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Non-Perturbative Infinities*

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ABSTRACT

We investigate non-perturbative contributions to ultra violet divergences in several field theories. Although we concentrate on finite theories with extended supersymmetry, we also present some results for asymptotically free theories. The N=2 supersymmetric σ models that we study at the end of the paper are important in the theory of superstrings. Unfortunately, while we are able to rule out instanton contributions to the β functions of these theories, we do not have a complete non-perturbative proof that they are conformally invariant. Sigma models with N=4 supersymmetry are shown to be both perturbatively and non-perturbatively finite.

Ultraviolet properties of renormalizable field theories are usually studied in perturbation theory. The modern theory of the renormalization group[1] justifies this treatment for asymptotically free theories and invalidates it for field theories defined by non-Gaussian fixed points. There remains an interesting class of theories for which the question of non-perturbative contributions to ultraviolet quantities remains unresolved. These are the so called finite theories in which coupling constant renormalization is absent to all orders in perturbation theory. In this note we will present some non-perturbative results about the ultraviolet behavior of such finite theories. We will show that similar questions arise (and can be resolved) in asymptotically free theories when we discuss the ultraviolet behavior of quantities which vanish to all orders in perturbation theory. We note that Wallace[2] has discussed non-perturbative infinities in the context of models in statistical mechanics.

Historically, the first example of a non-perturbative divergence is the divergence of the θ dependent part of the vacuum energy of the $O(3)$ non-linear σ model in two dimensions.[3] This is the paradigm for the effects we will discuss. The quantity in question vanishes to all orders in perturbation theory, and thus its leading non-perturbative contribution (for small coupling) is well defined and does not depend on how we choose to sum the perturbation series.[4] Furthermore, since the theory is asymptotically free we may hope to compute this divergent quantity exactly in the dilute instanton gas approximation.Monte Carlo calculations[5] show that this hope is justified and the deviation from the dilute gas behavior is ultra violet finite.

While the θ dependent vacuum energy is not a terribly interesting quantity, one may speculate[6] that similar non-perturbative divergences create an anomaly in the higher conservation laws which make the model completely integrable. This would explain why there is no Zamolodchikov S-matrix[7] for the σ model at non-zero θ.

A more interesting example of a calculable non-perturbative ultraviolet divergence can be constructed in four dimensions. Consider Quantum Chromodynamics coupled to a single massless fermion and a set of massive scalars. The fermion mass is zero to all orders in perturbation theory, but instantons give a non-zero contribution to it of the form*

$$m \propto \int_0^{\rho_c} \frac{d\rho}{\rho^2} e^{b_0 ln(\mu\rho)}$$

where

$$b_0 = \frac{33}{16\pi^2} - S$$

S is proportional to the number of scalar SU(3) multiplets, weighted by their Dynkin index. Clearly we can choose the scalars so that the integral diverges at $\rho = 0$ even though the theory is asymptotically free. Thus a non-perturbative renormalization of the fermion mass is necessary in this model. If a non-zero bare mass is present, it diverges logarithmically in perturbation theory while the non-perturbative divergence is stronger - power like. It is clear that this example is typical of a large class of similar phenomena. We mention in passing that this sort of mechanism may invalidate the axion solution of the strong CP problem if QCD is insufficiently asymptotically free at high energies.[8]

We now turn to the main subject of this note, a discussion of non-perturbative divergences in two dimensional σ models with extended supersymmetry (SUSY). We first discuss the N=4 models and argue that they have no non-perturbative divergences. We then proceed to N=2 models where our results are much less complete. We show that instantons do not produce infinities in the dilute gas approximation, but we are unable to make any sort of argument about other non-perturbative effects. This is a pity, for it is the conformally invariant N=2 σ models which provide candidate vacua for superstrings.[9]

⋆ We have inserted an arbitrary infrared cutoff in this expression, which does not affect its ultraviolet behavior. A physical infrared cutoff automatically appears if we study the system at high temperature.

