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PARTICLE CREATION IN INHOMOGENEOUS SPACETIMES*

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ABSTRACT

We study the creation of particles by inhomogeneous perturbations of spatially flat Friedmann-Robertson-Walker cosmologies. For massive scalar fields, the pair creation probability can be expressed in terms of geometric quantities (curvature invariants). The results suggest that inhomogeneities on scales up to the particle horizon will be damped out near the Planck time. Perturbations on scales larger than the horizon are explicitly shown to yield no created pairs. The results generalize to inhomogeneous spacetimes several earlier studies of pair creation in homogeneous anisotropic cosmologies.

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1. Introduction

In recent years, the study of quantum fields in curved spacetime[1] has had a profound impact on our understanding of cosmology. It is now recognized that both the effects of curvature on quantum fields and the influence of quantum field dynamics on the metric are likely to be important in determining the evolution of the early universe. Most recent work on the subject has concentrated on the dynamics of interacting gauge theories in curved space, with particular attention to such issues as asymptotic freedom, symmetry restoration, and the possibility of inflation.[2]

Yet, one of the most remarkable results in the subject remains Parker's discovery almost twenty years ago that the expansion of the universe can create pairs of particles. Parker's work[3] focused on particle production in the homogeneous, isotropic Friedmann-Robertson-Walker (FRW) models. In addition to establishing the possibility of pair creation, he showed that fields obeying conformally invariant wave equations (e.g., two-component neutrinos, massless Dirac particles, and photons in four dimensions) will not be produced, because the FRW models are conformally flat. Subsequently Zel'dovich and Starobinsky[4] considered particle creation in a broader class of homogeneous cosmologies and found that conformally invariant particles will be produced when the conformal symmetry of the FRW models is broken by anisotropy.

In this paper, we extend this work by considering the production of scalar particles due to inhomogeneous perturbations of conformally flat spacetimes. This calculation is of cosmological interest because, if inhomogeneity is present in the universe near the Planck time, it can act as an efficient source of relativistic particles; in particular, it may contribute significantly to the observed entropy of the microwave background and thus help explain the origin of the hot matter in the early universe. The possibility of particle creation is readily understood in field theoretic terms: whenever a quantum field couples to a classical time-dependent source, the breaking of time-translation invariance implies that the field energy need not be conserved; as a consequence, particles can be created. Thus, fields in a time-dependent inhomogeneous background should be excited. (Time-independent sources can create particle as well; however, see the discussion on this point in Section 4.) For weak inhomogeneities in a flat Minkowski background, the particle creation rate is negligible because the energy in the gravitational field is small. In the cosmological case, energy is provided by the expansion of the universe, while the inhomogeneity serves to break conformal symmetry.

Throughout, we shall work entirely in the external field approximation, that is, we take the classical perturbed metric to be given and study the production of matter fields in this fixed background. This is analogous to the usual treatment of Coulomb scattering in quantum electrodynamics (in which the vector potential is fixed) and is believed to be a consistent truncation of the theory when the backreaction of the quantum fields on the geometry is small. Whether it is a good approximation in considering particle creation in the very early universe is more doubtful. The work of many authors[4,5] on particle creation and vacuum polarization in homogeneous cosmological models shows that the backreaction can dramatically alter the evolution of the universe. In particular, any initial anisotropy in the expansion is rapidly damped out on the order of the Planck time. Parker has used these results to postulate a 'quantum gravitational Lenz's law' which states that "the reaction of the particle creation back on the gravitational field will modify the expansion in such a way as to reduce the creation rate".[6] This behavior is intuitively plausible in the quantum electrodynamics analogy: when external electric fields are strong, pairs are spontaneously produced which neutralize the charges which produce the external fields. It is thus precisely when particle creation becomes important that the external field approximation fails.

Applied to the present case, Parker's hypothesis strongly suggests that particle creation in an inhomogeneous cosmology will similarly tend to damp out the initial inhomogeneity. Cosmological particle creation may thus help account for the observed homogeneity and isotropy of the universe. If particle horizons are present, however, causality limits the damping of inhomogeneous perturbations to scales smaller than the horizon. As an indication of this, we will find that perturbations obeying Einstein's equations do not give rise to pair creation when their wavelengths are larger than the particle horizon. Unfortunately, in the standard FRW cosmology, during the epoch when particle creation can be significant, the comoving size of the present visible universe is much larger than the horizon, and particle creation alone cannot account for the observed homogeneity. However, it has been shown that vacuum polarization[5] can give rise to horizon-free models in the FRW case, and we expect the same to hold true for weakly perturbed models.[7]

Although the external field approximation is inadequate for the problem at hand, nevertheless it is the starting point for a systematic perturbation expansion in the case of weak fields. Our expression for the pair creation probability in terms of spacetime integrals of geometric invariants will be formally correct; the backreaction will determine quantitatively how these invariants evolve. Thus, in the homogeneous anisotropic case, this approach gives results for pair creation in agreement with those of the effective action approach,[8] which explicitly incorporates backreaction effects. A similar agreement will hold in the inhomogeneous case. It would be of interest to study the backreaction problem for inhomogeneous cosmology as well.[9] The first half of this problem has been solved by Horowitz and Wald,[10] who used an axiomatic approach to find the expectation value of the stress-energy tensor (the source in the semiclassical Einstein equations) of a conformally invariant scalar field for arbitrary perturbations around a conformally flat spacetime. However, a backreaction calculation requires one to postulate a dynamical theory of gravity near the Planck time. To date, such calculations have generally assumed semiclassical Einstein gravity, that is, classical general relativity modified only by the one loop quantum effects of matter fields. In leaving open the backreaction question, we may contemplate a broader range of possibilities.

