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ELECTROWEAK SYNCHROTRON RADIATION*

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ABSTRACT

We calculate the emission rate and spectrum for radiation of neutral electroweak bosons (Z^0) from an electron in a weak external homogeneous electromagnetic field satisfying $|\frac{1}{2}F_{\mu\nu}F^{\mu\nu}|^{1/2} \ll F_c \equiv m_e^2c^3/e\hbar$. The calculational method is based on the source theory formulation of quantum field theory introduced by Schwinger. In particular for ultra-relativistic electrons, we find that the Z^0 emission rate is exponentially suppressed relative to photon emission for values of the radiation parameter $\Upsilon = |\Pi_\mu F^{\mu\nu}\Pi^\lambda F_{\lambda\nu}|^{1/2}/m_e c F_c \ll (M_Z/m_e)^2 \sim 10^{10}$, where Π_μ is the electron mechanical momentum. This implies that the decay rate for $e^- \to W^- + \nu_e$ is also exponentially small under similar conditions. An application of these results to accelerator physics is discussed.

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The quantum mechanical problem of calculating the synchrotron radiation of photons from relativistic electrons in a homogeneous external magnetic field has been addressed by various authors.[1-3] Sokolov *et al.*[1] utilized the Dirac wave functions of an electron in a constant magnetic field to calculate synchrotron radiation. The transition amplitude for $e^- \rightarrow e^- + \gamma$ was computed by perturbation theory (*i.e.* to first order in the fine structure constant α) and the power spectrum obtained by squaring the amplitude and summing over final states. Recently Tsai and Yildiz[3] have presented a more efficient method for calculating radiation in external fields based on Schwinger's source theory formulation of quantum field theory.[4] This latter approach, which we use in this paper, eliminates the need for using wave functions by replacing the sum over final states by expectation values obtained directly from the Dirac equation.

The previous results for radiation in external magnetic fields are of course applicable in all Lorentz frames where $H^2 - E^2 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} > 0$ and $\bar{E} \cdot \bar{H} = \frac{1}{4} \widetilde{F}_{\mu\nu} F^{\mu\nu} = 0$.[5] The corresponding problem of radiation in a homogeneous electric field has received far less attention, possibly because of the well-known difficulty of the "Klein catastrophe," that is, spontaneous pair creation by an electric field.[6] In the weak field limit $|\frac{1}{2} F_{\mu\nu} F^{\mu\nu}|^{1/2} \ll F_c = m_e^2/e$ ($\approx 4.4 \times 10^{13}$ G $\approx 1.3 \times 10^{16}$ V/cm) that we consider, pair creation effects are negligible,[7] and we may treat fields with both positive and negative values of $H^2 - E^2$ and $\bar{E} \cdot \bar{H} = 0$ in the same manner.

In the standard unified electroweak theory,[8] the electron couples to the massive neutral weak boson, Z^0, as well as the photon. In this paper, the concept of synchrotron radiation is generalized to include the decay $e^- \rightarrow e^- + Z^0$ in an external electromagnetic field using the source theory method of Tsai and

Yildiz.[3] The novel aspects of the calculation are the distinct left and right handed couplings characteristic of the weak interactions and the finite mass of the Z^0.

We apply the method to first calculate the total Z^0 emission rate from an electron in an external homogeneous field, $F_{\mu\nu}$. The starting point is the action contribution associated with the exchange of a virtual Z^0, $-\frac{1}{2}\int (dx)\,(dx')\,\bar{\Psi}(x)\,M(x,x')\,\Psi(x')$, where the electron field is $\Psi = \frac{1}{2}(1+\gamma_5)\Psi + \frac{1}{2}(1-\gamma_5)\Psi \equiv \Psi_R + \Psi_L$. If we represent $M(x,x') = \langle x|M|x'\rangle$, and use the standard electron-weak boson coupling[9] we have (in the Feynman gauge)

