SLAC – PUB – 3836
November 1985
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SINGLE PARTICLE DYNAMICS AND NONLINEAR RESONANCES IN CIRCULAR ACCELERATORS*

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1. INTRODUCTION

The purpose of this paper is to introduce the reader to single particle dynamics in circular accelerators with an emphasis on nonlinear resonances. In several sections we follow Ref. 1 closely although the treatment given here is in some cases more general.

We begin with the Hamiltonian and the equations of motion in the neighborhood of the design orbit. In the linear theory this yields linear *betatron oscillations* about a *closed orbit*. It is useful then to introduce the action-angle variables of the linear problem.

Next we discuss the nonlinear terms which are present in an actual accelerator, and in particular, we motivate the inclusion of sextupoles to cure chromatic effects. To study the effects of the nonlinear terms, we next discuss canonical perturbation theory which leads us to nonlinear resonances. After showing a few examples of perturbation theory, we abandon it when very close to a resonance.

This leads to the study of an isolated resonance in one degree of freedom with a 'time'-dependent Hamiltonian. We see the familiar resonance structure in phase space which is simply closed islands when the nonlinear amplitude dependence of the frequency or 'tune' is included. To show the limits of the validity of the isolated resonance approximation, we discuss two criteria for the onset of chaotic motion.

Finally, we study an isolated coupling resonance in two degrees of freedom with a 'time'-dependent Hamiltonian and calculate the two invariants in this case. This leads to a surface of section which is a 2-torus in 4-dimensional phase space. However, we show that it remains a 2-torus when projected into particular 3-dimensional subspaces and thus can be viewed in perspective.

2. THE MOTION OF A PARTICLE IN AN ACCELERATOR

2.1 THE HAMILTONIAN AND THE EQUATIONS OF MOTION

The motion of a particle in a circular accelerator is governed by the Lorentz force equation,

$$\frac{d\mathbf{P}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \ , \tag{2.1}$$

where $\mathbf{P}$ is the relativistic kinetic momentum and $\mathbf{v}$ is the velocity. It is convenient to cast these equations in Hamiltonian form. If we introduce the vector and scalar potentials,

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \quad , \end{aligned} \tag{2.2}$$

* Work supported by the Department of Energy, contract DE-AC03-76SF00515 .
Invited paper presented at the Joint US/CERN School on
Particle Accelerators, Sardinia, Italy, January 31 – February 5, 1985.

then the Hamiltonian is given by

$$H = e\phi + c\left[m^2c^2 + (\mathbf{p} - e\mathbf{A}/c)^2\right]^{1/2} , \tag{2.3}$$

where $\mathbf{p}$ is the canonical momentum. In terms of the kinetic momentum and the vector potential

$$\mathbf{p} = \mathbf{P} + \frac{e}{c}\mathbf{A}(\mathbf{x}, t) . \tag{2.4}$$

The equations of motion can then be written in terms of Hamilton's equations,

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{x}} \quad , \quad \frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}} \quad . \tag{2.5}$$

It is useful to use a coordinate system based on a closed planar reference curve as shown in Fig. 1.1. This reference curve is taken to be the closed trajectory of a particle with some reference momentum p_0 in the guiding magnetic field. The coordinate system (x, s, y) is similar to a cylindrical system, however, the radius of curvature may vary along the curve. If $\mathbf{r}$ is the coordinate of a particle in space, and $\mathbf{r}_0$ is the point on the reference curve closest to $\mathbf{r}$, then

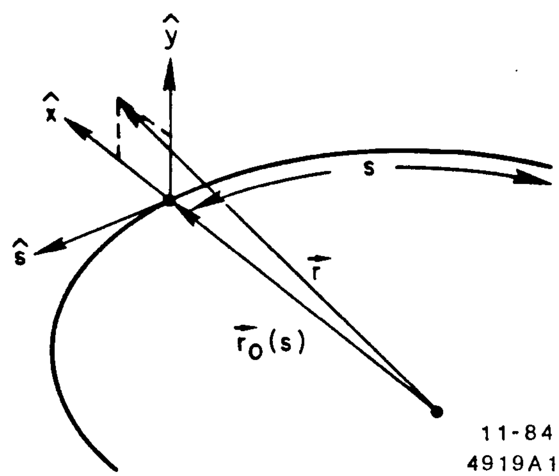

Fig. 1.1. The coordinate system.

s = distance along the curve to the point $\mathbf{r}_0$ from a fixed origin somewhere on the curve,
x = horizontal projection of the vector $\mathbf{r} - \mathbf{r}_0$,
y = vertical projection of the vector $\mathbf{r} - \mathbf{r}_0$,
ρ = local radius of curvature.

The Hamiltonian written in terms of these coordinates is[2]

$$H = e\phi + c\left[m^2c^2 + \frac{(p_s - \frac{e}{c}A_s)^2}{(1 + \frac{x}{\rho})^2} + \left(p_x - \frac{e}{c}A_x\right)^2 + \left(p_y - \frac{e}{c}A_y\right)^2\right]^{1/2} \tag{2.6}$$

where p_x and p_y are projections of $\mathbf{p}$ onto the x and y direction and

$$p_s = (\mathbf{p} \cdot \hat{s})\left(1 + \frac{x}{\rho}\right) \quad . \tag{2.7}$$

We will call the vector potential used in Eq. (2.6) the *canonical* vector potential since A_s, A_x, and A_y are defined analogously to the canonical momenta. In particular note that

$$A_s = (\mathbf{A} \cdot \hat{s})\left(1 + \frac{x}{\rho}\right) \quad . \tag{2.8}$$

Instead of using the Hamiltonian above, it is useful to change the independent variable to s rather than t. This can be done provided that s is monotonic in t. This is a standard transformation and can be accomplished by defining another Hamiltonian,

$$\mathcal{H} \equiv -p_s(x, p_x, y, p_y, t, -H) \ . \tag{2.9}$$

That is, we solve Eq. (2.6) for p_s. With this new Hamiltonian and new independent variable, Hamilton's equations become

$$\begin{aligned} \frac{dx}{ds} &= \frac{\partial \mathcal{H}}{\partial p_x} \ , & \frac{dp_x}{ds} &= -\frac{\partial \mathcal{H}}{\partial x} \\ \frac{dy}{ds} &= \frac{\partial \mathcal{H}}{\partial p_y} \ , & \frac{dp_y}{ds} &= -\frac{\partial \mathcal{H}}{\partial y} \\ \frac{dt}{ds} &= \frac{\partial \mathcal{H}}{\partial(-H)} \ , & \frac{d(-H)}{ds} &= -\frac{\partial \mathcal{H}}{\partial t} \ . \end{aligned} \tag{2.10}$$

Note that $(t, -H)$ now play the role of the third coordinate and conjugate momentum.

To be specific we will specialize to the case of no electric field and a constant magnetic field given by

$$\begin{aligned} B_y &= -B_0(s) + B_1(s)\ x + \cdots \\ B_x &= B_1(s)\, y + \cdots \ . \end{aligned} \tag{2.11}$$

The main bending field $B_0(s)$ is chosen so that a particle at the reference momentum p_0 will bend with a local radius of curvature $\rho(s)$. Thus, we set

$$B_0(s) = \frac{p_0 c}{e\rho(s)} \ . \tag{2.12}$$

$B_1(s)$ in Eq. (2.11) is simply the gradient of the magnetic field. It is conventional and useful to scale the gradient to obtain the focusing function,

$$K_1(s) = \frac{eB_1(s)}{p_0 c} \ . \tag{2.13}$$

Using Eqs. (2.12) and (2.13) the canonical vector potential which yields the above magnetic field is

$$A_s = -\frac{p_0 c}{e}\left[\frac{x}{\rho} + \left(\frac{1}{\rho^2} - K_1\right)\frac{x^2}{2} + \frac{K_1 y^2}{2}\right] + \cdots \ . \tag{2.14}$$

The new Hamiltonian from Eq. (2.9) is

$$\mathcal{H} = (-p_s) = \frac{-e\,A_s}{c} - \left(1 + \frac{x}{\rho}\right)\left[\frac{H^2}{c^2} - m^2c^2 - p_x^2 - p_y^2\right]^{1/2} \ . \tag{2.15}$$

Since there is no time dependence, H is a constant of the motion which we call E (the energy). In an actual accelerator the magnetic fields do change in time, and there are longitudinal electric fields to accelerate the particles. However, the acceleration process is slow and can be considered adiabatic for our purposes. In addition, the longitudinal electric fields cause longitudinal oscillations which are omitted here. These are discussed in Ref. 3 in these proceedings.

To continue we expand the square root in Eq. (2.15) and substitute the vector potential from Eq. (2.14) to obtain

$$\mathcal{H} = (p_0 - p)\frac{x}{\rho} + p_0\left[\left(\frac{1}{\rho^2} - K_1\right)\frac{x^2}{2} + K_1\frac{y^2}{2}\right] + \frac{p_x^2}{2p} + \frac{p_y^2}{2p} + \cdots \quad , \tag{2.16}$$

where p is the total kinetic momentum of the particle,

$$p = [E^2/c^2 - m^2c^2]^{1/2} \ , \tag{2.17}$$

which may be somewhat different from the reference momentum. The expansion of the square root is a good approximation provided that

$$\left|\frac{p_{x,y}}{p}\right| \ll 1 \ , \tag{2.18}$$

which is typically the case. From Hamilton's equations and the Hamiltonian in Eq. (2.16) we find

$$\begin{aligned} \frac{dx}{ds} &= \frac{p_x}{p} \quad , \quad \frac{dp_x}{ds} = -p_0\left(\frac{1}{\rho^2} - K_1\right)x + \frac{(p - p_0)}{\rho} \\ \frac{dy}{ds} &= \frac{p_x}{p} \quad , \quad \frac{dp_y}{ds} = -p_0K_1y \ . \end{aligned} \tag{2.19}$$

In terms of x and y Eqs. (2.19) become

$$\begin{aligned} x'' + \frac{p_0}{p}\left(\frac{1}{\rho^2} - K_1\right)x &= \frac{p - p_0}{p}\frac{1}{\rho} \\ y'' + \frac{p_0K_1}{p}y &= 0 \quad , \end{aligned} \tag{2.20}$$

where prime denotes differentiation with respect to s. Equations (2.20) yield the motion of particles near the reference orbit. Because K_1 and ρ are *periodically* dependent on s with period C , the circumference, these equations are Hill's equations.

2.2 BETATRON OSCILLATIONS

Before proceeding to discuss the nonlinear terms which have so far been neglected, it is useful to discuss the linear equations of motion. Since Eqs. (2.20) are inhomogeneous, we construct a general solution by a linear combination of a particular solution of the inhomogeneous equation and the general solution of the homogeneous equation. It is conventional and useful to take the particular solution to be the periodic solution or *closed orbit*.