A final, more speculative motivation for the study of cosmological particle creation is the light it may shed on the thermodynamic aspects of gravity. Although the entropy of the gravitational field has so far been defined only for spacetimes with event horizons, Penrose[11] and Hu[12] have discussed the possible meaning of gravitational entropy in a general cosmological context.[13] Penrose suggested the Weyl tensor C_{abcd} (which measures the deviation from conformal flatness) as a measure of the gravitational entropy and argued that the present 'low entropy' state of the universe (as compared to a universe full of black holes), and thus the arrow of time, could be explained by postulating $C_{abcd} = 0$ at the initial singularity (a condition presumably brought about by as yet unknown time asymmetric physical laws acting near the singularity). This definition is made plausible by the fact that, in general relativity with classical matter sources obeying an energy condition[14] (and zero cosmological constant), the universe becomes clumpier and more anisotropic as it evolves, so C_{abcd} grows with time.[15] Hu proposed that the matter entropy generated in cosmological particle production[16] be used as a measure of the change in the gravitational entropy. In this view, particle creation and backreaction damping of anisotropy act as a 'transducer' of gravitational entropy to matter entropy, leading from a wide class of initial conditions to a universe that nearly satisfied the Penrose hypothesis ($C_{abcd} = 0$) near the Planck time. In support of this picture, the total probability of producing a pair of massless conformally coupled scalar particles in a homogeneous anisotropic cosmology[4,5] (and thus the total matter entropy produced) is proportional to the spacetime integral of the square of the Weyl tensor $C_{abcd}C^{abcd}$. Soon after the Planck time, particle creation effects are negligible, and C_{abcd} again grows classically. The decrease of the gravitational entropy in quantum processes and its growth during 'classical' epochs is similar to the behavior of black hole entropy. In this paper, we find a similar form for the particle creation probability, which suggests that the above heuristic picture, if correct, can be extended to inhomogeneous spacetimes as well. This is not surprising, because the Weyl tensor gives a measure of inhomogeneity as well as anisotropy.

We now give a brief outline of our method of calculation. The excitation of free fields (i.e., fields with no nongravitational interaction) by a curved background is usually studied by means of a Bogoliubov transformation of the Heisenberg equations of motion, which gives an exact solution of the problem.[1] For spatially homogeneous metrics, this method is convenient because mode solution of the curved space field equation can be separated, and the time evolution of individual modes can be given exactly in favorable cases. For inhomogeneous spacetimes we resort to a perturbative treatment: we assume the geometry can be written as a flat Minkowski background plus a small perturbation, $g_{ab} = \eta_{ab} + h_{ab}$, and expand the scalar field Lagrangian in powers of h_{ab}. In Section 2 we carry out the expansion to lowest order and calculate the pair creation probability via the S-matrix. In Section 3, we generalize the result to perturbations around conformally flat metrics, $g_{ab} = a^2(\eta)\eta_{ab} + H_{ab}$, which are of cosmological interest (a is the Robertson-Walker scale factor, η is conformal time). Our conclusions follow in Section 4, and we relegate most of the technical details to the Appendices.

Here, we briefly mention the relation of this paper to previous work. Birrell and Davies[1] studied particle creation in homogeneous anisotropic spacetimes using a perturbative treatment of the Heisenberg equations of motion. The results of this paper include their work as a special case. The calculation of $\langle T_{ab} \rangle$ by Horowitz and Wald[10] includes vacuum polarization and particle creation effects to lowest order in h_{ab}, but the energy density of created particles considered here arises only in second order in h_{ab} and is not included in their computation.

2. Perturbations in Minkowski Space

To study particle creation by inhomogeneous perturbations of flat space, we consider the following idealized picture:[17] the metric is taken to be everywhere that of flat space with the exception of a compact region where the curvature is non-zero. This formulation has the advantage that in the Minkowskian 'in' ($t \rightarrow +\infty$) and 'out' ($t \rightarrow -\infty$) regions, particle states, and in particular the vacuum state, are physically well-defined: all inertial observers in the asymptotic regions will agree on the presence or absence of particles, because the 'in' and 'out' vacua are Poincaré-invariant. (We could replace the assumption of a compact perturbation with one in which the curvature falls off sufficiently rapidly, by defining adiabatic particle states.[1]) This situation is clearly analogous to the usual asymptotic treatment of scattering interactions in flat space field theory. We will develop this analogy further by evaluating the S-matrix in the interaction picture.