$$
\begin{aligned}
M = i\int \frac{dk}{(2\pi)^4}\ \Bigg\{ & g_R^2\,\frac{1}{2}\,(1-\gamma_5)\,\gamma^\mu\,\frac{1}{k^2+M_Z^2}\,\frac{1}{m+\gamma(\Pi-k)}\,\gamma_\mu\,\frac{1}{2}\,(1+\gamma_5) \\
& + g_R\,g_L\left[\frac{1}{2}\,(1-\gamma_5)\,\gamma^\mu\,\frac{1}{k^2+M_Z^2}\,\frac{1}{m+\gamma(\Pi-k)}\,\gamma_\mu\,\frac{1}{2}\,(1-\gamma_5)\right. \\
& \qquad \left. +\frac{1}{2}\,(1+\gamma_5)\,\gamma^\mu\,\frac{1}{k^2+M_Z^2}\,\frac{1}{m+\gamma(\Pi-k)}\,\gamma_\mu\,\frac{1}{2}\,(1+\gamma_5)\right] \\
& + g_L^2\,\frac{1}{2}\,(1+\gamma_5)\,\gamma^\mu\,\frac{1}{k^2+M_Z^2}\,\frac{1}{m+\gamma(\Pi-k)}\,\gamma_\mu\,\frac{1}{2}\,(1-\gamma_5)\Bigg\} \\
& + \text{c.t.}
\end{aligned}
\tag{1}
$$

where $g_R = e\tan\theta_W$, $g_L = \frac{1}{2}\,e(\tan\theta_W - \cot\theta_W)$, θ_W is the electroweak mixing angle, and $\Pi_\mu = -i\partial_\mu - eqA_\mu$, $q = \pm 1$. The contact terms (c.t.) are determined by requiring that when $F_{\mu\nu} = 0$, M and its first derivative with respect to $\gamma\Pi$ must vanish at $\gamma\Pi = -m$.

According to the optical theorem, the total decay rate, $\Gamma(e^- \to e^- + Z^0)$, is related to the imaginary part of the matrix element M by

$$\Gamma = -\left(\frac{2m}{\mathcal{E}}\right)\ \mathrm{Im}\ M \quad . \tag{2}$$

Since we are considering real Z^0 emission, $\gamma - Z^0$ interference does not affect Im M. The simplification of M in Eq. (1) using the proper-time technique and the replacement of the momentum integration with an algebraic procedure[10] proceeds exactly as in Ref. 3 for the case of photon emission, with the result

$$\begin{aligned} M = &-\frac{1}{(4\pi)^2}\int_0^\infty \frac{ds}{s}\int_0^1 du\,\left(\det\,\frac{2eqFs}{D}\right)^{1/2} e^{-is\Phi}\cdot\left\{\frac{1}{2}\,\left[(g_R^2+g_L^2)-(g_R^2-g_L^2)\gamma_5\right]\right. \\ &\times\left[(-4-\mathrm{tr}\,A+2i\sigma A)\gamma\,\frac{2(1-u)eqFs}{D}\,\Pi + 2\gamma(1+A^T)\,\frac{2(1-u)eqFs}{D}\,\Pi\right] \\ &\left. + g_R\,g_L\,(-4-\mathrm{tr}\,A+2i\sigma A)\ m\right\} + \mathrm{c.t.} \end{aligned} \tag{3}$$

where $A = \exp(2ueqFs) - 1$, $D = A + 2(1-u)eqFs$, $\Phi = u(\Pi^2 + m^2 - eq\sigma F) + (1-u)M_Z^2 + \Pi[-1/(2eqFs)\ln(-D/D^T)]\Pi$ and $\sigma A \equiv \frac{1}{2}\,\sigma_{\mu\nu}A^{\mu\nu}$. The contact terms have the form c.t. $= -m_c - \varsigma_c(m + \gamma\Pi)$, where

$$m_c = \frac{m}{(4\pi)^2}\int \frac{ds}{s}\ du\ e^{-is(m^2u^2+(1-u)M_Z^2)}\ \left[(g_R^2+g_L^2)(u-1) + 4g_R\,g_L\right] \quad . \tag{4}$$

The explicit form of ς_c is not needed since M will subsequently be approximated (to an accuracy of order g^2) by its expectation value between fields obeying the Dirac equation $(m + \gamma\Pi)\Psi = 0$.