Let us assume that we have this periodic solution to Eq. (2.20), and let us denote it by $[x_\epsilon(s), p_\epsilon(s)]$. The periodic solution in the y direction is simply $y = 0$. (In the presence of errors the vertical closed orbit is nonzero and must also be calculated.) Now perform a canonical transformation which shifts the origin of phase space to (x_ϵ, p_ϵ). The transformation $(x, p) \mapsto (x_\beta, p_\beta)$ can be performed with the generating function

$$F_2(x, p_\beta) = (x - x_\epsilon(s))(p_\beta + p_\epsilon(s)) \ , \tag{2.21}$$

which yields the transformation equations

$$\begin{aligned} x &= x_\beta + x_\epsilon(s) \\ p &= p_\beta + p_\epsilon(s) \\ \mathcal{H}_\beta &= \mathcal{H} + \partial F_2/\partial s \ , \end{aligned} \tag{2.22}$$

where the identity transformation for y and p_y has been suppressed. Substituting into $\mathcal{H}$, the new Hamiltonian is given by

$$\mathcal{H}_\beta = p_0\left[\left(\frac{1}{\rho^2} - K_1\right)\frac{x_\beta^2}{2} + K_1\frac{y^2}{2}\right] + \frac{p_\beta^2}{2p} + \frac{p_y^2}{2p} + \cdots \quad . \tag{2.23}$$

Thus, we are left a Hamiltonian with terms which are quadratic and higher order. In the nonlinear case a similar transformation can be performed; however, in this case we must use the periodic solutions to the full nonlinear equations.

The linear differential equations which are obtained from the Hamiltonian in Eq. (2.23) are of the form

$$z'' + K(s)z = 0 \ , \tag{2.24}$$

with

$$K(s) = K(s + C) \ , \tag{2.25}$$

where z stands for either x_β or y, and C is the circumference. The periodicity of K is that of the closed orbit; however, there may also be stronger periodicity imposed by design.

Equation (2.24) is Hill's equation and has a solution of the form

$$z = A\beta^{1/2}\cos(\psi(s) + \delta) \ , \tag{2.26}$$

where

$$\psi = \int_0^s \frac{ds'}{\beta(s')} \ , \tag{2.27}$$

and $\beta(s)$, the Courant-Snyder amplitude function,[2] is the periodic solution of

$$\beta''' + 4K\beta' + 2K'\beta = 0 \ , \tag{2.28}$$

with the additional condition

$$\beta\beta''/2 - (\beta')^2/4 + K\beta^2 = 1 \ . \tag{2.29}$$

Both A and δ are constants.

This solution is well known and constitutes a pseudo-harmonic oscillation with a periodically varying amplitude and wavelength. This motion is called *betatron oscillations* after the early betatron accelerators although in that case the transverse equations of motion reduced to two simple harmonic oscillator equations.

For stability, the tune ν,

$$\nu \equiv \frac{1}{2\pi}\int_0^C \frac{ds}{\beta(s)} \ , \tag{2.30}$$

must be non-integer. In the case of piecewise constant K, it is useful to use a matrix mapping technique to calculate both ν and $\beta(s)$.[2] This technique is used extensively in the design of magnetic lattices for circular accelerators.

2.3 Action-Angle Variables

To calculate the effects of higher-order nonlinear terms, it is useful to change variables to the action-angle variables of the linear problem. First assume that we have explicitly calculated $\beta(s)$. Then the transformation to action-angle variables, $(z, p_z) \mapsto (\phi, J)$, can be accomplished with the generating function

$$F_1(z,\phi) = -\frac{z^2}{2\beta(s)}\left[\tan\phi - \frac{\beta'(s)}{2}\right] \quad , \tag{2.31}$$

which yields the transformation equations

$$\begin{aligned} z &= \sqrt{2J\beta}\,\cos\phi \quad , \\ p_z &= -\sqrt{2J/\beta}\left(\sin\phi - \frac{\beta'}{2}\cos\phi\right) \quad , \\ H_1 &= H + \partial F_1/\partial s = J/\beta(s) \quad . \end{aligned} \tag{2.32}$$

where H has been scaled to make it dimensionless (see Section 3.1), and Eq. (2.29) has been used to simplify H_1. In these new coordinates the solution of the equations of motion is

$$\begin{aligned} J &= \text{constant} \quad , \\ \phi(s) &= \phi(0) + \int_0^s \frac{ds'}{\beta(s')} \quad . \end{aligned} \tag{2.33}$$

Note that we have explicitly constructed an invariant, J. Solving for the new action J in terms of the old variables we find

$$J = \frac{1}{2\beta}\left[z^2 + \left(\beta z' - \frac{\beta' z}{2}\right)^2\right] \quad . \tag{2.34}$$

Equation (2.34) is the equation of a torus in the extended phase space (z, p_z, s). If a particle has initial conditions which begin on some torus given by J_0, then the coordinates and momentum of that particle always stay on that torus. The usual way to view the torus is to take a surface of section at some s_0. The resulting curve is an ellipse in the phase space (z, p_z).

Alternatively, consider a single particle traversing the periodic focusing structure and plot its position and momentum in phase space each time it passes $s = s_0$. Then, the locus of those points is an ellipse in phase space. At points other than s_0, the ellipse so generated evolves according to Eq. (2.34).

The invariant J is simply related to the area enclosed by the ellipse,

$$\text{Area enclosed} = 2\pi J \quad . \tag{2.35}$$

In accelerator and storage ring terminology there is a quantity called the *emittance* which is closely related to this invariant. The emittance, however, is a property of a distribution of particles not a single particle. Consider a Gaussian distribution in amplitude. Then the (rms) emittance, ϵ, is given by

$$(y_{\text{rms}})^2 = \beta(s)\ \epsilon \quad . \tag{2.36}$$

In terms of the action variable, J, this can be rewritten

$$\epsilon = \langle J \rangle \quad , \tag{2.37}$$

where the bracket indicates an average over the distribution in J.

Finally note that the form of the new Hamiltonian is not precisely that of a harmonic oscillator in that the phase does not advance uniformly. This of course causes no difficulty in that both cases are trivial to solve. However, it is possible to perform another canonical transformation to coördinates which do have a uniformly advancing phase. This is accomplished with the canonical transformation $(\phi, J) \mapsto (\phi_1, J_1)$ with the generating function

$$
\begin{aligned}
F_2(\phi, J_1, s) &= J_1 \left[\frac{2\pi\nu s}{C} - \int_0^s \frac{ds'}{\beta(s')} \right] + \phi J_1 \quad , \\
\phi_1 &= \phi + \frac{2\pi\nu s}{C} - \int_0^s \frac{ds'}{\beta} \quad , \\
J_1 &= J \quad , \\
H_1 &= \frac{2\pi\nu}{C} J_1 \equiv \frac{\nu}{R} J_1 \quad .
\end{aligned}
\tag{2.38}
$$

In these new coordinates the oscillating part of the phase advance has been extracted leaving only the average phase advance. Either these coordinates or the previous set can be used to study nonlinear effects. We will use the second set in the section on canonical perturbation theory since no reference is made to a specific problem. In the examples of perturbation theory we will use the first coordinate set.

3. THE NONLINEAR TERMS

3.1 THE SOURCES OF NONLINEARITY AND CHROMATICITY

The nonlinear terms that have so far been neglected come from several sources. The so-called *geometric* terms arise from terms in the longitudinal vector potential which are higher than quadratic. These arise from both deliberate and inadvertent nonlinear magnetic fields. In addition, there are higher-order terms in the transverse components of the vector potential which are necessary to satisfy Maxwell's equations. There are also *kinematic* terms which come from the expansion of the square root in Eq. (2.15). Finally, in colliding beam storage rings there is the beam-beam force. A particle from one beam feels the electric and magnetic fields due to the collection of all the particles in the opposing beam. The beam-beam force is typically very strong, quite nonlinear, and of a different character than the others mentioned; therefore, it is usually treated separately. For useful reviews of the beam-beam effect see Refs. 4 and 5.

Aside from the beam-beam force, a dominant source of nonlinearity comes from the deliberate use of sextupoles to cure chromatic effects in storage rings. Before discussing the deleterious effects of sextupoles on the homogeneous equations, it is first useful to motivate their inclusion in the first place.

Let us first examine the Hamiltonian for betatron oscillations in Eq. (2.16). Since in all cases considered here p is a constant, it is first useful to scale the Hamiltonian with p to make it dimensionless. Defining the quantity

$$
\Delta \equiv \frac{p - p_0}{p} \quad , \tag{3.1}
$$

the effective Hamiltonian becomes

$$
\mathcal{H} = -\Delta \frac{x}{\rho} + (1 - \Delta) \left[\left(\frac{1}{\rho^2} - K_1 \right) \frac{x^2}{2} + K_1 \frac{y^2}{2} \right] + \frac{p_x^2}{2} + \frac{p_y^2}{2} + \cdots \tag{3.2}
$$

which is simply the Hamiltonian in Eq. (2.16) scaled appropriately. Note that in these new variables the canonical momenta are simply equal to the slopes dx_β/ds and dy/ds as is easily verified through Hamilton's equations. The quantity Δ measures the deviation of the actual momentum from the momentum on the reference orbit. It is clear from the Hamiltonian in Eq. (3.2) that the solutions of the linear equations of motion will depend on Δ as a parameter. Since all particle beams have a finite spread in momentum, this 'chromatic' dependence is undesirable. In addition, there are collective instabilities which are enhanced by these chromatic effects; thus, it is necessary to provide some chromatic correction.

3.2 Sextupoles for Chromatic Correction

To see the effects of sextupoles we must first include them in the Hamiltonian. The vector potential for a sextupole magnet is

$$eA_s/c = p_0\frac{S(s)}{6}(x^3 - 3xy^2)\ . \tag{3.3}$$

In terms of the magnetic field

$$S(s) = \frac{e}{p_0 c}\frac{d^2 B_y}{dx^2}\ . \tag{3.4}$$

$S(s)$ is a periodic function of s which is typically piecewise constant in the regions where the correction sextupoles are placed and zero elsewhere. If $S(s)$ comes from errors in magnetic field, then the strongest contribution is usually in the bending magnets which are typically pure dipole magnets.

The new Hamiltonian including sextupoles is

$$\hat{\mathcal{H}} = \frac{p_x^2}{2} + \frac{p_y^2}{2} - \Delta\frac{x}{\rho} + (1-\Delta)\left[-K_x\frac{x^2}{2} + K_1\frac{y^2}{2}\right] + (1-\Delta)\frac{S(s)}{6}(x^3 - 3xy^2) \tag{3.5}$$

where we have defined

$$K_x \equiv K_1 - \frac{1}{\rho^2} \tag{3.6}$$

in order to simplify the notation. Using Hamilton's equations, the differential equations for the motion are

$$\begin{aligned} x'' - (1-\Delta)K_x x + (1-\Delta)\frac{S}{2}(x^2 - y^2) &= \frac{\Delta}{\rho} \\ y'' + (1-\Delta)K_1 y - (1-\Delta)Sxy &= 0\ . \end{aligned} \tag{3.7}$$

The equations above may look slightly different from and somewhat simpler than others in the literature. The difference arises due to the definition of Δ chosen here.