In a general curved space, the Lagrangian for a real scalar field is taken to have the form (see Appendix A for conventions and notation)

$$L = \frac{1}{2}\sqrt{-g}\,(g^{ab}\partial_a\phi\partial_b\phi - (m^2 + \xi R)\phi^2) \tag{1}$$

where R is the Ricci scalar and ξ is a dimensionless constant. (For $\xi = 0$, the field is said to be minimally coupled to the metric; for $\xi = 1/6$, the curved space Klein Gordon equation is conformally invariant in the massless limit.) To write this in the form $L = L_0 + L_I$, where

$$L_0 = \frac{1}{2}\,(\eta^{ab}\partial_a\phi\partial_b\phi - m^2\phi^2) \tag{2}$$

is the Lagrangian in flat space and L_I describes the interaction with the external gravitational field, we expand the scalar field action in a functional Taylor series about flat space. The first order term is well known to be $\delta S = \frac{1}{2}\int d^4x T^M_{ab}\delta g^{ab}$; the interaction Lagrangian is then $L_I = -\frac{1}{2}T^M_{ab}h^{ab}$, where we have used the fact that, to first order in the perturbation, $g^{ab} = \eta^{ab} - h^{ab}$. The Minkowski stress tensor of the scalar field is

$$T^M_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}\eta_{ab}(\eta^{cd}\partial_c\phi\partial_d\phi - m^2\phi^2) - \xi(\partial_a\partial_b - \eta_{ab}\partial^c\partial_c)\phi^2\ . \tag{3}$$

In the interaction picture, the field operators satisfy the flat space Klein-Gordon equation derived from the 'free' Lagrangian (2), with the usual plane wave solutions $\phi_{in}(x)$. Although L_I has the form of a derivative interaction, it is straightforward to show that the canonical interaction Hamiltonian $H_I(\phi_{in}) = -L_I(\Phi_{in})$, independent of representation. From Eq. (3), we can write the Feynman rule for the pair creation vertex shown in Fig. 1. (Parentheses on indices denote symmetrization.) Note that we are treating h^{ab} as a classical c-number source, so we only need evaluate matrix elements of the stress tensor. Also, the scattering vertex is obtained by letting $k \to -k$. For the total pair creation probability (in this case also the expectation value of the number operator in the 'out' region), we find,[18] using Appendix B and the definitions of Appendix A,

$$\begin{aligned} P &= \int \frac{d^4q\, d^3p\, d^3k}{2\omega_k 2\omega_p}\, \delta^4(q-p-k)\, |M(q,k,p)|^2 \\ &= \frac{\pi^3}{60}\int d^4q\, \theta(q^2-4m^2)\left(1-\frac{4m^2}{q^2}\right)^{1/2} \\ &\times \left[|R(q)|^2 \left\{ 60\left(\xi-\frac{1}{6}\right)^2 - 40\,\frac{m^2}{q^2}\left(\xi-\frac{1}{6}+\frac{m^2}{6q^2}\right)\right\} \right. \\ &\left. + |C_{abcd}(q)|^2 \left(1-\frac{4m^2}{q^2}\right)^2 \right] . \end{aligned} \tag{4}$$

As required, this expression is manifestly gauge and Lorentz-invariant. The total emitted energy in the 'out' region is just Eq. (4) with a factor $2q^0\theta(q^0)$ inserted in the integrand.

There are several features to note about expression (4). First, in this approximation time-independent sources do not create particles, because the amplitude

$$S_{fi} \sim \int d^4x\, h^{ab}(\vec{x})\, e^{i(k+p)\cdot x} \sim 2\pi\delta(k^0+p^0) .$$

Second, there is no particle creation for Ricci-flat perturbations, i.e., for solutions satisfying the vacuum linearized Einstein equations, e.g., gravitational waves. (To this order in perturbation theory we expect the graviton to be stable anyway because there is no phase space for it to decay.) Third, the threshold for massive particles roughly implies that creation occurs only if the curvature varies over scales less than the particle Compton wavelength, in agreement with dimensional arguments. Needless to say, perturbations due to macroscopic sources today, e.g., stellar pulsations, have negligible power in sub-Compton wavelengths. For example, the collapse of a protostar of solar mass releases $\sim 10^{48}$ ergs in the form of heat but only $\lesssim 10^{-35}$ ergs in direct particle creation. By power counting, the pair creation probability is ultraviolet-finite for sources which fall off faster than $h_{ab} \sim q^{-4}$ at large momentum. For massless particles, there is no infrared catastrophe if $h_{ab} \gtrsim q^{-4}$ at small momentum. As in the electromagnetic case, however, there are sources for which P diverges but for which the emitted energy is finite. In the massless case, for sources which satisfy $|R_{ab}(q)|^2 = |R_{ab}(q)|^2\theta(q^2)$, we can use Parseval's theorem to rewrite Eq. (4) as

$$P_{m=0} = \frac{1}{960\pi} \int d^4x \left[60 \left(\xi - \frac{1}{6} \right)^2 R^2 + C_{abcd} C^{abcd} \right] . \tag{5}$$

For conformally invariant scalars ($\xi = 1/6$, $m = 0$), this expression is conformally invariant, so we expect it to hold in a conformally flat background as well.

3. Cosmological Perturbations

We next consider the more physically interesting case of inhomogeneity in an expanding background. We assume that the unperturbed metric has the form of a spatially flat FRW model[19]

$$g^{(0)}_{ab}\,dx^a\,dx^b = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) = a^2(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2) \quad (6)$$

where the conformal time $\eta = \int^t dt'/a(t')$, and $a(t)$ is the FRW scale factor. When $a(\eta)$ is time-dependent, in general there is no privileged definition of the vacuum state as there is in Minkowski space, and the notion of particles is inherently ambiguous. (This is partially a reflection of the fact that the expansion can create particles.) To obtain meaningful results, we must restrict the form of the expansion such that the vacuum state can be defined in the asymptotic regions (see discussion following Eq. (10)).