We do not pretend to be able to make a true non-perturbative construction of models with N=4 SUSY. Thus our "proof" of the absence of non-perturbative infinities in these theories will rely on certain plausible assumptions about properties of their exact solutions. Firstly we assume that N=4 SUSY is a true symmetry of the model. In particular, if we consider two of the supercharges (say Q^m_α) then the energy momentum tensor $\theta_{\mu\nu}$ is in an N=2, Q^m_α-supermultiplet with an axial vector current J^5_μ. Tracelessness of $\theta_{\mu\nu}$ is equivalent to conservation of J^5_μ.

In addition, we assume that the SU(2) symmetry which rotates the four supercharges into each other is a symmetry of the exact theory. This symmetry is vectorlike and unlikely to suffer from anomalies. There exist non-perturbative regulators which preserve it (the same cannot be said for SUSY).

In the classical limit, the current J^5_μ is part of a non-abelian extension of the supercharge automorphism group. The larger group is $SU(2)XSU(2)$ and it is chiral. The vectorlike group is the diagonal subgroup of $SU(2)XSU(2)$. J^5_μ is thus part of a vector multiplet of SU(2). The crux of our argument is the claim that the non-abelian symmetry guarantees that J^5_μ is anomaly free. Sohnius and West[10] have given a similar argument for N=4 gauge theories in four dimensions.

The Lagrangian of the theory is

$$\mathcal{L} = \frac{1}{2} g_{mn}(\phi)\partial^\mu \phi^m \partial_\mu \phi^n + \frac{i}{2} g_{mn} \psi^m \gamma^0 \gamma^\mu D_\mu \psi^n + \frac{1}{12} R_{mnkl} \psi^m \gamma^0 \psi^k \psi^n \gamma^0 \psi^l$$

$$D_\mu \psi^m = \partial_\mu \psi^m + \Gamma^m_{nk} \partial_\mu \phi^n \psi^k$$

ϕ^m are coordinates on the manifold, which is assumed to be Hyper-Kahler, and have 4n real dimensions. Thus, the tangent space indices have a natural splitting of the form (a, A) where A is 2n dimensional and a is an SU(2) doublet index. In this basis the curvature tensor has the form $\epsilon_{ab}\epsilon_{cd}\Omega_{ABCD}$ We will use fermion fields adapted to this basis. ψ^{aA} is for each A and a a one component left moving Weyl spinor. χ^{aA} is the corresponding right moving spinor. The $SU(2)XSU(2)$

group consists of separate SU(2) rotations on ψ and χ. The scalar fields are singlets. In terms of ψ and χ the four fermion interaction is

$$\epsilon_{ab}\epsilon_{cd}\Omega_{ABCD}\psi^{aA}\psi^{bB}\chi^{cC}\chi^{dD}$$

This is invariant under $SU(2)XSU(2)$, and it is easily verified that the fermion kinetic term is as well.

We have assumed that N=4 SUSY and the vectorlike SU(2) symmetry are properties of the exact theory. This implies that finiteness of the theory is equivalent to conservation of the axial current J^5_μ Furthermore this current is part of an SU(2) vector multiplet. We will now argue that the divergence of this current is zero. First we claim that the divergence must be an operator of dimension two. To prove this we will have to make one more assumption, namely that if the theory is not finite, it is at least asymptotically free. Any operator of dimension other than two in the current divergence must be multiplied by an appropriate power of the renormalization group invariant mass scale M. A negative power of M (corresponding to an operator of dimension greater than two) would imply that the divergence did not vanish as the coupling went to zero, which is inconsistent. A positive power of M would imply that the divergence (and thus the β function) vanished like $exp-(\int \frac{dx}{\beta(x)})$. This consistency condition for the β function at small coupling is satisfied only if it is linear in the coupling, a possibility which is ruled out by explicit perturbative calculations.

We can now classify all dimension two operators that could appear in the divergence of the axial current. There are operators with no fermions, two derivatives and any number of bose fields, operators with two fermions, one derivative, and any number of bose fields, and operators with four fermions, no derivatives and any number of bose fields. Note that Lorentz invariance requires that the two fermion terms contain only left movers, or only right movers, while the four fermion terms must contain two left movers and two right movers. It is therefore convenient to classify bilinears in purely left moving or right moving fields. The product of two left moving fermions is a linear combination of the operators

$$V_i^{AB} = \psi^{aA}\psi^{bB}\sigma_i\sigma_2{}^{ab}$$

and

$$S^{AB} = \psi^{aA}\psi^{bB}\epsilon^{ab}$$

The corresponding operators for right movers will be called W_i^{AB} and T^{AB}.