At this point it is necessary to calculate the periodic solution to Eq. (3.7) above. This will give us the closed orbit for an off momentum particle in the full nonlinear field. By inspection we can see that once again the vertical closed orbit simply vanishes. In the horizontal direction it is conventional and useful to introduce the 'dispersion' function D. If we let the periodic solution be $x_\epsilon(s)$, then

$$D(s) \equiv x_\epsilon(s)/\Delta \tag{3.8}$$

where, of course, $D(s)$ is a periodic function of s. Writing the equation for the horizontal dispersion we find

$$D'' - (1-\Delta)K_x D + (1-\Delta)\frac{S}{2}D^2 = \frac{1}{\rho}\ . \tag{3.9}$$

$D(s)$ is the periodic solution to Eq. (3.9). With this definition, D depends upon Δ; however, since Δ is typically quite small, the dependence is weak. The more familiar linear dispersion function D_0 is obtained by setting Δ and S to zero in Eq. (3.9). D can be thought of as the exact dispersion function for the Hamiltonian in Eq. (3.5).

Now we would like to perform a canonical transformation to place the periodic orbit just calculated at the center of phase space. This transformation $(x, p_x) \mapsto (x_\beta, p_\beta)$ can be accomplished with the generating function

$$F_2(x, p_\beta) = (x - \Delta D(s))(p_\beta + \Delta D'(s)) \ , \tag{3.10}$$

which yields the transformation equations

$$\begin{aligned} x &= x_\beta + \Delta D(s) \\ p_x &= p_\beta + \Delta D'(s) \\ \hat{\mathcal{H}}_\beta &= \hat{\mathcal{H}} + \partial F_2/\partial s \ . \end{aligned} \tag{3.11}$$

Substituting using the Hamiltonian in Eq. (3.5) yields the new Hamiltonian

$$\begin{aligned} \hat{\mathcal{H}}_\beta = &\frac{p_\beta^2}{2} + \frac{p_y^2}{2} - K_x\frac{x_\beta^2}{2} + K_1\frac{y^2}{2} + \frac{S}{6}(x_\beta^3 - 3x_\beta y^2) \\ &- \Delta\left[(SD(s) - K_x)\frac{x_\beta^2}{2} - (SD(s) - K_1)\frac{y^2}{2} + \frac{S}{6}(x_\beta^3 - 3x_\beta y^2)\right] \\ &+ \Delta^2\frac{SD(s)}{2}(x_\beta^2 - y^2) \quad . \end{aligned} \tag{3.12}$$

Examining the linear chromatic terms, we find that sextupoles contribute to the linear differential equations at points where the dispersion D is nonzero. Thus, by adjusting $S(s)$ one can cancel many of the chromatic effects. In particular, one can cancel the linear variation of the tune with momentum.

Unfortunately, in the process of cancelling the chromatic effects, we add nonlinear terms to the equations of motion. To begin the study of the effects of these nonlinear terms on the motion, in the next section we discuss canonical perturbation theory.

4. CANONICAL PERTURBATION THEORY

In this section we seek a method to study nonlinear effects perturbatively. We do this by attempting to find a canonical transformation which makes the new Hamiltonian a function of the new momenta alone. This is just the approach which yields the Hamiltonian-Jacobi equation; however, in perturbation theory the new Hamiltonian may depend upon the coordinates and time in higher order.

Suppose that the problem can be described by a Hamiltonian

$$H = H_0(\mathbf{J}) \ + \ V(\mathbf{\Phi}, \mathbf{J}, \theta) \tag{4.1}$$

where H has been written in terms of action-angle variables of the unperturbed problem and bold face characters denote d-dimensional vectors. The unperturbed Hamiltonian H_0 includes nonlinear terms which depend only on $\mathbf{J}$; thus, the unperturbed tune may depend upon amplitude. In the absence of the perturbation, the action variables are invariant and the motion is confined to a $(d+1)$-dimensional torus in the extended phase space $(\mathbf{J}, \mathbf{\Phi}, \theta)$. In the following we look for the distortions of this torus due to the nonlinear perturbation.

Note that in this section we have scaled the independent variable from s to θ so that the Hamiltonian is 2π periodic in both the angle variables $\mathbf{\Phi}$ and the independent variable θ. In particular, the nonlinear perturbing term $V(\mathbf{\Phi}, \mathbf{J}, \theta)$ is a periodic function of θ and $\mathbf{\Phi}$ and has zero average with respect to them, *i.e.*,

$$\int_0^{2\pi} d\theta \int_0^{2\pi} d\mathbf{\Phi}\; V(\mathbf{\Phi}, \mathbf{J}, \theta) = 0 \quad . \tag{4.2}$$

If V has a nonzero average, the average value of V can be absorbed into $H_0(\mathbf{J})$.

Consider a canonical transformation $(\mathbf{J}, \mathbf{\Phi}) \mapsto (\mathbf{J}_1, \mathbf{\Phi}_1)$ with a generating function of the following form:

$$F_2(\mathbf{\Phi}, \mathbf{J}_1, \theta) = \mathbf{\Phi} \cdot \mathbf{J}_1 + G(\mathbf{\Phi}, \mathbf{J}_1, \theta) \quad . \tag{4.3}$$

The above transformation is close to the identity provided that G is small. The new coordinates and Hamiltonian are given by

$$\begin{aligned} \mathbf{\Phi}_1 &= \mathbf{\Phi} + G_{\mathbf{J}_1} \\ \mathbf{J} &= \mathbf{J}_1 + G_{\mathbf{\Phi}} \\ H_1 &= H + G_\theta \end{aligned} \tag{4.4}$$

where the subscripts indicate partial differentiation.

The new Hamiltonian after substituting the transformed variables is

$$H_1 = H_0(\mathbf{J}_1 + G_{\mathbf{\Phi}}) + V(\mathbf{\Phi}, \mathbf{J}_1 + G_{\mathbf{\Phi}}, \theta) + G_\theta \quad . \tag{4.5}$$

Note that we have substituted so that the Hamiltonian is a function of the same variables as G, the old coordinates and the new momenta. Eventually we must complete the substitution; however, for the moment it is more convenient to work with the mixed variables. Equation (4.5) can be rewritten in the interesting form

$$\begin{aligned} H_1 = H_0(\mathbf{J}_1) &+ [H_0(\mathbf{J}_1 + G_{\mathbf{\Phi}}) - H_0(\mathbf{J}_1) - \boldsymbol{\nu}(\mathbf{J}_1) \cdot G_{\mathbf{\Phi}}] \\ &+ [V(\mathbf{\Phi}, \mathbf{J}_1 + G_{\mathbf{\Phi}}, \theta) - V(\mathbf{\Phi}, \mathbf{J}_1, \theta)] \\ &+ \boldsymbol{\nu}(\mathbf{J}_1) \cdot G_{\mathbf{\Phi}} + G_\theta + V(\mathbf{\Phi}, \mathbf{J}_1, \theta) \;, \end{aligned} \tag{4.6}$$

where $\boldsymbol{\nu}(\mathbf{J}_1)$ is the *vector* frequency as a function of amplitude of the unperturbed problem,

$$\boldsymbol{\nu}(\mathbf{J}) \equiv \frac{\partial H_0(\mathbf{J})}{\partial \mathbf{J}} \quad . \tag{4.7}$$

If we can find a solution to the equation

$$\boldsymbol{\nu}(\mathbf{J}_1) \cdot G_{\mathbf{\Phi}} + G_\theta + V(\mathbf{\Phi}, \mathbf{J}_1, \theta) = 0 \;, \tag{4.8}$$

G will be a quantity of order V. All other parts of the new Hamiltonian are either independent of the coordinates and time or are of order V^2. To see this more easily we can expand for small G to obtain

$$H_1 = H_0(\mathbf{J}_1) + \boldsymbol{\nu}(\mathbf{J}_1) \cdot G_{\mathbf{\Phi}} + G_\theta + V(\mathbf{\Phi}, \mathbf{J}_1, \theta) + [G_{\mathbf{\Phi}} \cdot \boldsymbol{\nu}_{\mathbf{J}_1} \cdot G_{\mathbf{\Phi}}/2 + V_{\mathbf{J}_1} \cdot G_{\mathbf{\Phi}}] + \cdots \quad . \tag{4.9}$$

Since we are looking for the distortions of the invariant torus, we must find the *periodic* solution to Eq. (4.8); however, in order for a periodic solution to exist, the average value of V must vanish. This was anticipated by our earlier requirement in Eq. (4.2).

Since both V and G are periodic functions of $\mathbf{\Phi}$, they can be Fourier analyzed,

$$\begin{aligned} V(\mathbf{\Phi}, \mathbf{J}_1, \theta) &= \sum_{\mathbf{m}} v_{\mathbf{m}}(\mathbf{J}_1, \theta)\, e^{i\mathbf{m}\cdot\mathbf{\Phi}} \\ G(\mathbf{\Phi}, \mathbf{J}_1, \theta) &= \sum_{\mathbf{m}} g_{\mathbf{m}}(\mathbf{J}_1, \theta)\, e^{i\mathbf{m}\cdot\mathbf{\Phi}} \quad . \end{aligned} \tag{4.10}$$

Then the equation to be solved for G becomes

$$\left[i\mathbf{m}\cdot\boldsymbol{\nu}(\mathbf{J}_1) + \frac{\partial}{\partial\theta}\right] g_{\mathbf{m}} = -v_{\mathbf{m}} \quad , \tag{4.11}$$

which has the periodic solution

$$g_{\mathbf{m}} = \frac{i}{2\sin(\pi\mathbf{m}\cdot\boldsymbol{\nu})} \int_{\theta}^{\theta+2\pi} e^{i\mathbf{m}\cdot\boldsymbol{\nu}(\theta'-\theta-\pi)}\, v_{\mathbf{m}}(\mathbf{J}_1, \theta')\, d\theta' \quad . \tag{4.12}$$

Finally, the full expression for G is given by

$$G = \sum_{\mathbf{m}} \frac{i}{2\sin(\pi\mathbf{m}\cdot\boldsymbol{\nu})} \int_{\theta}^{\theta+2\pi} e^{i\mathbf{m}\cdot[\mathbf{\Phi}+\boldsymbol{\nu}(\theta'-\theta-\pi)]} v_{\mathbf{m}}(\mathbf{J}_1, \theta')\, d\theta' \quad . \tag{4.13}$$

Sometimes it is desirable to make use of the fact that V is a periodic function of θ to expand it as a 'double' Fourier series

$$V = \sum_{\mathbf{m},n} v_{\mathbf{m}\,n}(\mathbf{J}_1)\, e^{i(\mathbf{m}\cdot\mathbf{\Phi}-n\theta)} \quad . \tag{4.14}$$

This leads to an alternative expression for the generating function in Eq. (4.13),

$$G = i \sum_{\mathbf{m},n} \frac{v_{\mathbf{m}\,n}(\mathbf{J}_1)\, e^{i(\mathbf{m}\cdot\mathbf{\Phi}-n\theta)}}{\mathbf{m}\cdot\boldsymbol{\nu} - n} \quad . \tag{4.15}$$