We now specialize to the two cases of radiation in a pure magnetic field and a pure electric field. For a magnetic field in the z direction, $F_{12} = -F_{21} = H$, Eq. (3) can be simplified by taking its expectation value between fields obeying $(m + \gamma\Pi)\Psi = 0$ (assuming $\Pi_3 = 0$ without loss of generality). The expectation values of the various operators not involving γ_5 may be found in the Appendix of Ref. 3. The expectation values of all operators containing γ_5 vanish. The resulting matrix element is

$$M = \frac{m}{(4\pi)^2} \int_0^\infty \frac{dx}{x} \int_0^1 du \exp\left(-i\,\frac{m^2}{eH}\,ux - i\,\frac{M_Z^2}{eH}\,\frac{1-u}{u}\,x\right)$$

$$\times \left\{\Delta^{-1/2} \exp\{-i[\beta - (1-u)x](\mathcal{E}^2 - m^2)/eH\} \left[(g_R^2 + g_L^2)\left(e^{-i\varsigma(\beta+x)}(u-1)\right.\right.\right.$$

$$\left. + (1-u)\left(\frac{\mathcal{E}^2 - m^2}{m^2}\right)\left(\frac{1-u}{\Delta}\cos(\beta - x) + \frac{u}{\Delta}\,\frac{\sin x}{x}\cos\beta - \cos(\beta + x)\right)\right)$$

$$\left.\left. + 2g_R\, g_L\left(e^{-i\varsigma(\beta-x)} + e^{-i\varsigma(\beta+x)}\right)\right] - \left[(g_R^2 + g_L^2)(u-1) + 4g_R\, g_L\right]\right\} \quad , \qquad (5)$$

where $\Delta = \det[D/(2eqFs)] = (1-u)^2 + u(1-u)\sin 2x/x + u^2(\sin x/x)^2$, $\varsigma = q\sigma_3$, $x = eHus$, $\tan\beta = (1-u)\sin x/[(1-u)\cos x + u\sin x/x]$, $\mathcal{E}^2 = m^2 + (2n + 1 - \varsigma')eH$ is the energy eigenvalue of the Dirac equation, $\varsigma' = \pm 1$ is the eigenvalue of $\gamma^0\varsigma$, and $n = 0, 1, 2, \ldots$.

In the case of a pure electric field in the z direction, $F_{30} = -F_{03} = E$, Eq. (3) is again simplified by taking its expectation value between fields obeying $(m + \gamma\Pi)\Psi = 0$, assuming that $E \ll F_c$ so that spontaneous pair creation is negligible.[11] Using the fact that the eigenvalues of the matrix $F^\mu{}_\nu$ are ± 0, $\pm E$ for an electric field instead of ± 0, $\pm iH$ for a magnetic field, the corresponding matrix element in an electric field is obtained from Eq. (5) with the following straightforward substitutions: $H \to iE$, $x \to ix = ieEus$, $\beta \to i\beta$, $\varsigma \to i\varsigma = iq\sigma^{03}$ and $\mathcal{E}^2 - m^2 \to -(p_\perp^2 + m^2)$, where $p_\perp$ is the eigenvalue of Π_2. Because of the similarity of this matrix element to Eq. (5), it is unnecessary to write down the explicit form.

Returning to Eq. (5) for Z^0 emission in a magnetic field, we will evaluate the imaginary part of M in the high-energy ($\mathcal{E}/m \gg 1$) and weak field ($eH/m^2 \ll 1$) limit. Because of the overall exponential factor under the integrand, M will be vanishingly small unless the x integration is dominated by small x. Examination of the exponential structure $\exp\{-i(m^2ux+M_Z^2(1-u)x/u+[\beta-(1-u)x]\mathcal{E}^2)/eH\}$ for small x shows that the important range of x occurs when $x \sim (M_Z/\mathcal{E})(1-u+u^2m^2/M_Z^2)^{1/2}/(u(1-u))$. In order that $x \ll 1$, we must have $\mathcal{E}/M_Z \gg 1$ since $0 \leq u \leq 1$. The u integration can now be divided into three regions: (i) $0 \leq u < u_0$, with $1 \gg u_0 \gg M_Z/\mathcal{E}$, (ii) $u_0 < u < 1-\epsilon$ and (iii) $1-\epsilon < u \leq 1$. Here $\epsilon \gg (M_Z/\mathcal{E})^2$ if $\mathcal{E}/m \ll (M_Z/m)^2$, and $\epsilon \gg m/\mathcal{E}$ if $\mathcal{E}/m \gg (M_Z/m)^2$. With u_0 and ϵ so specified, the contribution from regions (i) and (iii) are negligible compared to that from region (ii) where small x dominates the integration.