The basic tool that we will use to restrict the possible combinations that can occur in the divergence of J_5^μ is the Wess-Zumino consistency condition[11] These conditions can be stated in the following operator form. If T^m are the generators of an internal symmetry group, and D^m are the divergences of the corresponding currents, then:

$$[T^m, D^n] - [T^n, D^m] = f^{mnk} D^k$$

where f^{mnk} are the structure constants of the group. These relations follow from the usual postulates of current algebra and are valid in any local quantum field theory. In the case at hand the group is $SU(2)XSU(2)$ and the diagonal currents have zero divergence. The relations then say that the divergence of the axial current is an SU(2) vector, and that the commutator of the divergence of the axial current with the axial charge is symmetric in its SU(2) vector indices.

It is now easy to see that terms in the divergence of J_5^μ which have no fermi fields in them must be absent because they cannot transform like vectors. Furthermore, the two fermion terms, which must be proportional to V and/or W are also absent, since the commutator of the axial charge with V or W is antisymmetric in SU(2) indices. There are three types of possible four fermion terms, schematically: VW,SW, and VT. The SW and VT terms transform like V and W under axial transformations and so these terms must be absent. The only remaining term is:

$$D^i = \epsilon^{ijk} V_j^{AB} W_k^{CD} F_{ABCD}$$

where F_{ABCD} is an arbitrary function of bose fields. The commutator of this with the axial charge T^i is indeed symmetric in i and j. We will need another argument to eliminate this term. Note that if the axial charge was a conserved operator then this term could not appear in the axial current divergence. The divergence of the axial current should be (3,1)+(1,3) under $SU(2)XSU(2)$ while the VW term transforms as (3,3). This argument would be sufficient to rule out a divergence for the axial current in perturbation theory, where we would expect the anomaly to be a total divergence, and not to violate conservation of the axial charge. In order to prove that there is a conserved axial charge without recourse to perturbation theory we will have to be a bit more specific about how we propose to define the theory

The Lagrangian that we have written contains four fermi interactions. We can introduce auxiliary fields to make the Lagrangian quadratic in fermi fields. This can be done in various ways, all of which share the property that the auxiliary fields are $SU(2)XSU(2)$ singlets. Now define the theory by some regularization of the functional integral of the Lagrangian with auxiliary fields. Assume that the regulator preserves the vectorlike diagonal SU(2) subgroup, and consider the divergence of the axial current. For fixed values of the external fields the divergence of the current depends only on the external fields. But all of the Bose fields are singlets under the vectorlike SU(2) while the divergence is an SU(2) vector. Thus the divergence vanishes. Note that it vanishes for finite values of the cutoff, not just in the limit as the cutoff goes to infinity. It therefore vanishes even after integration over the Bose fields.

We thus see that there are many regularizations of the functional integral which preserve the axial symmetry. We must assume that the theory with N=4 SUSY can be defined as the limit of a regularized theory of this sort. The regulator will probably not preserve SUSY, but we can recover a supersymmetric limit by adding appropriate counterterms to the action. We assume that this can be done without violating the vectorlike SU(2). Furthermore we assume that the necessary counterterms do not explicitly violate SU(2)XSU(2). That is, possible violations of this axial symmetry are assumed to arise, as usual, only from regularization of the fermion determinant. SU(2) invariance then shows that such violations do not occur. It is thus very plausible that the theory with N=4 SUSY has a conserved multiplet of axial currents.

Of course, since the above method of constructing the theory makes SUSY very obscure, we have no way of knowing whether the conserved current we have constructed is in a multiplet with the energy momentum tensor. However, it gives us a conserved chiral charge with which to classify operators which can appear in the divergence of the axial current which *is* in the supermultiplet of the energy momentum tensor. This chiral charge generates the $SU(2)XSU(2)$ rotations on ψ and χ, and it is sufficient (as we have seen above) to show that the divergence of the superpartner of the energy momentum tensor is zero. This implies that the theory is finite. Note that previous proofs of finiteness of this theory[12][13] eliminated perturbative infinities only and assumed that the hyper-Kahler manifold is compact.

