\widetilde{\mathbf{K}} = \mathbf{L} + \mathbf{I}_\phi$ and $\widetilde{\mathbf{K}}' = \mathbf{L}' + \mathbf{I}_\psi$. This hybrid angular momentum is, in turn, added to the meson's spin to form states $|K, K_z>$ and $|K', K'_z>$. The symmetry of the unrotated skyrmion then implies $K = K'$ and $K_z = K'_z$. We thus have:

$$
\begin{aligned}
< \psi^b_j(x')\phi^a_i(x) >_0 = & \sum_{LML'M'} < \psi(x')|L'M' >< LM|\phi(x) > \\
& \times \sum_{\widetilde{K}\widetilde{K}_z\widetilde{K}'\widetilde{K}'_z} < \mathbf{L}'\mathbf{I}_\psi|\widetilde{\mathbf{K}}' >< \widetilde{\mathbf{K}}|\mathbf{L}\mathbf{I}_\phi > \sum_{KK_z} < \widetilde{\mathbf{K}}'\mathbf{S}_\psi|\mathbf{K} >< \mathbf{K}|\widetilde{\mathbf{K}}\mathbf{S}_\phi > \\
& \times G_{K\widetilde{K}\widetilde{K}'LL'}(\phi U \rightarrow \psi U)
\end{aligned}
\tag{6}
$$

where $< \widetilde{\mathbf{K}}|\mathbf{L}\mathbf{I}_\phi >$ is shorthand for the Clebsch $< \widetilde{K}\widetilde{K}_z L I_\phi | L I_\phi M a >$, *etc.* The Green's function G is the "reduced" amplitude for the process; apart from its indices, it depends only on energy.

It is easy to generalize this formula to the case when the skyrmion, instead of being in its canonical orientation, has been rotated through an angle A, as in Eq. (5). By isospin invariance, the 2-point function simply becomes

$$
< \psi^b_j \phi^a_i >_0 \longrightarrow < \psi^b_j \phi^a_i >_A = \mathcal{D}^{(I_\psi)}(A)_{bd} < \psi^d_j \phi^c_i >_0 \mathcal{D}^{(I_\phi)}(A)^\dagger_{ca} \,. \tag{7}
$$

We are finally in a position to consider the *physical* scattering process $\phi B \longrightarrow \psi B'$, which requires a superposition of all values of A. The relevant expression for the 2-point function is naturally given by

$$\int\limits_{SU(2)} dA\, \chi'(A)^\dagger < \psi_j^b \phi_i^a >_A \chi(A) \tag{8}$$

where $\chi(A)$ and $\chi'(A)$ are the wavefunctions appropriate to B and B', respectively. The A-integration in Eq. (8) can, in fact, be carried out in closed form,[6] thanks to the explicit expression for the wavefunctions[♯2]

$$\chi^R_{i_z s_z}(A) = \frac{i}{\pi}\sqrt{\tfrac{1}{2}(2R+1)}\left(\epsilon^{(R)}\mathcal{D}^{(R)}(A)^\dagger\right)_{s_z i_z} \tag{9}$$

and to the identities

$$\mathcal{D}^{(R_1)}(A)_{ab}\mathcal{D}^{(R_2)}(A)_{cd} = \sum_{\widetilde{R}} \mathcal{D}^{(\widetilde{R})}(A)_{a+c,b+d} \quad \times$$
$$< R_1 R_2 ac | \widetilde{R}, a+c, R_1 R_2 >< \widetilde{R}, b+d, R_1 R_2 | R_1 R_2 bd > \tag{10}$$

and

$$\int dA\, \mathcal{D}^{(R_1)}(A)_{ab}\mathcal{D}^{(R_2)}(A)^\dagger_{cd} = \frac{2\pi^2}{2R_1+1}\,\delta_{R_1 R_2}\delta_{bc}\delta_{ad}. \tag{11}$$

To compare with experiment, some further massaging is in order. We first restrict the incoming and outgoing mesons to partial waves L and L', respectively. The initial and final meson-baryon systems are then projected onto states $|I_{\rm tot} I_{{\rm tot}z} J_{\rm tot} J_{{\rm tot}z} S_{\rm tot}>$ and $|I'_{\rm tot} I'_{{\rm tot}z} J'_{\rm tot} J'_{{\rm tot}z} S'_{\rm tot}>$ of definite total isospin, angular momentum, and spin. Together with Eqs. (6) and (10), this projection leaves

♯2 The index R in Eq. (9) gives the spin/isospin representation of the baryon, *i.e.*, $R = \frac{1}{2}$ for nucleons and $\frac{3}{2}$ for Δ's.

us with a product of 14 Clebsches! Fortunately, upon summation, our expression simplifies enormously, and we find:

$$< \psi\phi >_{\rm physical} = \delta_{I_{\rm tot}I'_{\rm tot}}\delta_{I_{{\rm tot}z}I'_{{\rm tot}z}}\delta_{J_{\rm tot}J'_{\rm tot}}\delta_{J_{{\rm tot}z}J'_{{\rm tot}z}} \sum_{K\widetilde{K}\widetilde{K}'} \eta\eta' G_{K\widetilde{K}\widetilde{K}'LL'}(\phi U \to \psi U). \tag{12}$$