The calculation of the total decay rate $\Gamma(e^- \to e^- + Z^0)$ can now be performed in analogy with Ref. 3 for photon emission. If we expand the integrand of M for small x and define two new variables by $x = [(m/\mathcal{E})(1+(M_Z/m)^2(1-u)/u^2)^{1/2}/(1-u)]z$ and $\xi = 2u(1+(M_Z/m)^2(1-u)/u^2)^{3/2}/(3\Upsilon(1-u))$, where $\Upsilon = (\mathcal{E}/m)(eH/m^2)$, we obtain a result for the decay rate quite similar in form to that of photon emission,

$$\Gamma(e^- \to e^- + Z^0) = \frac{2m^2}{\sqrt{3}\,(4\pi)^2\mathcal{E}} \int_0^1 du \left\{ \left[(g_R^2+g_L^2)(u-1)+4g_R\,g_L\right] \int_\xi^\infty K_{5/3}(\eta)\,d\eta \right.$$

$$\left. + \left[(g_R^2+g_L^2)\left(\frac{4+2u^2-16u/3}{1-u}+\frac{2-4u/3}{u^2}\frac{M_Z^2}{m^2}\right)-8g_R\,g_L\right]K_{2/3}(\xi)\right.$$

$$+\varsigma' \left[(g_R^2 + g_L^2)(u-2) + 4 g_R\, g_L\right] \left(1 + \frac{1-u}{u^2} \frac{M_Z^2}{m^2}\right)^{1/2} K_{1/3}(\xi) \Bigg\} \,, \tag{6}$$

where $K_\nu(\eta)$ is the modified Bessel function of the second kind.

We note that by setting $M_Z = 0$ and $g_R = g_L = e$ in Eq. (6), the correct decay rate for photon emission in a weak magnetic field is recovered, where $\Upsilon = (\mathcal{E}/m)(eH/m^2)$ characterizes the quantum mechanical nature of the radiation.[3] Because of the similarity of the matrix elements in a magnetic field and an electric field, the decay rate formulas for both photon and Z^0 emission by a relativistic electron in a magnetic field are identical to those in an electric field E with the replacement $\Upsilon = (p_\perp/m)(eE/m^2)$ if $p_\perp/m \gg 1$.[12] For an electron with its acceleration and momentum nearly parallel (*i.e.* $p_\perp/m \ll 1$), the radiation is always negligible in weak electric fields.

Although Eq. (6) is mathematically valid for all Υ, the behavior of $\Gamma(e^- \to e^- + Z^0)$ for $\Upsilon \ll (M_Z/m)^2$ is particularly interesting. For $\Upsilon \ll (M_Z/m)^2$, we have $\xi \gg 1$, and the Bessel functions may be approximated by their asymptotic forms ($\sim (\pi/2\xi)^{1/2} e^{-\xi}$). The u integration can then be done in the steepest descent approximation with $\xi(u)$ having a minimum value $\sqrt{3}(M_Z^2/m^2)/\Upsilon$ when $u \simeq 1 - 2m^2/M_Z^2$. The decay rate is found to be exponentially small,

$$\Gamma\left(e^- \to e^- + Z^0\right) = \frac{m^4}{4\sqrt{3}\,\pi \mathcal{E} M_Z^2}\, (g_R^2 + g_L^2)\ \Upsilon\ \exp\{-\sqrt{3}(M_Z^2/m^2)/\Upsilon\} \tag{7}$$

where higher order terms in m^2/M_Z^2 have been neglected. This expression is valid for $\mathcal{E}/m \gg (M_Z/m)^2$ since the steepest descent point is then in region (ii) of the u integration. If $\mathcal{E}/m \ll (M_Z/m)^2$, this point is in region (iii) where the small x expansion is invalid and the decay rate will be negligible compared to Eq. (7).