Field theories with exact conformal invariance are particularly interesting in two dimensions, where they enable one to construct classical solutions of string theories.[14] However, models with N=4 SUSY do not lead to particularly promising string phenomenology. N=2 models with vanishing Ricci tensor, however, appear to be the most likely candidates for string theories of the real world. Unfortunately, the arguments we have presented so far do not apply when N=2. The non-abelian symmetry groups which played such a prominent role in our argument, are replaced by abelian groups. These theories were recently shown to be perturbatively finite.[15] Here we will show that the leading non-perturbative effect is ultra violet finite.

If we assume that these models make sense as interacting quantum field theories, they are either finite or asymptotically free. We can therefore use weak coupling techniques to study their short distance behavior. The leading non-perturbative effect is then found in the semi-classical approximation. Since Π_2 is non-trivial for Kahler manifolds, these models have instantons. For weak coupling they can be treated in the dilute gas approximation. Our examples of non-perturbative infinities teach us that if the perturbative beta function vanishes, the integral over the instanton scale size might diverge in the ultra violet. It might then lead to a non-zero beta function. We will now show that due to the presence of fermionic zero modes, the ρ integral converges in the ultra violet. No infinities are generated to all perturbative orders around the instanton.

Let us compute the instanton contribution to the coefficient of a term in the effective action. The coefficient of a term of dimension δ has dimension $2-\delta$. Since the perturbative beta function vanishes, the integral over scale size behaves in the ultra violet as

$$\int \frac{d\rho}{\rho^{-3+\delta}}$$

It converges for $\delta > 2$ and diverges for $\delta \leq 2$. We will show that $\delta > 2$.

We should first determine the number of fermionic zero modes.[16] In general, the axial anomaly $\partial^\mu J^5_\mu \propto R_{mn}$ gives us an index theorem. In our case $R_{mn} = 0$ and hence $\Delta Q_5 = 0$ (Q_5 is the axial charge) in the background of the instanton and one might think that there are no fermionic zero modes at all. A closer analysis shows that there are at least eight fermionic zero modes (four left movers and four right movers). If the instanton is neither self-dual nor anti-self-dual, these eight zero modes are generated by applying the four SUSY charges and the four superconformal charges on the classical solution.

If the instanton is self-dual or anti-self-dual (holomorphic or anti- holomorphic) only four zero modes are generated this way. Two of the SUSY charges and two of the superconformal charges annihilate the classical solution.* These four zero modes contribute $\Delta Q_5 = 4$. Since the index theorem demands $\Delta Q_5 = 0$, at least four more zero modes should exist (there may, of course, be more than these eight zero modes). The occurrence of more zero modes than are implied by the index theorem is familiar from the example of SUSY QCD.[18] We conclude that to leading order instantons contribute to the coefficient of an operator with at least eight fermions and possibly some derivatives. For such an operator $\delta \geq 4$ and the ρ integral is clearly ultra violet finite. It is interesting to note that unlike other cases, this operator is invariant under all the classical symmetries of the Lagrangian - $U(1)XU(1)$. Therefore its coefficient is generically non-zero already in perturbation theory.

One might worry that since there is no index theorem here, some of these zero modes might be tied together by loops - higher order perturbative corrections around the instanton. Then instantons might contribute to the coefficient of lower dimension operators and ultra violet divergences might arise. This does not happen. Two SUSY collective coordinates can be introduced for the two SUSY zero modes. Since the theory is perurbatively finite, two superconformal collective coordinates can also be introduced (because before the ρ integral is performed the theory is conformally covariant to all perturbative orders around the instanton). The "index" theorem $\Delta Q_5 = 0$ then guarantees the existence of the other zero modes. For instantons which are not self-dual these other zero modes are also related to SUSY and collective coordinates can be introduced for them as well. We conclude that to all (perturbative) orders around the instanton we do not lose the zero modes. Instantons contribute only to terms in the effective action with very high dimension and their contribution is finite.

⋆ A similar phenomenon happens in SUSY gauge theory in four dimensions. For a discussion on the two dimensional σ model in a particular case see e.g.[17]

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