Recall that our original purpose was to transform the Hamiltonian into a form which is approximately independent of the coordinates and the time. The new Hamiltonian in Eq. (4.9) is now given by

$$\begin{aligned} H_1 &= H_0(\mathbf{J}_1) \;+\; [V_{\mathbf{J}_1}\cdot G_{\mathbf{\Phi}} + G_{\mathbf{\Phi}}\cdot\boldsymbol{\nu}_{\mathbf{J}_1}\cdot G_{\mathbf{\Phi}}/2 + \cdots] \\ &\equiv H_0(\mathbf{J}_1) \;+\; V'(\mathbf{J}_1, \mathbf{\Phi}_1, \theta) \quad . \end{aligned} \tag{4.16}$$

The remaining nonlinear term can be separated into a part which depends only on the new action variable and into another part which involves $\mathbf{J}_1$, $\mathbf{\Phi}_1$ and θ but which has zero average value. This oscillatory term is the object of the next canonical transformation, whereas the term which is a function of the new action variable $\mathbf{J}_1$ leads to a change of frequencies with amplitude. The latter term is given by

$$\langle V' \rangle \equiv \frac{1}{(2\pi)^{d+1}} \int_0^{2\pi} d\theta \int_0^{2\pi} d\mathbf{\Phi} \, [V_{\mathbf{J}_1} \cdot G_{\mathbf{\Phi}} + G_{\mathbf{\Phi}} \cdot \nu_{\mathbf{J}_1} \cdot G_{\mathbf{\Phi}}/2 + \cdots] \quad . \tag{4.17}$$

Separating the average value, the new Hamiltonian can be written

$$\begin{aligned} H_1 &= \underbrace{[\, H_0(\mathbf{J}_1) + \langle V'(\mathbf{J}_1) \rangle \,]} + \underbrace{[\, V' - \langle V' \rangle \,]} \\ &\equiv \quad H_{01}(\mathbf{J}_1) \quad + \quad V_1(\mathbf{\Phi}_1, \mathbf{J}_1, \theta) \end{aligned} \tag{4.18}$$

and the new frequency becomes

$$\boldsymbol{\nu}_1(\mathbf{J}_1) = \frac{\partial H_{01}}{\partial \mathbf{J}_1} = \boldsymbol{\nu}(\mathbf{J}_1) + \frac{\partial \langle V' \rangle}{\partial \mathbf{J}_1} \quad . \tag{4.19}$$

Note that if we examine the new perturbing term V_1, it is second order in the strength of the perturbation. In addition it is higher order in $\mathbf{J}_1$. If the original perturbation has a lowest-order contribution of order J_1^b, then the new term is of order $J_1^{(2b-1)}$. Therefore , for sufficiently small J_1, we can neglect V_1. If this is done, we have a new Hamiltonian which depends only upon the new momenta. Therefore, these new momenta are (approximate) constants of the motion, and from Eq. (4.4) for $J(\mathbf{\Phi}, J_1, \theta)$ the motion is restricted to a $(d+1)$-dimensional torus in phase space.

To proceed to higher order in perturbation theory there are two approaches. In the first approach we return to the generating function in Eq. (4.3) and express it as a power series in the strength of the perturbation. Then upon substitution into the Hamiltonian in Eq. (4.5), we obtain a hierarchy of equations as we cancel the perturbing terms order by order. In this approach if ϵ is the strength of the perturbing term, after the n^{th} step we are left with a perturbing term of order $\epsilon^{(n+1)}$.

In the second approach we begin where we left off and make successive canonical transformations which are formally identical to the first one. This method is called superconvergent perturbation theory and was first introduced in this context by Kolmogorov in his proof of the KAM theorem.[6] It is called superconvergent because on the n^{th} step the remaining perturbing term is of order ϵ^{2^n}. Despite the name, however, the method need not converge! If the procedure does converge, then it does so much faster than the first method.

Unfortunately these methods do not always work. Everything would be fine if G were always small; however, a quick inspection of Eq. (4.13) shows that this is not the case for arbitrary ν. There are resonances whenever

$$\mathbf{m} \cdot \boldsymbol{\nu} = \text{integers} \quad . \tag{4.20}$$

This happens because we have required periodic solutions to the equation for G. It is straightforward to see that if the resonance condition is satisfied, there are no periodic solutions to Eq. (4.11). In fact the amplitude of the solution grows linearly in θ.

Thus, in the neighborhood of a resonance one must abandon perturbation theory at least insofar as it applies to the resonance. We can continue to use perturbation theory for the nonresonant terms, but we must isolate the resonant term for special treatment. Before beginning the study of isolated resonances, it is first useful to apply perturbation theory to a few simple cases.

5. LINEAR PERTURBATIONS

It is interesting and useful to apply the canonical perturbation theory developed in the previous section to linear perturbations. In these cases we can solve the perturbed problems exactly; however, it is quite useful to have analytic formulae which describe the effect of a small perturbation. First consider the perturbation of the quadrupole gradient in one degree of freedom.

5.1 Quadrupole Gradient Perturbation

In this case, the Hamiltonian we consider is

$$H = \frac{p^2}{2} + \frac{K(s)z^2}{2} + \frac{k(s)z^2}{2} , \tag{5.1}$$

where $k(s)$, the coefficient of the linear perturbation, is considered small. The transformation to the action-angle variables of the unperturbed linear problem yields

$$H' = \frac{J}{\beta(s)} + \frac{Jk(s)\beta(s)}{2} [1 + \cos(2\phi)] . \tag{5.2}$$

Before proceeding it is necessary to include the average part of the perturbation in H_0,

$$H_0 \equiv J\,[1/\beta(s) + k(s)\beta(s)/2] . \tag{5.3}$$

This yields the shift of the phase advance to first order in the strength of the perturbation,

$$\phi_0(s) \equiv \psi(s) = \psi(0) + \int_0^s \frac{ds'}{\beta(s')} + \frac{1}{2}\int_0^s k(s')\beta(s')\,ds' . \tag{5.4}$$

The tune shift due to this additional phase advance is thus given by

$$\Delta\nu = \frac{1}{4\pi}\int_0^C k(s')\beta(s')\,ds' , \tag{5.5}$$

where C is the circumference.

Eq. (5.5) above is the well known formula for the tune shift due to a small quadrupole perturbation. In canonical perturbation theory it is obtained simply by averaging the Hamiltonian to obtain H_0 before proceeding to the first step of perturbation theory.

To calculate the first order distortions of the invariant curves it is only necessary to use the formula for the generating function in Eq. (4.13) to obtain

$$G = \frac{-J_1}{4\sin(2\pi\nu)}\int_s^{s+C} k(s')\beta(s')\sin 2(\phi + \psi(s') - \psi(s) - \pi\nu)\,ds' \quad , \tag{5.6}$$

where ν is the tune which *includes* the shift in Eq. (5.5). Note that the phase advance $\psi(s)$ from Eq. (5.4) appears in Eq. (5.6) rather than $\nu\theta$ as in Eq. (4.13). The approximate invariant curves are given by

$$J = J_1 + G_\phi(\phi, J_1, s) \tag{5.7}$$

with

$$J_1 = \text{constant} + O(k^2)\ . \tag{5.8}$$

From Eq. (5.6) we have explicitly

$$J = J_1 - \frac{J_1}{2\sin(2\pi\nu)} \int_s^{s+C} k(s')\beta(s') \cos 2(\phi + \psi(s') - \psi(s) - \pi\nu)\ ds'\ . \tag{5.9}$$

In standard accelerator physics literature one usually finds the distortions of the β function calculated rather than the invariant curves. This is simply related to the variation in amplitude of the invariant curve at $\phi = 0$. Identifying the new beta function $\beta_1(s)$, we find

$$\frac{\beta_1(s) - \beta_0(s)}{\beta_0(s)} = \frac{-1}{2\sin(2\pi\nu)} \int_s^{s+C} k(s')\beta_0(s') \cos 2(\psi(s') - \psi(s) - \pi\nu)\ ds'\ . \tag{5.10}$$

This form is somewhat different than usual in that it is the perturbed tune which appears in the formula.

5.2 Weak Linear Coupling

It is also interesting to apply canonical perturbation theory to the case of weak linear coupling. The perturbed Hamiltonian is given by

$$H = \frac{p_x^2}{2} + \frac{p_y^2}{2} - \frac{K_x(s)x^2}{2} + \frac{K_1(s)y^2}{2} + M(s)xy\ , \tag{5.11}$$

where $M(s)$ is the skew focusing function defined by

$$M(s) = \frac{e}{p_0 c}\frac{\partial B_y}{\partial y}\ . \tag{5.12}$$

In this case the transformation to the action-angle variables of the unperturbed linear problem yields

$$H_1 = \frac{J_1}{\beta_1(s)} + \frac{J_2}{\beta_2(s)} + 2M(s)(\beta_1\beta_2)^{1/2}(J_1 J_2)^{1/2}\cos(\phi_1)\cos(\phi_2)\quad . \tag{5.13}$$

Now if we treat the last term above as a perturbation, we can use the perturbation theory developed previously.

From Eq. (4.13) the generating function in this case is

$$\begin{aligned} G = &\frac{-(I_1 I_2)^{1/2}}{2\sin\pi(\nu_1+\nu_2)} \int_s^{s+C} M(s')[\beta_1(s')\beta_2(s')]^{1/2} \sin\Psi_+(\phi_1,\phi_2,s,s')ds' \\ &- \frac{(I_1 I_2)^{1/2}}{2\sin\pi(\nu_1-\nu_2)} \int_s^{s+C} M(s')[\beta_1(s')\beta_2(s')]^{1/2} \sin\Psi_-(\phi_1,\phi_2,s,s')ds'\ . \end{aligned} \tag{5.14}$$

where the subscripts 1 and 2 refer to x and y, I_1 and I_2 are the new action variables, and the phase factors in the integral are given by

$$\Psi_{\pm}(\phi_1,\phi_2,s,s') \equiv (\phi_1 + \psi_1(s') - \psi_1(s) - \pi\nu_1) \pm (\phi_2 + \psi_2(s') - \psi_2(s) - \pi\nu_2) \quad , \tag{5.15}$$

where

$$\psi_{1,2}(s) \equiv \int_0^s \frac{ds'}{\beta_{1,2}(s')} \quad . \tag{5.16}$$

To calculate the invariant surfaces we simply use Eq. (4.4) to obtain

$$\begin{aligned} J_1 &= I_1 + G_{\phi_1}(\phi_1,\phi_2,I_1,I_2,s) \\ J_2 &= I_2 + G_{\phi_2}(\phi_1,\phi_2,I_1,I_2,s) \quad , \end{aligned} \tag{5.17}$$

where I_1 and I_2 are constant.

In this case the distorted invariant surface is a 3-torus in the extended 5-dimensional phase space. If we make a surface of section at some s_0, then we remain with a 2-torus in 4-dimensional phase space. In the uncoupled case this torus is simply the direct product of the two ellipses from the horizontal and vertical phase spaces; however, in the case of coupling this is no longer true. There are at least two different ways to view the invariant surface. One can make another surface of section, say at $\phi_2 = \phi_0$, and view the resulting curve in (J_1, ϕ_1) phase space. Alternatively, one can project the surface onto a three dimensional subspace, (ϕ_1, ϕ_2, J_1) or (ϕ_1, ϕ_2, J_2). If we examine Eq. (5.17), we find that in these 3-dimensional subspaces the invariant surface remains a 2-torus. This surface can be viewed in perspective in each of the subspaces mentioned above. This latter method will be discussed in detail in Section 8.