If we write the perturbed metric as $g_{ab} = g^{(0)}_{ab} + H_{ab} = a^2(\eta)(\eta_{ab} + h_{ab})$ and define $L^{(0)}$ as the scalar Lagrangian devaluated at $g^{(0)}_{ab}$, then a similar argument to that of Section 2 gives $L_I = -\frac{1}{2}\sqrt{-g_{(0)}}\,H^{ab}T^{(0)}_{ab}$, where $g_{(0)}$ is the determinant of $g^{(0)}_{ab}$ and $T^{(0)}_{ab}$ is the scalar field energy-momentum in the FRW background,

$$\begin{aligned} T^{(0)}_{ab} &= \partial_a\phi\partial_b\phi - \frac{1}{2}\,g^{(0)}_{ab}(g^{cd}_{(0)}\partial_c\phi\partial_d\phi - m^2\phi^2) \\ &\quad - \xi\left(\nabla_a\partial_b - g^{(0)}_{ab}\nabla^c\nabla_c + R^{(0)}_{ab} - \frac{1}{2}\,R^{(0)}g^{(0)}_{ab}\right)\phi^2 \;. \end{aligned} \quad (7)$$

Here ∇_a is the covariant derivative with respect to $g^{(0)}_{ab}$ and the d'Alembertian $\nabla^c\nabla_c = (-g_{(0)})^{1/2}\partial_a\left(\sqrt{-g_{(0)}}\,g^{ab}_{(0)}\partial_b\right)$.

In the interaction picture, the field operator satisfies the Klein-Gordon equation in the background spacetime (derived from $L^{(0)}$)

$$(\nabla^a \nabla_a + m^2 + \xi R^{(0)})\phi = 0 \tag{8}$$

which has the solutions[1]

$$\phi(x) = \int d^3k (a_k f_k + a_k^+ f_k^*)$$

$$f_k(x) = \frac{e^{i\vec{k}\cdot\vec{x}} \chi_k(\eta)}{(2\pi)^{3/2} a(\eta)} \tag{9}$$

$$[a_k, a_k^+] = \delta^3(\vec{k} - \vec{k}')$$

where

$$\chi_k'' + \left[|\vec{k}|^2 + m^2 a^2 + \left(\xi - \frac{1}{6} \right) \frac{6a''}{a} \right] \chi_k = 0 \tag{10}$$

(A prime denotes $d/d\eta$). From Eq. (10), in order to obtain well-defined asymptotic vacua, we must restrict the expansion rate as follows:[1,3] for $\xi \neq 1/6$, we require $a''/a \to 0$ as $\eta \to \pm\infty$; for $m \neq 0$, the expansion must be asymptotically static, i.e., $\lim_{\eta \to -\infty} a(\eta) = a_1$, $\lim_{\eta \to +\infty} a(\eta) = a_2$. (As before, we are assuming the inhomogeneous perturbation vanishes as $\eta \to \pm\infty$.) For the general massive case, $f_k(x)$ is assumed to be a pure positive frequency flat space mode in the distant past:

$$\lim_{\eta \to -\infty} f_k(x) = \frac{e^{i\vec{k}\cdot\vec{x} - i\omega_{in}\eta}}{(2\pi)^{3/2} \sqrt{2\omega_{in}}\, a_1} \tag{11}$$

where $\omega_{in} = (|\vec{k}|^2 + m^2 a_1^2)^{1/2}$. However, the interaction picture field does not have the form (11) for all times, because the homogeneous expansion mixes positive and negative frequency modes. If the form of $a(\eta)$ is known, and the mode solutions of Eq. (10) can be found, then the pair creation by the background can be described by a Bogliubov transformation in the usual way.[1]

For a general expansion factor $a(\eta)$, the particle contribution from the inhomogeneous perturbation will depend on integrals over time of products of the unknown mode functions $\chi_k^*(\eta)\chi_p^*(\eta)$ and their derivatives. Instead of considering specific functional forms for $a(\eta)$ for which the $\chi_k(\eta)$ are known, we shall study the dependence on the scalar field parameters.

First, for conformally invariant scalars ($m = 0$, $\xi = 1/6$), the wave equation (10) reduces to

$$\chi_k'' + |\vec{k}|^2 \chi_k = 0 \tag{12}$$

which is just the mode equation in flat space. The normalized positive-frequency solutions for all η are