The Z^0 power spectrum, $P(\omega)$, where ω is the Z^0 energy $(M_Z \leq \omega \leq \mathcal{E})$, can be obtained by a simple modification of the method used to calculate the decay rate. By inserting a unit factor $1 = \int_{-\infty}^{\infty} d\omega\ \delta(\omega - k^0) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} (d\tau/2\pi)$ $\exp\ i(\omega - k^0)\tau$ into the matrix element M of Eq. (1), the spectrum $P(\omega)$ is identified from the ω-integrand before performing the momentum integration. The procedure is essentially identical to that given in Ref. 3 for photon emission in a homogeneous magnetic field with the result[13]

$$
\begin{aligned}
P(\omega) = & -\frac{2m}{\mathcal{E}}\,\omega\,\mathrm{Im}\left(\frac{m}{(4\pi)^2}\int_0^\infty \frac{ds}{s}\int_0^1 du\ \exp\{-is[m^2u^2 + M_Z^2(1-u)]\}\right.\\
& \times \Big\{\Delta^{-1/2}\exp\{-i[\beta - (1-u)x](\mathcal{E}^2 - m^2)/eH\}\\
& \times \left[(g_R^2 + g_L^2)\left(e^{-i\varsigma(\beta+x)}(u-1) + (1-u)\left(\frac{\mathcal{E}^2 - m^2}{m^2}\right)\right.\right.\\
& \times \left(\frac{1-u}{\Delta}\cos(\beta - x) + \frac{u}{\Delta}\frac{\sin x}{x}\cos\beta - \cos(\beta + x)\right)\\
& \left.\left. + \frac{i}{2ms\Pi^0}\,e^{-i\varsigma(\beta+x)}\frac{d}{du}\gamma^0\right) + 2g_R g_L\left(e^{-i\varsigma(\beta-x)} + e^{-i\varsigma(\beta+x)}\right)\right]\\
& - \left[(g_R^2 + g_L^2)\left(u - 1 + \frac{i}{2ms\Pi^0}\,\frac{d}{du}\gamma^0\right) + 4g_R g_L\right]\Big\}\\
& \left.\times \int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\,e^{i(\omega - u\Pi^0)\tau}\,e^{-i(\tau^2/4s)}\right)\quad ,
\end{aligned}
\tag{8}
$$

where all variables have been previously defined.

In the high-energy and weak-field limit, the τ-dependent term is $\int_{-\infty}^{\infty} (d\tau/2\pi)$ $\exp\{i(\omega - u\Pi^0)\tau - i\tau^2/4s\} \simeq \delta(\omega - u\mathcal{E})$ since the Gaussian function is close to unity. The u-integration is then trivial, and the z-integration yields

$$P(\omega) = \frac{2m^2}{\sqrt{3}(4\pi)^2\mathcal{E}} \frac{\omega}{\mathcal{E}} \left\{ \left[(g_R^2 + g_L^2) \left(\frac{M_Z^2}{2m^2} - 1 \right) + 4g_R g_L \right] \int_{\xi'}^{\infty} K_{5/3}(\eta)\, d\eta \right.$$

$$+ \left[(g_R^2 + g_L^2) \left(4 + \left(\frac{\omega}{\mathcal{E}}\right)^2 \left(1 - \frac{\omega}{\mathcal{E}}\right)^{-1} + 2\left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \frac{M_Z^2}{m^2} \right) - 8g_R g_L \right]$$

$$\times K_{2/3}(\xi') + \varsigma' \left[(g_R^2 + g_L^2) \left(\frac{\omega}{\mathcal{E}} - 2 \right) + 4g_R g_L \right]$$

$$\left. \times \left[1 + \left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \frac{M_Z^2}{m^2} \right]^{1/2} K_{1/3}(\xi') \right\} \quad , \tag{9}$$

where $\xi' = 2(\omega/\mathcal{E})[1 + (M_Z/m)^2(\omega/\mathcal{E})^{-2}(1 - (\omega/\mathcal{E}))]^{3/2}/[3\Upsilon(1 - (\omega/\mathcal{E}))]$. Again by setting $M_Z = 0$ and $g_R = g_L = e$ in Eq. (9), the correct power spectrum for photon emission in a weak magnetic field is recovered. The same formula applies in a weak electric field, E, with the replacement $\Upsilon = (p_\perp/m)(eE/m^2)$.