Finally, in the linear coupling case, it is possible to return to the Hamiltonian in Eq. (5.11) to find the eigenvectors which decompose the torus into the direct product of two circles by directly solving the linear differential equations. However, these do not project as simple curves in the original phase spaces.

6. A SEXTUPOLE PERTURBATION IN ONE DEGREE OF FREEDOM

In this section we apply perturbation theory to a sextupole perturbation in one degree of freedom. Since there are also coupling terms in the Hamiltonian in Eq. (3.12), one should actually use 2-dimensional perturbation theory. However, for the sake of brevity, we treat only one degree of freedom here; the extension to two degrees of freedom is quite straight forward by following the previous section.

From Eq. (3.12) we consider the non-chromatic part of the Hamiltonian for horizontal motion,

$$H = \frac{1}{2}(p^2 + K(s)x^2) + \frac{S(s)}{6}x^3 \quad . \tag{6.1}$$

Recall that $S(s)$ is periodic with period C (the circumference) but may have stronger periodicity imposed by design. Transforming to the action-angle variables introduced in Eq. (2.32) we obtain the new Hamiltonian

$$
\begin{aligned}
H &= J/\beta(s) + \frac{\sqrt{2}}{3} S(s)(J\beta)^{3/2} \cos^3\phi \\
&\equiv J/\beta(s) + V(\phi, J, s) \quad .
\end{aligned}
\tag{6.2}
$$

From Eq. (6.2) the perturbing term is

$$
V(\phi, J, s) = \frac{1}{6\sqrt{2}} S(s)(J\beta(s))^{3/2}[\cos 3\phi + 3\cos\phi] \quad ,
\tag{6.3}
$$

and using Eq. (4.13) the generating function is

$$
\begin{aligned}
G = -\frac{J_1^{3/2}}{\sqrt{2}} \Bigg\{ &\frac{1}{4\sin\pi\nu} \int_s^{s+C} ds' S(s')\beta(s')^{3/2} \sin[\phi + \psi(s') - \psi(s) - \pi\nu] \\
&+ \frac{1}{12\sin 3\pi\nu} \int_s^{s+C} ds' S(s')\beta(s')^{3/2} \sin 3[\phi + \psi(s') - \psi(s) - \pi\nu] \Bigg\} \quad .
\end{aligned}
\tag{6.4}
$$

Note that since the phase of betatron motion does not advance uniformly like a harmonic oscillator, the factor of $\nu\theta$ in Eq. (4.13) is replaced in Eq. (6.4) by $\psi(s)$ where

$$
\psi(s) \equiv \int_0^s \frac{ds'}{\beta(s')} \quad .
\tag{6.5}
$$

Next we can evaluate the average of the new perturbing term in Eq. (4.17). V_{J_1} and G_ϕ are given by

$$
\begin{aligned}
V_{J_1} &= \frac{\partial V}{\partial J_1} = \frac{1}{4\sqrt{2}} S(s)(J_1)^{1/2}\, \beta(s)^{3/2}[\cos 3\phi + 3\cos\phi] \\
G_\phi &= \frac{\partial G}{\partial \phi} = -\frac{J_1^{3/2}}{\sqrt{2}} \Bigg\{ \frac{1}{4\sin\pi\nu} \int_s^{s+C} ds' S(s')\beta(s')^{3/2} \cos[\phi + \psi(s') - \psi(s) - \pi\nu] \\
&\qquad + \frac{1}{4\sin 3\pi\nu} \int_s^{s+C} ds'\, S(s')\beta(s')^{3/2} \cos 3[\phi + \psi(s') - \psi(s) - \pi\nu] \Bigg\} \quad .
\end{aligned}
\tag{6.6}
$$

First we average over ϕ to get rid of the cross term and then average over s to obtain

$$
\begin{aligned}
\langle V_{J_1}\, G_\phi \rangle = -\frac{J_1^2}{64C} &\int_0^C ds\, \beta(s)^{3/2}\, S(s) \int_s^{s+C} \beta(s')^{3/2} S(s') ds' \\
&\times \left\{ \frac{3\cos(\psi(s') - \psi(s) - \pi\nu)}{\sin\pi\nu} + \frac{\cos 3(\psi(s') - \psi(s) - \pi\nu)}{\sin 3\pi\nu} \right\} \quad .
\end{aligned}
\tag{6.7}
$$

If the actual distribution of sextupoles is known, the integral in Eq. (6.7) can be evaluated. If we drop the fluctuating term, the new Hamiltonian is given by

$$H_1 = J_1/\beta(s) + \langle G_\phi \, V_{J_1} \rangle + \cdots \quad . \tag{6.8}$$

The new tune is then obtained by integrating the phase advance through one turn

$$\begin{aligned} \nu_1(J_1) &= \frac{1}{2\pi} \int_0^C \left(\frac{1}{\beta(s)} + \frac{\partial \langle G_\phi V_{J_1} \rangle}{\partial J_1} \right) ds \\ &= \nu + \frac{C}{2\pi} \frac{\partial \langle G_\phi V_{J_1} \rangle}{\partial J_1} \end{aligned} \quad . \tag{6.9}$$

Since the additional term in the new Hamiltonian in Eq. (6.8) is of order J^2, the tune in Eq. (6.9) varies linearly with J. This is similar to the first-order effect of an octupole perturbation ($\sim x^4$); therefore, a sextupole perturbation in second order produces an octupole-like nonlinear frequency shift with amplitude.

Finally, the approximate invariant torus is given by

$$J = J_1 + G_\phi(J_1, \phi, s) \quad , \tag{6.10}$$

with $J_1 =$ constant. As the tune approaches $n/3$ the phase space curves obtained at some surface of section $s = s_0$ develop the characteristic 3^{rd} harmonic distortion of the third integer resonance. However, when the tune is too close to a third integer resonance, G is not small and perturbation theory is not appropriate. In the next sections we confront this problem for general nonlinear resonances.

7. A NONLINEAR RESONANCE IN ONE DEGREE OF FREEDOM

In Section 4 we discovered that there were resonances whenever

$$\mathbf{m} \cdot \boldsymbol{\nu} = n \, . \tag{7.1}$$

Perturbation theory is not the appropriate method for studying the behavior in the neighborhood of such a resonance. In this section we study an isolated nonlinear resonance in one degree of freedom in detail, that is, a 2-dimensional phase space with a 'time' dependent Hamiltonian. We suppose that we are close to a resonance and that all other nonresonant terms in the Hamiltonian can be neglected. Thus, we are left with the truncated Hamiltonian,

$$H_T = \nu J \; + \; \alpha(J) \; + \; f(J) \cos(m\phi - n\theta) \quad . \tag{7.2}$$

Note that we have separated H_0 into a linear and nonlinear part, and that $f(J)$ is taken to be positive in the region of interest.

This problem can be solved exactly by using a canonical transformation to a rotating system in phase space. The generating function for the transformation $(J, \phi) \mapsto (J_1, \phi_1)$ is

$$F_2(\phi, J_1) = (\phi - n\theta/m) J_1 \quad , \tag{7.3}$$

which yields the transformation equations

$$\phi_1 = \phi - n\theta/m \, , \quad J_1 = J \quad . \tag{7.4}$$

The new Hamiltonian is then given by

$$H_1 = H_T - n/m \; J_1 = \delta \; J_1 + \alpha(J_1) \; + \; f(J_1) \cos m\phi_1 \quad , \tag{7.5}$$

where

$$\delta = \nu - n/m \quad . \tag{7.6}$$

The Hamiltonian has been cast in a form explicitly independent of the 'time' variable θ; thus, it is a constant of the motion.

7.1 Fixed Points

In the phase space (ϕ_1, J_1) we can find a set of points where the trajectories are stationary. These *fixed points* can be obtained by the conditions

$$\frac{\partial H_1}{\partial J_1} = 0 , \quad \frac{\partial H_1}{\partial \phi_1} = 0 \quad , \tag{7.7}$$

which yield

$$\begin{aligned} &\sin m\phi_1 = 0 \\ &\delta + \alpha'(J_1) + f'(J_1)\cos m\phi_1 = 0 \quad , \end{aligned} \tag{7.8}$$

where the prime above indicates differentiation with respect to J_1.

In the polar coordinates $(\sqrt{J_1}, \phi_1)$, these form a string of points surrounding the origin, as shown in Fig. 7.1. In fact when $\sin m\phi_1 = 0$, $\cos m\phi_1 = \pm 1$ and for different signs of $\cos m\phi_1$ the characteristics of the fixed points are different. The trajectories surrounding stable fixed points, SFP, are closed (ellipses), while those surrounding unstable fixed points, UFP, are open (hyperbolic). Those fixed points where $\cos m\phi_1 = -1$ $(+1)$ are stable (unstable) since the potential has a minimum (maximum) there.

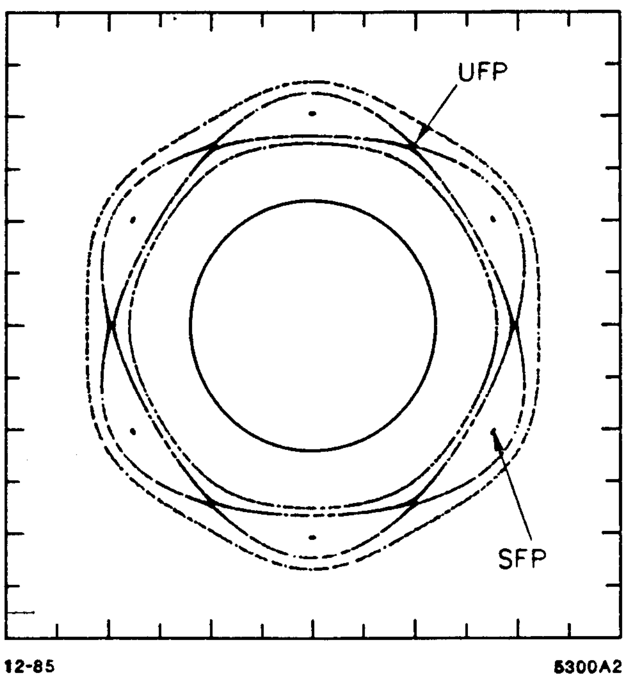

Fig. 7.1. Phase space for a sixth order resonance with a width of $\Delta J \simeq .2J_r$.

Suppose we define J_r as that amplitude which yields an oscillation frequency at resonance, *i.e.*,

$$\nu + \alpha'(J_r) = n/m \quad , \tag{7.9}$$

then Eq. (7.8) becomes

$$\alpha'(J_1) - \alpha'(J_r) + f'(J_1)\cos m\phi_1 = 0 \tag{7.10}$$

or expanding for J_1 close to J_r

$$(J_1 - J_r) \simeq -\frac{f'(J_r)}{\alpha''(J_r)}\cos m\phi_1 \quad . \tag{7.11}$$

Therefore, provided that f'/α'' is positive, the amplitude of the UFP is slightly less than J_r while the amplitude of the SFP is slightly larger than J_r.