$$\chi_k = \frac{e^{-ik\eta}}{\sqrt{2k}} \tag{13}$$

where $k \equiv |\vec{k}|$. It remains to evaluate the vacuum-to-two particle matrix elements of Eq. (7), using the modes of Eq. (13). The task is simplified by exploiting conformal invariance. Under a conformal transformation $g_{ab}^M \rightarrow g_{ab}^{(0)} = a^2(\eta, \vec{x})\, g_{ab}^M$, for conformally invariant fields, the stress-energy tensor transforms as[17] $T_{ab}^M \rightarrow T_{ab}^{(0)} = a^{-2} T_{ab}^M$ provided its trace vanishes, $T \equiv T_a^a = 0$. In curved space, the vacuum expectation value of T is not zero, due to the trace anomaly.[1] However, it is clear from the form of Eq. (7) that the $0 \rightarrow 2$ matrix element of $T_{ab}^{(0)}$ is finite (unlike its vacuum expectation value), while the trace anomaly arises from the fact that conformal invariance is broken when the theory is regularized (e.g., in dimensional regularization, the effective action is not conformally invariant in $d \neq 4$). The anomaly thus does not contribute to this matrix element, and $\left\langle 2|T^{(0)}_{ab}|0\right\rangle$ is conformally related to the Minkowski value. It follows that the first-order S-matrix element is conformally invariant, $S_{fi} = -(i/2)\int d^4x\sqrt{-g_{(0)}}\,H^{ab}\left\langle 2|T^{(0)}_{ab}|0\right\rangle = -(i/2)\int d^4x\,h^{ab}\left\langle 2|T^{M}_{ab}|0\right\rangle$, and from Eq. (5) (with the same condition on $|R_{ab}|^2$) we find

$$P = \frac{1}{960\pi}\int d^4x\, C^{M}_{abcd}C^{abcd}_{M} \tag{14}$$

where C^{M}_{abcd} is the Weyl tensor calculated with the metric $h_{ab} = a^{-2}H_{ab}$. From the conformal invariance of C^{d}_{abc}, this can also be written in terms of the Weyl tensor of the cosmological perturbed metric g_{ab},

$$P = \frac{1}{960\pi}\int \sqrt{-g_{(0)}}\; d^4x C_{abcd}C^{abcd} \quad \text{for} \quad m=0\ ,\xi=\frac{1}{6} \tag{15}$$

which agrees with the form found in the homogeneous anisotropic model[1,4,5] and includes it as a special case.[20]

As an important example, we consider cosmological density perturbations obeying Einstein's equations as a possible source of particles. The FRW solution for the scale factor is $a(\eta) \sim \eta^{2/(1+3\nu)}$, where $\nu = p/\rho$ describes the equation of state ($\nu = 0$ for a matter-dominated universe; $\nu = 1/3$ for radiation). Expanding the perturbation in plane waves, at sufficiently early times the wavelength of the perturbation is larger than the instantaneous Hubble radius $(\dot{a}/a)^{-1} = a^2/a'$. On scales outside the Hubble radius, a calculation in synchronous gauge[21] ($h_{00} = h_{0i} = 0$) shows that the density perturbation grows as $(\delta\rho/\rho) \sim \eta^2$, and that the metric perturbation

$$h_{ij}(\eta) \sim \left(a\eta^{-\left(\frac{3+3\nu}{1+3\nu}\right)}\right)^2\left(\frac{\delta\rho}{\rho}\right) \sim \text{const.}, \tag{16}$$

independent of the equation of state. Substituting Eq. (16) into Eq. (14), we find $P = 0$ for these perturbations. This is just a reflection of the fact, noted earlier, that static sources do not create particles. Although this result was derived in the synchronous gauge, the statement that $P = 0$ is gauge-invariant (see Eq. (14)). As confirmation of this, there exists a gauge invariant measure of the perturbation which is time independent for this mode outside the Hubble radius.[22] Thus, growing mode perturbations outside the Hubble radius satisfying the classical Einstein equations do not create conformally invariant particles.

For non-conformally coupled particles, we can make progress with a further approximation; namely, by also treating the homogeneous expansion of the universe as a perturbation around flat space. The simplest way to implement this is to impose the condition $a_2 = a_1$ and the requirement that $a(\eta) - a_1$ be small. From Eq. (10), we see that this can alternatively be interpreted as a perturbation around the conformally invariant limit $\xi = 1/6$, $m = 0$. The interaction picture field now satisfies the flat space Klein-Gordon equation as in Section 2. First consider the case of expansion without inhomogeneity. The metric is $g_{ab} = a^2\eta_{ab} = \eta_{ab} + \widetilde{h}_{ab}$, where $\widetilde{h}_{ab} = (a^2 - a_1)\eta_{ab} = (1/4)\widetilde{h}\eta_{ab}$. This has the form of a perturbation around flat space which is localized in time (but not in space), and we can simply transpose the results of Section 2. Since in this case $\widetilde{h}_{ab}$ corresponds to a local conformal transformation, the Weyl tensor vanishes. Therefore the particle creation probability due to the expansion is

$$P = \frac{\pi^3}{60} \int d^4q\, \theta(q^2 - 4m^2) \left(1 - \frac{4m^2}{q^2}\right)^{1/2} |R_{(0)}(q)|^2 \times \left[60\left(\xi - \frac{1}{6}\right)^2 - 40\,\frac{m^2}{q^2}\left(\xi - \frac{1}{6} + \frac{m^2}{6q^2}\right)\right] \tag{17}$$

where $R_{(0)}$ is the Ricci scalar of the expanding metric. As a check, we note that in the massless case, we can make a transformation to coordinate space, as in Eq. (5), to obtain

$$P_{m=0} = \frac{\left(\xi - \frac{1}{6}\right)^2}{16\pi} \int d^4x \sqrt{-g_{(0)}}\, R_{(0)}^2 ,$$

in agreement with the results of Birrell[23] and Hartle.[24] The coordinate space version of Eq. (17) was obtained by Birrell and Davies (Eq. (5.114) of Ref. 1) by expanding around the conformally invariant limit. For inhomogeneous perturbations of the expanding metric, we have, to lowest order in both perturbations,

$$g_{ab} = \eta_{ab} + \widetilde{h}_{ab} + h_{ab} \equiv \eta_{ab} + \overline{h}_{ab} \tag{18}$$

which yields Eq. (4), but with the curvature invariants now evaluated using $\overline{h}_{ab}$ (see Appendix A). Only the inhomogeneous part h_{ab} contributes to the Weyl term, but the Ricci scalar term will have interfering contributions from both the expansion and the inhomogeneity; clearly, this expression is only appreciable when the curvature grows large, i.e., near the Planck time.