Here η and η' are group-theoretic coefficients characterizing the entering and exiting channels, respectively; they are given in terms of $9j$-symbols by:

$$\eta = [(2K+1)(2\widetilde{K}+1)(2R+1)(2S_{\rm tot}+1)]^{\frac{1}{2}} \begin{Bmatrix} L & I_\phi & \widetilde{K} \\ S_{\rm tot} & R & S_\phi \\ J & I & K \end{Bmatrix} \tag{13a}$$

and

$$\eta' = [(2K+1)(2\widetilde{K}'+1)(2R'+1)(2S'_{\rm tot}+1)]^{\frac{1}{2}} \begin{Bmatrix} L' & I_\psi & \widetilde{K}' \\ S'_{\rm tot} & R' & S_\psi \\ J & I & K \end{Bmatrix} . \tag{13b}$$

As a reassuring check on our formalism, note that conservation of isospin and angular momentum has emerged in the Kronecker-δ's of Eq. (12).

These expressions generalize the formalism for $\pi N \to \pi N$ and $\pi N \to \pi\Delta$ presented in Refs. 6-8 to the case when the initial and/or final meson has nonzero spin. Recall that the comparable expression for pions involves $6j$-symbols. Indeed, if one plugs $S_\phi = S_\psi = 0$ and $I_\phi = I_\psi = 1$ into (13), then the $9j$-symbols collapse into $6j$-symbols, and we recover the previous formula.

Eq. (12) embodies a neat separation of dynamics and group theory, represented by the G's and η's, respectively. To make maximum use of the formula, one should write down an effective Lagrangian, solve for the soliton, and extract the G's numerically from a phase-shift analysis. (Such is the approach of Refs.

7-9 and Ref. 10 for the special case of elastic πN scattering in the 2- and 3-flavor Skyrme models, respectively.) This would enable us to compare theory to experiment for a wide variety of 2-body processes, and would be a crucial test of the more "realistic" skyrmion models currently being constructed.

Alternatively, one can derive *model-independent* relations by finding those linear combinations of physical amplitudes for which the right-hand side of (12) precisely cancels out. These relations for $\pi N \to \pi N$ and $\pi N \to \pi\Delta$ were the subject of a detailed analysis in Ref. 6; on the whole, they were surprisingly well obeyed by the experimental partial-wave amplitudes. In the remainder of this paper, we shall concentrate on the model-independent predictions of Eq. (12) for the processes $\pi N \to \rho N$ and $\pi N \to \omega N$.

Let us denote the independent amplitudes for this process by the conventional notation $\rho_{2S_{\rm tot}}(LL')_{2I_{\rm tot},2J_{\rm tot}}$. When $L' = L \pm 2$, Eq. (12) can be shown to imply a simple proportionality between the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ amplitudes in the same partial wave:

$$\begin{aligned} \rho_3(L, L+2)_{3,2L+1} &= -\tfrac{1}{2}\rho_3(L, L+2)_{1,2L+1}\,; \\ \rho_3(L, L-2)_{3,2L-1} &= -\tfrac{1}{2}\rho_3(L, L-2)_{1,2L-1}\,. \end{aligned} \tag{14}$$

Note that these are *energy-independent* relations. In this respect, they go well beyond predictions based on traditional algebraic coupling schemes such as $SU(6)$, which apply only at resonant energies.

Similarly, for the more complicated case where $L' = L$, Eq. (12) implies that the four independent isospin-$\frac{3}{2}$ amplitudes can be expressed as linear combinations of the four isospin-$\frac{1}{2}$ amplitudes. We find:

$$\begin{bmatrix} \rho_1(LL)_{3,2L-1} \\ \rho_1(LL)_{3,2L+1} \\ \rho_3(LL)_{3,2L-1} \\ \rho_3(LL)_{3,2L+1} \end{bmatrix} = \frac{1}{4L+2} \begin{bmatrix} -\alpha^2 & -\beta^2 & -\beta\gamma & -\alpha^{-1}\beta^2\delta \\ -\alpha^2 & -\beta^2 & \alpha^2\beta^{-1}\gamma & \alpha\delta \\ -\beta\gamma & \beta\gamma & -2 & \alpha^{-1}\beta\gamma\delta \\ -\alpha\delta & \alpha\delta & \alpha\beta^{-1}\gamma\delta & 2 \end{bmatrix} \begin{bmatrix} \rho_1(LL)_{1,2L-1} \\ \rho_1(LL)_{1,2L+1} \\ \rho_3(LL)_{1,2L-1} \\ \rho_3(LL)_{1,2L+1} \end{bmatrix} \tag{15}$$

with $\alpha = \sqrt{L}$, $\beta = \sqrt{L+1}$, $\gamma = \sqrt{2L-1}$, and $\delta = \sqrt{2L+3}$.