7.2 RESONANCE ISLAND WIDTH

The boundaries of the stable islands shown in Fig. 7.1 are formed by curves joining the unstable fixed points. These curves are separatrices and their equation can be easily found by the fact that the new Hamiltonian H_1 is a constant on the curve.

From Eqs. (7.5) and (7.8), we have

$$\delta J_1 + \alpha(J_1) + f(J_1)\cos m\phi_1 = \delta J_u + \alpha(J_u) + f(J_u) \quad , \tag{7.12}$$

where J_u is the action at the unstable fixed point. Expanding for J close to J_u and recalling that $J_r \simeq J_u$, we find that on the separatrix

$$(J - J_u)^2 \simeq \frac{2f(J_r)(1-\cos m\phi_1)}{\alpha''(J_r)} \quad . \tag{7.13}$$

From Eq. (7.13) we find the maximum separation or island width

$$\Delta J = \pm 2\sqrt{\frac{f(J_r)}{\alpha''(J_r)}} \quad , \tag{7.14}$$

where $\alpha''(J_r)$ has been assumed positive for simplicity. Keep in mind that this is only valid when $\Delta J \ll J_r$. In addition, the other resonances which have so far been neglected must be far away. If the widths calculated using the isolated resonance assumption are such that neighboring resonances overlap each other, then it is clearly incorrect to consider the resonances isolated.

To summarize the phase space portrait shown in Fig. 7.1, at small amplitude the motion is relatively unaffected by the resonance. Near the resonance the circles are distorted. Finally, at the resonant amplitude there is a string of stable islands with widths determined (approximately) by Eq. (7.14).

7.3 ISLAND SEPARATION AND THE CHIRIKOV CRITERION

It has been observed that if the main resonance islands have widths which are close to their separation, there is chaotic behavior in the overlap region. This has been investigated extensively by B. Chirikov[7] and is used as a criterion to estimate the onset of stochastic instability. To apply the *Chirikov criterion* it is first necessary to calculate the spacing of the resonance islands.

To find the distance to a neighboring resonance, we first find the spacing in frequency and then convert that to amplitude. Near J_r the amplitude dependence of the tune is nearly linear. Therefore, two resonances with a spacing of $\Delta\nu$ are separated in amplitude by

$$\delta J = \Delta\nu/\alpha''(J_r). \tag{7.15}$$

To estimate the separation in frequency consider two *neighboring* resonances n/m and n'/m' such that $nm' - n'm = \pm 1$. (The sense in which these are close is discussed in Refs. 8 and 9.) Then in this case the spacing is given by

$$\delta J = \frac{\Delta\nu}{\alpha''(J_r)} \simeq \frac{1}{m'm\alpha''(J_r)} \quad . \tag{7.16}$$

To avoid chaotic behavior we require that the island width be much less than the island spacing. Using Eqs. (7.14) and (7.16) this becomes

$$\sqrt{\alpha''(J_r)f(J_r)} \ll \frac{1}{4m'm} \quad . \tag{7.17}$$

Equation (7.17) sets a limit to the validity of the isolated resonance analysis. This condition requires that the nonlinear detuning, α'', not be too large since in this case the resonances do not separate. On the other hand if α'' is small, the widths of the islands get large. Unfortunately, as we increase α'' the island width decreases more slowly than the separation. Thus, if we increase the nonlinear detuning we eventually get island overlap and stochastic instability. This leads one to select a moderate nonlinear detuning to avoid chaotic behavior.

7.4 Island 'Tune' and Greene's Residue Criterion

Having understood the phase space structure in general, we can study a particular island. Consider a small island width. In this case it is useful to expand the Hamiltonian in Eq. (7.5) for small deviations about J_r,

$$H_T \simeq \frac{\alpha''(J_r)}{2}(J - J_r)^2 + f(J_r)\cos m\phi_1 + \cdots \quad . \tag{7.18}$$

We have dropped constant terms and used the resonance condition in Eq. (7.9) for simplification. The Hamiltonian above is that for a pendulum; from Hamilton's equations we find

$$\frac{d^2\phi_1}{d\theta^2} + \alpha''(J_r) m f(J_r)\sin m\phi_1 = 0 \quad . \tag{7.19}$$

This is the equation of motion for a pendulum with familiar phase space structure shown in Fig. 7.2.

When the amplitude is small, the small amplitude oscillation frequency or 'tune' Ω can be obtained from Eq. (7.19) by approximating

$$\sin m\phi_1 \simeq m\phi_1 \quad , \tag{7.20}$$

which yields

$$\Omega^2 = \alpha''(J_r) f(J_r) m^2 \quad . \tag{7.21}$$

Using this frequency an alternate expression for the overlap condition can be derived.

J. Greene has established that the last invariant curve which separates two neighboring island chains survives provided that the 'residue' of the neighboring stable fixed points is less than about 1/4.[10] The residue R of resonance treated here is simply

$$R = \sin^2(\pi m\Omega) \quad . \tag{7.22}$$

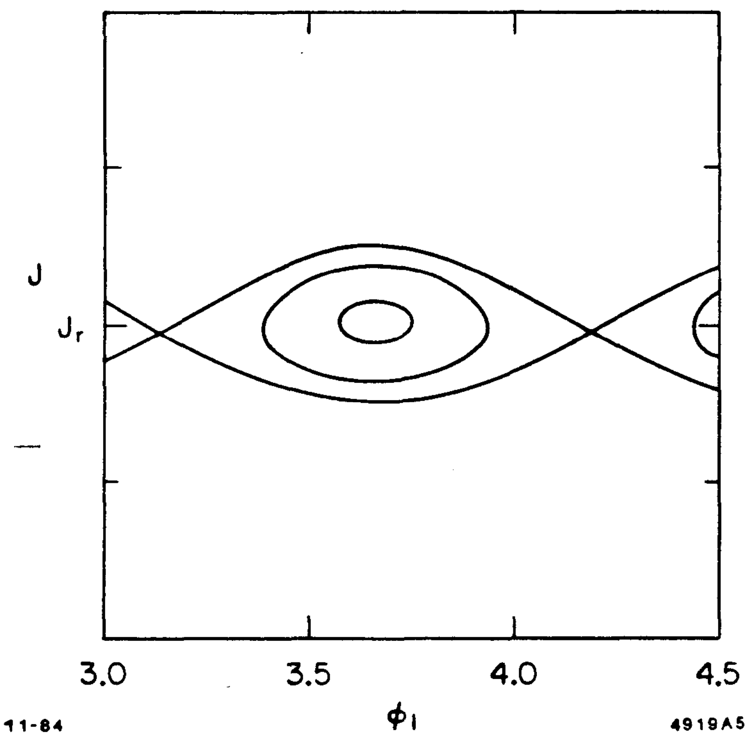

Fig. 7.2. Pendulum-like phase space structure.

If we rewrite the residue condition in terms of the frequency calculated above, it becomes

$$m\Omega < \frac{1}{6} \quad , \tag{7.23}$$

which yields

$$\sqrt{\alpha''(J_r)f(J_r)} < \frac{1}{6m^2} \quad . \tag{7.24}$$

At this point the region between the two island chains is typically quite stochastic. Thus, to avoid large scale stochastic behavior, the inequality in Eq. (7.24) should be strongly satisfied. Notice that the residue criterion and the overlap criterion are essentially identical in form although the residue criterion is a much more precise statement. Many more details of the residue criterion can be found in Refs. 8 and 9.

7.5 UNBOUNDED MOTION

So far we have treated cases in which the frequency of the unperturbed problem is a function of amplitude. This is important in that it yields finite island widths. However, if the unperturbed Hamiltonian is simply linear, then an isolated resonance causes unbounded motion. This case is particularly important for particle accelerators since the amplitude dependence of the tune is typically quite weak and in many cases can be neglected. To illustrate this consider a sextupole induced third order resonance with the Hamiltonian

$$H_T = \nu J + \epsilon J^{3/2} \cos(3\phi - \theta) \quad . \tag{7.25}$$

If we transform to the rotating system in phase space, we find the new invariant Hamiltonian

$$H_1 = \delta J_1 + \epsilon J_1^{3/2} \cos(3\phi_1) = \text{constant} \quad , \tag{7.26}$$

where in this case

$$\delta = \nu - 1/3 \quad . \tag{7.27}$$

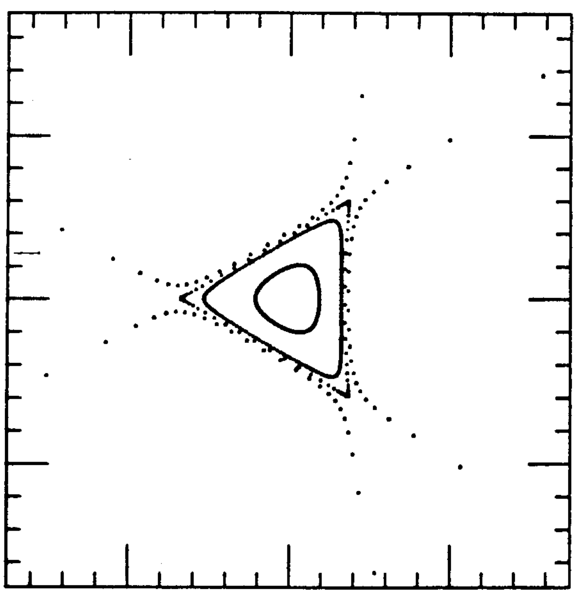

Fig. 7.3. Phase space near a third order resonance with $\alpha = 0$.

For δ nonzero the motion in phase space is shown in Fig. 7.3. The curves shown correspond to four different values of the invariant H_1. At small amplitude the circles are distorted and are described well by the first order perturbation theory in Section 6. For larger amplitude the curves approach a triangular shape with three unstable fixed points at the points of the triangle. Finally, at sufficiently large amplitude the motion is unbounded. As δ is decreased to zero, the stable area inside the triangle goes to zero. This effect is quite well known in accelerator physics literature since it is used as a mechanism for driving particles in a beam to large amplitude to extract them from circular accelerators.[11]

Unfortunately, sextupoles provide not only the cubic term which yields the resonance structure shown in Fig. 7.3, but also a coupling term $\sim xy^2$ as shown in Eq. (3.5). This leads us to the next section to consider coupling resonances.

8. AN ISOLATED RESONANCE IN TWO DEGREES FREEDOM

It is interesting and useful to consider an isolated resonance in 2 degrees of freedom (with a time dependent Hamiltonian). In a particle accelerator this corresponds typically to the coupling of the two transverse degrees of freedom; however, it could involve one transverse and the longitudinal degree of freedom. We will consider the former case here. In this case the resonance condition becomes

$$m_1\nu_1 + m_2\nu_2 = n \quad . \tag{8.1}$$

where m_1, m_2 and n are integers, and ν_1 and ν_2 are the tunes in the two transverse degrees of freedom. In the previous section we found resonances at all rational values of the tune, that is, at a set of points in tune space. In this case the resonances consist of lines in 2-dimensional tune space (ν_1, ν_2). In Fig. 8.1 we illustrate this with several examples. Note that as we include higher-order resonances the tune space rapidly fills up. Thus, to avoid resonances it is necessary to carefully place the two tunes.