For the special case $\Upsilon \ll (M_Z/m)^2$, the spectrum (9) can be written as

$$P(\omega) = \frac{2m^2}{\sqrt{3}(4\pi)^2\mathcal{E}} \frac{\omega}{\mathcal{E}} \left\{ (g_R^2 + g_L^2) \right.$$

$$\times \left[3 + \frac{M_Z^2}{2m^2} + \left(\frac{\omega}{\mathcal{E}}\right)^2 \left(1 - \frac{\omega}{\mathcal{E}}\right)^{-1} + 2\left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \frac{M_Z^2}{m^2} \right]$$

$$- 4g_R g_L + \varsigma' \left[(g_R^2 + g_L^2) \left(\frac{\omega}{\mathcal{E}} - 2 \right) + 4g_R g_L \right] \tag{10}$$

$$\left. \times \left[1 + \left(\frac{\omega}{\mathcal{E}}\right)^{-2} \left(1 - \frac{\omega}{\mathcal{E}}\right) \frac{M_Z^2}{m^2} \right]^{1/2} \right\} \left(\frac{\pi}{2\xi'}\right)^{1/2} e^{-\xi'} \quad .$$

The spectrum is sharply peaked near $\omega/\mathcal{E} \simeq 1 - 2m^2/M_Z^2$. Since the number spectrum $N(\omega)$ is related to the power spectrum by $P(\omega) = \omega N(\omega)$, one finds in the steepest descent approximation, that the total power is related to the decay rate Γ by $P \simeq \mathcal{E}\Gamma$ with Γ given by Eq. (7).

Because of the large mass of the neutral weak boson, synchrotron radiation by Z^0 emission is exponentially suppressed relative to photon emission when $\Upsilon \ll (M_Z/m)^2 \sim 10^{10}$. The decay $e^- \to W^- + \nu_e$ in an external field should also be exponentially small under similar conditions. We caution that when Υ is large, electroweak synchrotron radiation will be modified by vacuum polarization effects. It is known that photon synchrotron radiation changes to a new synergetic synchrotron-Cerenkov radiation for $\Upsilon > 10^5$ when the vacuum, modified by the external electromagnetic fields, acts like a dielectric medium.[14] Electroweak synchrotron radiation should be similarly affected, but we will not address this problem in the present paper.

Customarily synchrotron radiation is calculated under the simplifying assumptions that the electromagnetic fields are homogeneous and of infinite extent. The applicability of these assumptions to actual fields of finite extent can be quantified by two inequalities. Suppose that a field has longitudinal and transverse (relative to the electron velocity vector) scale lengths $\sigma_{\parallel}$ and $\sigma_{\perp}$, respectively. An electron with a local radius of curvature R in this field radiates a quantum over a characteristic longitudinal distance R/γ and is transversely deflected a distance R/γ^2. Provided that $R/\gamma \ll \sigma_{\parallel}$ or $\Upsilon \gg \lambda\!\!\!^{-}_c\gamma/\sigma_{\parallel}$, the field can be treated as homogeneous and of infinite extent longitudinally. Similarly if $R/\gamma^2 \ll \sigma_{\perp}$ or $\Upsilon \gg \lambda\!\!\!^{-}_c/\sigma_{\perp}$, the field is essentially homogeneous and of inifinite extent transversely.

A particularly interesting application of synchrotron radiation in locally intense electromagnetic fields is beamstrahlung from colliding electron-positron beams. When an electron beam and a positron beam from an accelerator collide, the particles in each beam emit radiation due to their interaction with the fields generated by the opposite beam. Each relativistic beam has longitudinal and transverse dimensions $\sigma_{\parallel}$ and $\sigma_{\perp}$, but the approximately transverse and mutually perpendicular $\bar{E}$ and $\bar{H}$ fields of each beam ($E \simeq H$) are ultimately due to the individual electrons and positrons. Provided that the characteristic radiation length R/γ is much greater than the inverse longitudinal beam density $\sigma_{\parallel}/N$, the beam field may be approximated as continuous for the purpose of calculating radiation. This condition can be expressed by the inequality $\Upsilon \ll N\gamma \lambda\!\!\!^{-}_c/\sigma_{\parallel}$.

The Stanford Linear Collider (SLC) uses 50 GeV ($\gamma = 10^5$) e^+e^- beams with $\sigma_{\perp} = 1$ micron, $\sigma_{\parallel} = 1$ mm and $N = 5 \times 10^{10}$ particles. Typically $\Upsilon \sim 10^{-3}$ at the SLC collision point so the synchrotron radiation is classical. Since $N\gamma \lambda\!\!\!^{-}_c/\sigma_{\parallel} \sim 2 \times 10^6 \gg \Upsilon$, the continuous beam field approximation is well satisfied, and the field homogeneity conditions are valid except at the extreme center and edges of the beams where little radiation is emitted.