4. Conclusions

In this paper, we have calculated the probability for pair creation by small amplitude perturbations of FRW cosmological models. For conformally invariant and nearly conformally invariant scalar fields, the pair production probability is expressed entirely in terms of gauge-invariant geometrical quantities. This result reduces to the homogeneous anisotropic expressions found previously by several authors in the limit that h_{ab} is space-independent. By analogy with those results, we believe that sub-horizon perturbations will be strongly damped at early times. We should point out that other processes may damp inhomogeneities at early times as well; for example, in asymptotically free theories, near the Planck time the mean free path is larger than the particle horizon and particles can free-stream out of overdense regions. In addition, large-amplitude perturbations may develop into black holes which subsequently evaporate by the Hawking process. The inflationary scenario is premised on the initial condition of homogeneity on horizon scales, a condition which may require such damping mechanisms to achieve.

We mention here some limitations of our calculation. First, we have confined our study to free fields; although the effects of interactions (e.g., a $\lambda\phi^4$ term) on particle creation are interesting,[1] they could be included in a straightforward way and constitute an inessential complication to a first study of the problem. (Although we have focused on scalar fields, the methods and results for higher spin fields are similar.) Second, and more important, our perturbative treatment is limited to time-dependent sources. This is because in perturbation theory, static sources cannot transfer energy to the fields. Particle creation by static sources is a nonperturbative tunneling effect which vanishes faster than any power of $\hbar$. Therefore, to study radiation by static black holes, one must use the Bogoliubov technique; fortunately, the Schwarzschild metric has enough symmetry to make this tractable. The advantage gained by the perturbative treatment is that, for a Minkowski background, we can exploit the global Lorentz invariance of the interaction-picture field equations. The perturbative method is particularly useful for calculations of particle creation in higher dimensional theories.[25]

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APPENDIX A. Results from Linearized Gravity

In this Appendix we set out our notation and conventions and some useful results from linearized relativity.[17] We follow Bjorken and Drell[26] in normalizing field operators, states, and commutation relations. Four-dimensional Fourier transforms are defined with the normalization $f(q) = (2\pi)^{-4} \int d^4x\, e^{iq\cdot x} F(x)$. For the metric and curvature, our sign conventions are $(- - -)$ in the terminology of Misner, Thorne, and Wheeler,[27] so the metric has signature -2 and on-shell squared momenta are positive. We use units in which $\hbar = c = 1$.

In linearized relativity, the metric is $g_{ab} = \eta_{ab} + h_{ab}$, and the inverse metric is an expansion in the perturbation, $g^{ab} = \eta^{ab} - h^{ab} - h^a_c h^{cb} + \cdots$; the determinant of the metric is given by $\sqrt{-g} = 1 + h/2 + \cdots$, where $h \equiv \eta^{ab} h_{ab}$. Indices are raised and lowered with η_{ab}. The connection coefficients are

$$\Gamma^c_{ab} = \frac{1}{2} \eta^{cd} \left[\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}\right] . \tag{A.1}$$

To lowest order, the Riemann curvature is

$$R_{abcd} = \eta_{af}(\partial_d \Gamma^f_{bc} - \partial_c \Gamma^f_{bd}) = \frac{1}{2} \partial_c \partial_{[b} h_{a]d} + \frac{1}{2} \partial_d \partial_{[a} h_{b]c} \tag{A.2}$$

where brackets denote antisymmetrization. The Ricci curvature is

$$R_{ab} = R^c_{acb} = -\frac{1}{2} \partial^c \partial_c h_{ab} + \frac{1}{2} \partial^c \partial_{(a} h_{b)c} - \frac{1}{2} \partial_a \partial_b h \tag{A.3}$$

and the Ricci scalar

$$R = g^{ab} R_{ab} = \partial^c \partial_c h - \partial^c \partial^a h_{ac} . \tag{A.4}$$

We would like to express the pair creation probability in terms of geometric invariants. Since the S-matrix is $\sim h^{ab}(q) O(q^2)$, the probability $\sim h^{ab}(q) h^{cd}(-q) O(q^4)$, since h^{ab} is real. From the form of Eqs. (A.2)-(A.4) and the requirements of local gauge invariance and global Lorentz invariance, the choices are the curvature-squared terms $R(q)\,R(-q)$, $R_{ab}(q)\,R^{ab}(-q)$, $R_{abcd}(q)\,R^{abcd}(-q)$. Now, from (A.2)-(A.4), it is easy to read off the Fourier transforms, e.g.,

$$R(q) = q^a q^c h_{ac}(q) - q^2 h(q) \tag{A.5}$$

and the invariants of interest are

$$\begin{aligned} |R(q)|^2 = R(q)\,R(-q) &= q_a q_c q_b q_d h^{ac}(q)\,h^{bd}(-q) + q^4 h(q)\,h(-q) \\ &\quad - q^2 q_b q_d [h^{bd}(-q)\,h(q) + h^{bd}(q)\,h(-q)] \end{aligned} \tag{A.6}$$