8.1 CALCULATION OF THE INVARIANTS

Now consider two tunes which are close to one of the lines with finite slope in Fig. 8.1 but far from the intersection of any two lines. Thus, the system is close to an isolated *coupling* resonance. As in the previous section truncate the Hamiltonian so that only the dominant resonant term is retained. This yields

$$H_T = \nu_1 J_1 + \nu_2 J_2 + f(J_1, J_2)\cos(m_1\phi_1 + m_2\phi_2 - n\theta) \quad , \tag{8.2}$$

where for simplicity we have taken the unperturbed Hamiltonian to be that for uncoupled linear oscillation. Once again the truncated problem above can be solved exactly by transforming to a rotating system in phase space. The generating function for the transformation $(\phi_i, J_i) \mapsto (\psi_i, K_i)$ is

$$F_2(\phi_i, K_i, \theta) = (m_1\phi_1 + m_2\phi_2 - n\theta)K_1 + \phi_2 K_2 \quad . \tag{8.3}$$

$$\begin{aligned} \psi_1 &= m_1\phi_1 + m_2\phi_2 - n\theta \qquad & J_1 &= m_1 K_1 \\ \psi_2 &= \phi_2 & J_2 &= m_2 K_1 + K_2 \quad , \end{aligned} \tag{8.4}$$

and the new Hamiltonian becomes

$$H_1 = (m_1\nu_1 + m_2\nu_2 - n)K_1 + \nu_2 K_2 + \tilde{f}(K_1, K_2)\cos\psi_1 \quad , \tag{8.5}$$

where

$$\tilde{f}(K_1, K_2) = f(m_1 K_1, m_2 K_1 + K_2) \quad . \tag{8.6}$$

Since the Hamiltonian above is independent of the independent variable, it is a constant of the motion. In addition, however, it is independent of ψ_2. Therefore, the new action K_2 is also an invariant. Thus, we have

$$(m_1\nu_1 + m_2\nu_2 - n)K_1 + \nu_2 K_2 + \tilde{f}(K_1, K_2)\cos\psi_1 = \text{constant} \tag{8.7}$$

$$K_2 = \text{constant} \quad . \tag{8.8}$$

In terms of the old coordinates this becomes

$$\nu_1 J_1 + \nu_2 J_2 - \frac{n}{m_1}J_1 + f(J_1, J_2)\cos(m_1\phi_1 + m_2\phi_2 - n\theta) = \text{constant} \tag{8.9}$$

$$J_2 - \frac{m_2}{m_1}J_1 = \text{constant} \quad . \tag{8.10}$$

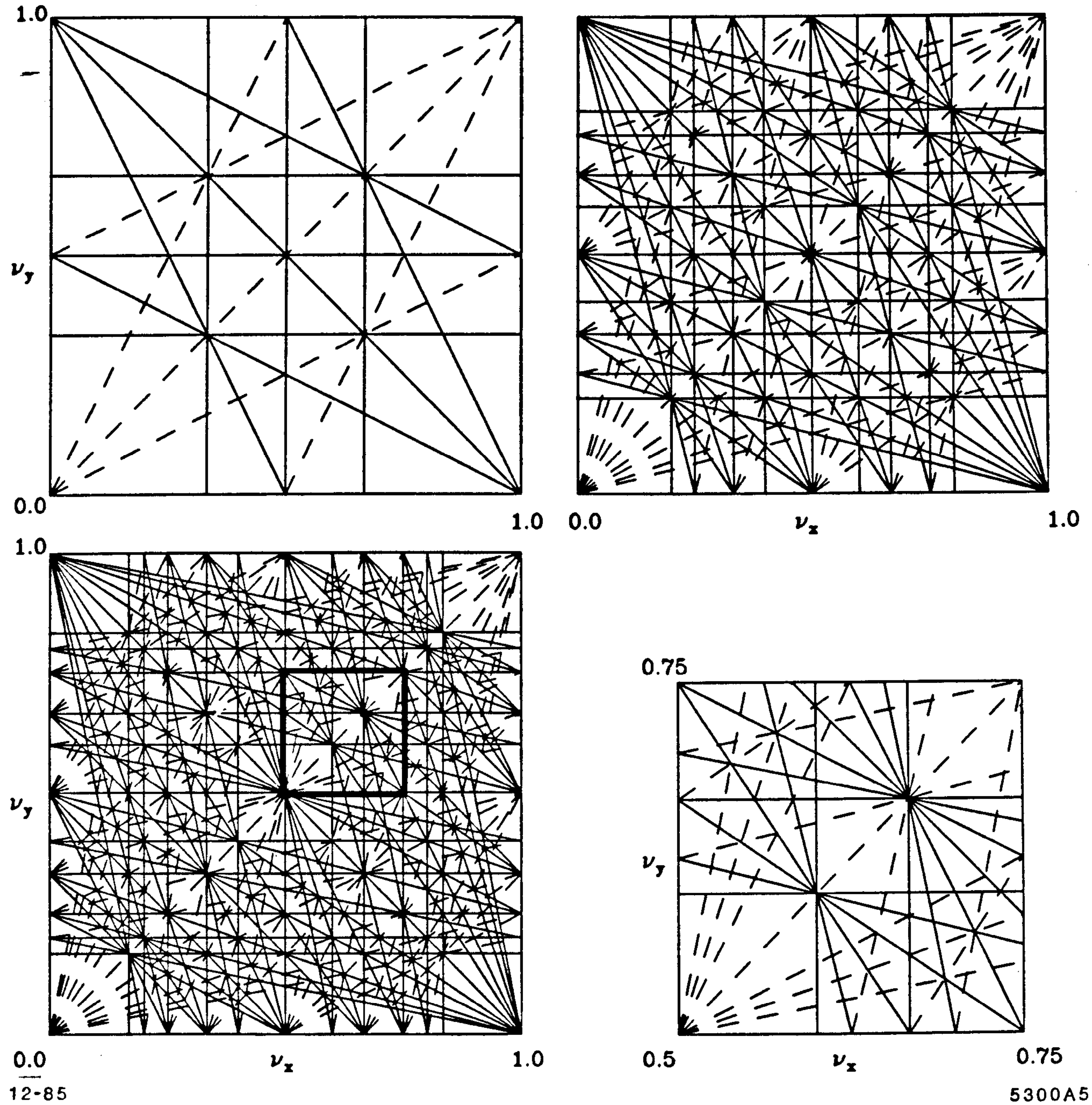

Fig. 8.1. Resonance lines in tune space. The figures include $|m_1| + |m_2| \leq 3$, 5, and 6. The outlined area in the sixth order case is blown up to show detail.

From Eq. (8.10) there are two distinct cases. In the case of a sum resonance, [sign(m_1) = sign(m_2)], stability is not guaranteed. However, in the case of a difference resonance [sign(m_1) = −sign(m_2)], stability is guaranteed since the weighted *sum* of the actions is a constant. In this second case there can be 'emittance' exchange; however, the overall motion is bounded.

8.2 VIEWING COUPLED MOTION[12]

As in the case discussed in Section 5.2, the motion near a coupling resonance is confined to a 3-torus in the extended phase space $(\phi_1, \phi_2, J_1, J_2, \theta)$. If we take a surface of section at some θ_0, then the resulting figure is a 2-torus in 4-dimensional phase space. We can view the 2-torus by taking yet another surface of section at $\phi_1 = \phi_0$ which yields a curve in (ϕ_1, J_1) space, or we could set $\phi_2 = \phi_0$ and view the resulting curve in (ϕ_2, J_2) space.

There is, however, another alternative as mentioned previously in Section 5.2. We can project the 2-torus onto a 3-dimensional subspace (ϕ_1, ϕ_2, J_1) or (ϕ_1, ϕ_2, J_2). In these subspaces we obtain a 2-torus imbedded in 3-dimensional space which can be viewed in perspective. This method is especially powerful if we are comparing theory and numerical experiments. In numerical experiments it is quite difficult to take a second surface of section mentioned above because there are so few points on it. The first surface of section does not suffer from this difficulty since it simply corresponds to the integration of the equations of motion through multiples of 2π.

To illustrate the technique first consider a system with 2-degrees of freedom far from a coupling resonances but close to a resonance $\nu_1 \simeq 1/3 \bmod(1)$. In this case the motion is nearly that corresponding to one degree of freedom. In Fig. 8.2 we show three equivalent ways of viewing the motion. In 8.2(a) you see the phase space $(J_1^{1/2}\cos\phi_1, -J_1^{1/2}\sin\phi_1)$ which would yield a circle for the case of uncoupled harmonic oscillation. The points are plotted at multiples

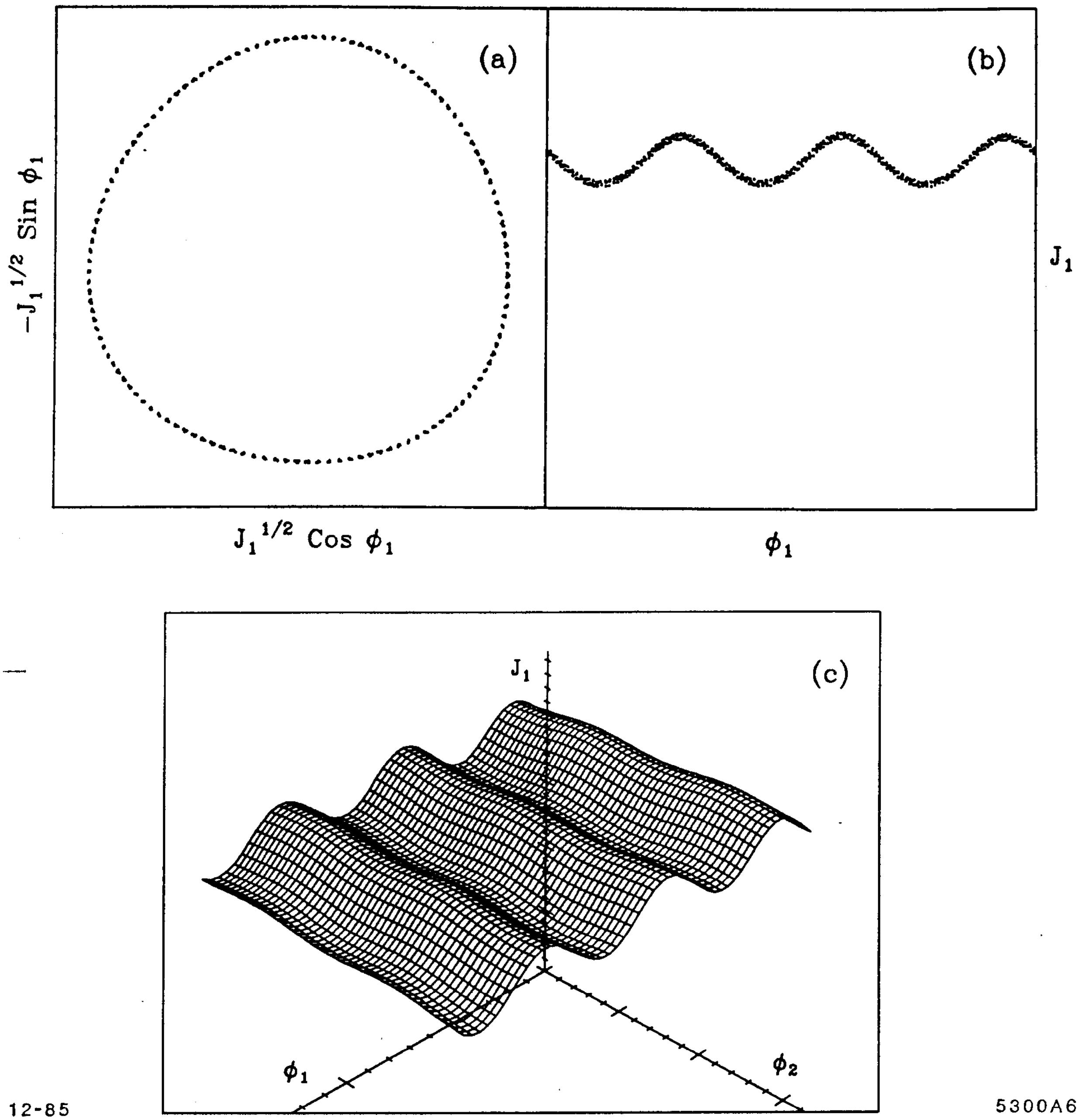

Fig. 8.2. Surface of section near a third integer resonance ($\nu_1 = 5.331$, $\nu_2 = 5.144$).