For a hypothetical 5 TeV + 5 TeV e^+e^- collider with $\sigma_{\perp} = 10^{-3}$ microns, $\sigma_{\parallel} = 10^{-3}$ mm and $N = 4 \times 10^8$ particles,[15] the radiation parameter would be $\Upsilon \sim 10^3$ at the collision point. The field continuity and homogeneity conditions for radiation are again satisfied, even though the individual particle fields (opening angle $1/\gamma$) within the beam do not overlap enough longitudinally for the resulting beam field to be spatially continuous, *i.e.* $\sigma_{\perp}/\gamma \ll \sigma_{\parallel}/N$. This discreteness of the field can in principle act like a wiggler resulting in a broad second peak in the synchrotron spectrum near $\omega_d \sim \gamma^2 N/\sigma_{\parallel} \gg \omega_c \sim \gamma^3/R$. However since $\Upsilon \gg 1$ we have $\omega_d > \mathcal{E}$ so the second peak cannot occur kinematically. The synchrotron radiation is extremely quantum mechanical in this case but vacuum polarization effects are not yet important. Obviously Z^0 synchrotron radiation and beam decay into charged boson-neutrino pairs will be negligible effects in linear colliders for the foreseeable future.

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11. Without loss of generality, we choose $\Pi_1 = 0$ and $\Pi_2 = p_\perp$. The wave function Ψ (where $\int \bar{\Psi}\ \Psi(dx) = 1$) is a simultaneous eigenstate of the Hamiltonian H, momentum Π_2 and $\gamma_2\varsigma$, where $\varsigma = \ q\sigma^{03}$ (*i.e.* $\gamma_2\varsigma\Psi = \varsigma'\Psi$; $\varsigma' = \pm 1$). Decomposing Ψ into subspaces $\Psi_\pm$ such that $i\gamma_2\Psi_\pm = \pm\Psi_\pm$

and $-i\varsigma\Psi_{\pm} = \pm\varsigma'\Psi_{\pm}$, one obtains the eigenvalue equation $\Pi_{\parallel}^2\Psi_{\pm} = -(p_{\perp}^2 + m^2 + \varsigma eE)\Psi_{\pm}$, where $\Pi_{\parallel}^2 = \Pi_0\Pi^0 + \Pi_3\Pi^3$. The following expectation values are then used to simplify Eq. (3): $\langle\gamma_2\rangle = p_{\perp}/m$, $\langle\varsigma\rangle = -\langle\gamma_2\rangle\varsigma'$, $\langle\Pi_{\parallel}^2\rangle = -(p_{\perp}^2 + m^2 + \langle\varsigma\rangle eE)$, $\langle\gamma\cdot\Pi_{\parallel}\rangle = -(p_{\perp}^2 + m^2)/m$, $\langle\varsigma\gamma\cdot\Pi_{\parallel}\rangle = 0$, where $\gamma\cdot\Pi_{\parallel} = \gamma_0\Pi^0 + \gamma_3\Pi^3$. The expectation values of all operators containing γ_5 vanish.

12. Within the relativistic approximations made, the parameter Υ in both the magnetic and electric cases can be written in a Lorentz invariant form $\Upsilon = |\Pi_\mu F^{\mu\nu}\Pi^\lambda F_{\lambda\nu}|^{1/2}/(mF_c)$, where Π_μ is the electron mechanical momentum. The expectation value ς' in both cases can be written as $q\epsilon_{\alpha\beta\mu\nu}\Pi^\alpha S^\beta F^{\mu\nu}/(mF_c\Upsilon)$, where S^β is the electron four-spin.

13. In the case of a weak homogeneous electric field, Eq. (8) is modified by the same substitutions noted for Eq. (5) in the decay rate calculation. In the high energy and weak electric field limit $\langle\Pi^0\rangle = \langle\mathcal{E} - V\rangle \simeq \mathcal{E}$, $\langle\gamma^0\rangle = \mathcal{E}/m$ and $\langle i\varsigma\gamma^0\rangle = \langle\gamma_3\rangle = \langle\Pi_3\rangle/m \ll \mathcal{E}/m$.

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