$$\begin{aligned} |R_{ab}(q)|^2 &= \frac{1}{2}\, q_a q_c q_b q_d h^{ac}(q)\,h^{bd}(-q) - \frac{q^2}{2}\, q_c q^a h_{ab}(q)\,h^{bc}(-q) \\ &\quad + \frac{1}{4}\, q^4 h^{ab}(q)\,h_{ab}(-q) \\ &\quad - \frac{1}{4}\, q^2 q_a q_c [h^{ac}(q)\,h(-q) + h^{ac}(-q)\,h(q)] \end{aligned} \tag{A.7}$$

$$\begin{aligned} |R_{abcd}|^2 &= \frac{1}{4}\, q^4\, h^{ab}(q)\,h_{ab}(-q) - 2q^2 q_a q^b\, h_{bd}(q)\,h^{ad}(-q) \\ &\quad + q_a q_c q_b q_d h^{ac}(q)\,h^{bd}(-q)\ . \end{aligned} \tag{A.8}$$

In the linearized theory, h^{ab} transforms as a Lorentz tensor under global Lorentz transformations, so these expressions are Lorentz invariant, as required. Also, under local gauge transformations (infinitesimal coordinate transformations) $x^a \to x^a + \xi^a$, the metric perturbation transforms as $h_{ab} \to h_{ab} - \partial_b \xi_a - \partial_a \xi_b$, and the Riemann tensor (and its contractions) are gauge invariant.

We note from Eqs. (A.6)-(A.8) that

$$|R_{abcd}|^2 - 4|R_{ab}|^2 + |R|^2 = 0 \tag{A.9}$$

so $|R_{abcd}|^2$ is not an independent invariant. As a check, we could also have guessed this from the Gauss-Bonnet theorem, which states that in four dimensions, the quantity

$$G[g_{ab}] = \int d^4x\sqrt{-g}\,(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2)$$

is a topological invariant, i.e., its variation with respect to the metric vanishes. Thus,

$$G[\eta_{ab} + h_{ab}] \sim \int d^4q\,\left(|R_{abcd}|^2 - 4|R_{ab}|^2 + |R|^2\right) = G[\eta_{ab}] = 0\ .$$

In four dimensions, the Weyl conformal tensor is defined as

$$C_{abcd} = R_{abcd} + (g_{b[c}R_{d]a} - g_{a[c}R_{d]b}) + \frac{1}{3}\,Rg_{a[d}g_{c]b} \qquad (A.10)$$

and the absolute value-squared of its Fourier transform is

$$|C_{abcd}(q)|^2 = |R_{abcd}|^2 - 2|R_{ab}|^2 + \frac{1}{3}\,|R|^2\ . \qquad (A.11)$$

Using (A.9), we can express this as

$$C_{abcd}(q)\,C^{abcd}(-q) = 2R_{ab}(q)\,R^{ab}(-q) - \frac{2}{3}\,R(q)\,R(-q) \qquad (A.12)$$

in the linearized case. The conformal tensor vanishes in conformally flat metrics and thus provides a measure of the deviation from isotropy and inhomogeneity. Physically, it is often thought of as the part of the curvature which propagates tidal forces.

APPENDIX B. Evaluation of Pair Production

In this Appendix, we outline the calculation of the pair production probability, Eq. (4). Since we are treating the inhomogeneity as the source of an ordinary perturbative interaction in Minkowski space, the answer must be Lorentz invariant, and this property greatly simplifies the calculation. We can evaluate all quantities in the center of momentum frame, in which

$$p = (E, \vec{p}) \ , \quad k = (E, -\vec{p}) \ , \quad q = p + k = (2E, 0) = (\sqrt{q^2}, 0) \ .$$

This gives the Lorentz-invariant four-momentum products

$$p \cdot k = \frac{q^2}{2} - m^2 \ , \quad p \cdot q = k \cdot q = \frac{q^2}{2} \ . \tag{B.1}$$

Substituting (B.1) into the momentum space Feynman rule of Fig. 1, the pair creation probability of Eq. (4) becomes

$$\begin{aligned} P = \pi^2 \int d^4q \Bigg\{ & (1-4\xi)^2 q^4 \, \frac{I_M}{4} \\ & + \left(I_{(ab)(cd)} + 4\xi^2 I_M q_a q_b q_c q_d\right) h^{ab}(q)\, h^{cd}(-q) \\ & + q^2(1-4\xi)\left(\xi q_a q_b I_M - \tfrac{1}{2} I_{(ab)}\right)\Big(h^{ab}(q)\, h(-q) + h^{ab}(-q)\, h(q)\Big) \\ & - 2\xi q_a q_b I_{(cd)} \Big(h^{ab}(q)\, h^{cd}(-q) + h^{ab}(-q)\, h^{cd}(q)\Big) \Bigg\} \end{aligned} \tag{B.2}$$

where we have used the reality of h^{ab} to set $h^{ab}(q)^* = h^{ab}(-q)$. The integrals appearing in (B.2) are

$$I_M = \int \frac{d^3k \, d^3p}{2\omega_k 2\omega_p} \, \delta^4(q-k-p) \tag{B.3}$$

$$I_{(ab)} = \int \frac{d^3k\, d^3p}{2\omega_k 2\omega_p}\, \delta^4(q-k-p)\, k_{(a}p_{b)} \tag{B.4}$$

$$I_{(ab)(cd)} = \int \frac{d^3k\, d^3p}{2\omega_k 2\omega_p}\, \delta^4(q-k-p) k_{(a}p_{b)} k_{(c}p_{d)}\ . \tag{B.5}$$