Figure 1 illustrates Eqs. (14) and (15) as applied to the experimental partial-wave $\pi N \to \rho N$ amplitudes, drawn from the recent comprehensive analysis by Manley *et al.*[11] The channels depicted are, unfortunately, the only ones for which sufficient experimental data is currently available for comparison. One should bear in mind that the curves do not represent the data directly, but result from a delicate, model-dependent analysis of the $\pi\pi N$ final state. As such, they should not be taken as definitive.[♯3] Although the shapes of the curves in the left- and right-hand columns are not in particularly good correspondence, the agreement in the *signs* of the appreciably-coupled resonances (*i.e.*, whether the curve lies in the upper or lower half of the circle at a resonance energy) is completely nontrivial. This in itself can be regarded as quite promising—especially when contrasted to the fact that $SU(6)$ (in both its "unbroken" and "l-broken" versions) makes several incorrect (relative) sign predictions for the resonances shown.[11]

Of course, Eq. (12) implies similar model-independent linear relations that are straightforward to derive for such experimentally accessible processes as $\pi N \to \rho\Delta$, $\pi N \to fN$, and $\pi N \to \omega N$. For example, in the latter case, one

♯3 In particular, the large "tail" on the SD_{31} curve actually exceeds the unitarity bounds imposed by the elastic partial-wave amplitudes (*cf.* Fig. 3 of Ref. 11); similarly for the FP_{35}.

can show:

$$\begin{aligned}\omega_1(LL)_{1,2L-1}-&\omega_1(LL)_{1,2L+1}\\&=\sqrt{\frac{2L-1}{L+1}}\cdot\omega_3(LL)_{1,2L-1}+\sqrt{\frac{2L+3}{L}}\cdot\omega_3(LL)_{1,2L+1}\end{aligned}\tag{16}$$

When reliable experimental low-energy partial-wave data for such processes become available, they will constitute further important tests of the skyrmion approach to hadron physics.

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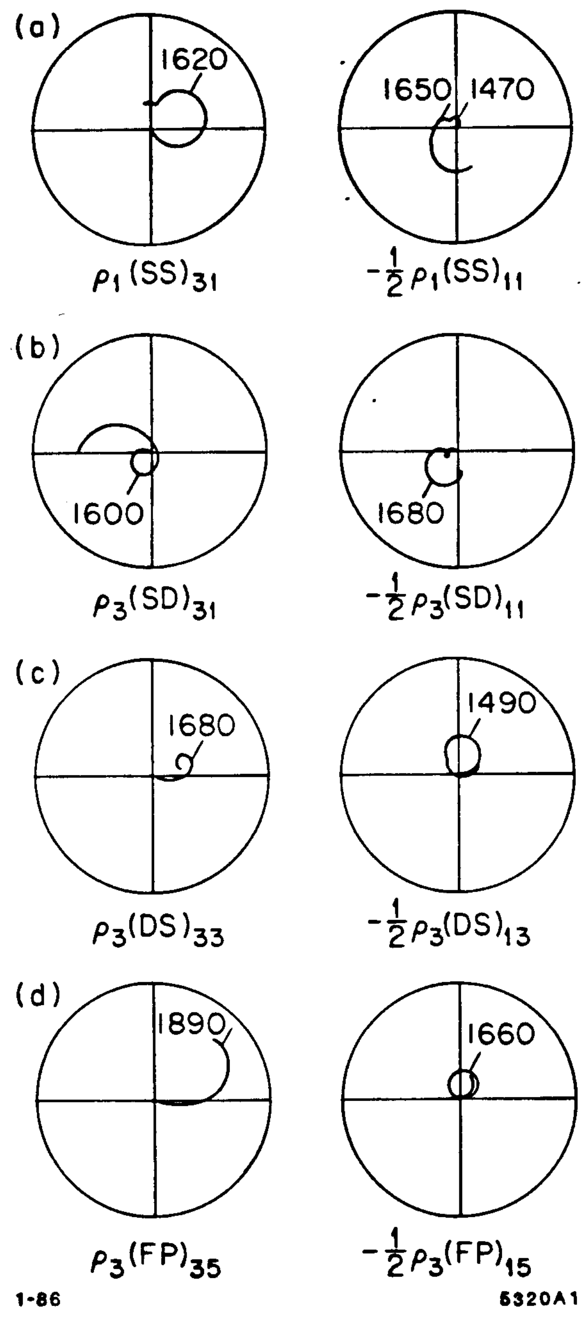

Fig. 1. Comparison of experimental isospin-$\frac{3}{2}$ $\pi N \rightarrow \rho N$ amplitudes (left-hand column) with the appropriate multiple of the experimental isospin-$\frac{1}{2}$ amplitudes (right-hand column) to which they should correspond to leading order in $1/N_c$. The indicated resonance masses are derived from peaks in the partial-wave cross-sections (Fig. 4 of Ref. 11). The energy range is from threshold to 1930 MeV.