I_M is a standard phase space integral for two-body decays. It is most easily evaluated by putting it in manifestly covariant form and subsequently evaluating in the COM frame. The result is

$$I_M = \frac{\pi}{2}\left(1 - \frac{4m^2}{q^2}\right)^{1/2} \theta(q^0)\, \theta(q^2 - 4m^2)\ . \tag{B.6}$$

To evaluate the remaining integrals, we use symmetry and Lorentz covariance. $I_{(ab)}$ is a symmetric Lorentz-covariant tensor which only depends on m^2 and q, so it must be of the form $I_{(ab)} = I_1\eta_{ab} + I_2 q_a q_b$ where I_1 and I_2 are Lorentz-invariant. Contracting with η_{ab} and $q^a q^b$ gives two simultaneous algebraic equations for I_1 and I_2; evaluating the solutions in the COM frame gives the well-known result

$$I_{(ab)} = \frac{I_M}{6}\left[2q_a q_b\left(1 + \frac{2m^2}{q^2}\right) + q^2\eta_{ab}\left(1 - \frac{4m^2}{q^2}\right)\right]\ . \tag{B.7}$$

To evaluate $I_{(ab)(cd)}$, we employ the same principles. Using symmetry under interchange $a \leftrightarrow b$, $c \leftrightarrow d$ and under exchange of the first and second index pairs $(ab) \leftrightarrow (cd)$, we can write

$$\begin{aligned} I_{(ab)(cd)} = {} & J_1\eta_{ab}\eta_{cd} + J_2(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc}) \\ & + J_3 q_a q_b q_c q_d + J_4(\eta_{ab}q_c q_d + \eta_{cd}q_a q_b) \\ & + J_5(\eta_{ac}q_b q_d + \eta_{bd}q_a q_c + \eta_{ad}q_b q_c + \eta_{bc}q_a q_d)\ . \end{aligned}$$

We again contract this to form Lorentz-invariants which can be evaluated in terms of I_M and the quantities in (B.1). Solving the resulting five simultaneous equations for $J_1, \ldots, J_5$ yields

$$I_{(ab)(cd)} = \frac{I_M}{240}\Bigg\{(q^2-4m^2)^2(\eta_{ab}\eta_{cd}+\eta_{ac}\eta_{bd}+\eta_{ad}\eta_{bc})$$
$$+\,8q_aq_bq_cq_d\left(1+\frac{2m^2}{q^2}+\frac{6m^4}{q^4}\right)$$
$$+\,\frac{4}{q^2}(q^2-4m^2)(q^2+m^2)(\eta_{ab}q_cq_d+\eta_{cd}q_aq_b) \qquad (B.8)$$
$$-\,\frac{(q^2-4m^2)^2}{q^2}\,(\eta_{ac}q_bq_d+\eta_{bd}q_aq_c+\eta_{ad}q_bq_c+\eta_{bc}q_aq_d)\Bigg\}\ .$$

We substitute (B.6)-(B.8) into (B.2) and obtain

$$P = \frac{\pi^2}{30}\int d^4q\, I_M\Bigg\{\left[3-40\xi+120\xi^2+\frac{m^2}{q^2}(16-80\xi)+\frac{8m^4}{q^4}\right]$$
$$\times\left[q^4h(q)\,h(-q)-q^2q_aq_b(h^{ab}(q)\,h(-q)+h^{ab}(-q)\,h(q))\right]$$
$$+\left(1-\frac{4m^2}{q^2}\right)^2\left(q^4h^{ab}(q)\,h_{ab}(-q)-2q^2q^aq_bh_{ac}(q)\,h^{bc}(-q)\right) \qquad (B.9)$$
$$+\left[4\left(1+\frac{2m^2}{q^2}+\frac{6m^4}{q^4}\right)-40\xi+120\xi^2-80\xi\,\frac{m^2}{q^2}\right]$$
$$\times\left[q_aq_bq_cq_dh^{ab}(q)\,h^{cd}(-q)\right]\Bigg\}\ .$$

Now, using Eqs. (B.6) and (A.6)-(A.12), we can finally write this in the form given in Eq. (4), Section 2. We note that since the integrand is even in q, we can replace $\theta(q^0)$ with a factor 1/2, with the integral now unrestricted.

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particles will be characterized by the entropy of a mixed state.

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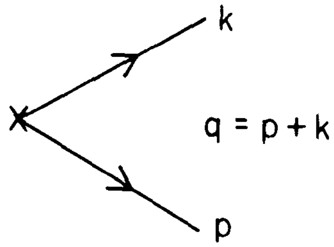
FIGURE CAPTION

Pair creation vertex for scalar particles.



$$M = i\pi h^{ab}(q)\left[p_{(a}k_{b)} - (1-4\xi)(m^2 + k\cdot p)\eta_{ab} - 2\xi q_a q_b\right]$$

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