of 2π in θ without regard to J_2 or ϕ_2. The locus of the points has the characteristic distortion of a 1/3 integer resonance superimposed onto basically circular motion. In Fig. 8.2(b) we unfold 8.2(a) and plot J_1 *vs.* ϕ_1 to see the modulation due to the resonance more clearly. Notice that although the motion is very nearly in one degree of freedom, there is still a small coupling which leads to a band of motion rather than a curve. Finally in Fig. 8.2(c) you see the 2-torus in (ϕ_1, ϕ_2, J_1) space as calculated from first order perturbation theory. The influence of the 1/3 resonance is shown as the dominant wave on the torus. Notice that if we project the surface onto the (J_1, ϕ_1) plane, we obtain a figure essentially identical to 8.2(b). The coupling causes small ripples in the 2-torus which give rise to the band of motion in 8.2(b).

To view a coupling resonance with this technique consider the sextupole-induced resonance

$$2\nu_2 - \nu_1 = \text{integer} \quad . \tag{8.11}$$

First let us view the motion by numerical integration of the equations of motion. In Fig. 8.3 we plot (ϕ_1, J_1) and (ϕ_2, J_2) at $\theta = \theta_0 \bmod(2\pi)$ which in the case of simple linear motion would yield straight lines. In both plots we see a wide band of motion; however, this scattering of points does *not* indicate chaotic motion. To see this clearly we turn to the perspective method just described. In Fig. 8.4 we show the surface of section $\theta = \theta_0 \bmod(2\pi)$ near the coupling resonance. In 8.4(a) and 8.4(b) we plot the 2-torus as calculated with perturbation theory. Below in 8.4(c) and 8.4(d) we again plot *all* the data points obtained by numerical integration. The data fall nicely on the torus obtained by perturbation theory. Notice that near a coupling resonance the surface is similar to that in Fig. 8.2; however, the ripples no longer run parallel to one of the axes.

Using this technique it is possible in numerical experiments to separate chaotic motion from mere coupling. Chaotic motion is shown as departures from a surface similar to the departures from closed curves for the case of chaotic motion in one degree of freedom.

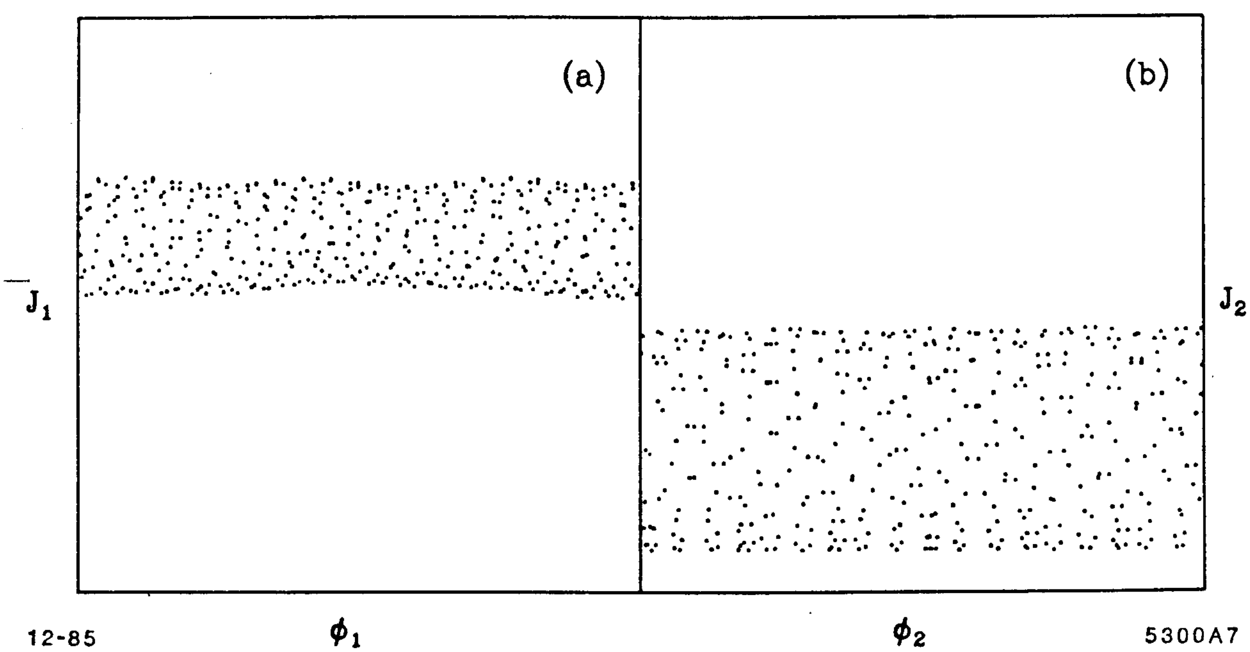

Fig. 8.3. The two phase space projections of coupled motion ($\nu_1 = 5.317$, $\nu_2 = 5.164$).

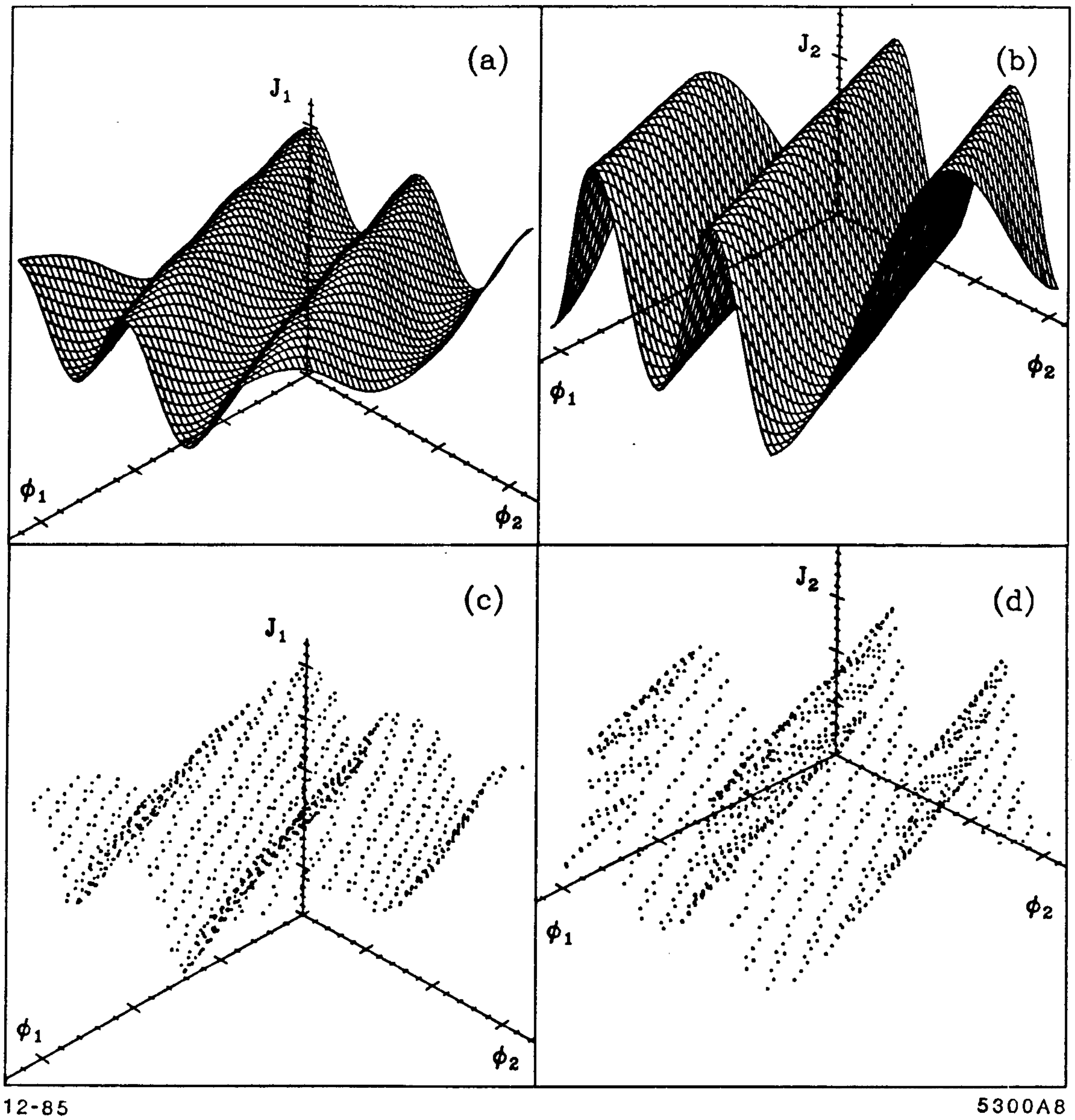

Fig. 8.4. Surface of section near a coupling resonance ($\nu_1 = 5.317,\ \nu_2 = 5.164$).

9. CONCLUDING REMARKS

This paper has attempted to cover the ground which lies between the relativistic equations of motion for a particle in an accelerator and the nonlinear resonances which affect particle motion near the reference orbit. The basic principles and techniques have been emphasized with a few examples for illustration. The treatment has been necessarily brief although most of the topics discussed deserve a much more thorough discussion.

Some topics have been completely omitted here. In particular, the *dynamic aperture* and methods for its determination in a particle accelerator have not been discussed; however, this gap is filled in Refs. 13 and 14 which also appear in these proceedings. In addition, the beam-beam effect was not treated here although many of the methods discussed are quite useful for that purpose. The reader is referred to Refs. 4 and 5 for useful reviews of the beam-beam effect. Finally, we included no discussion of 'synchro-betatron' resonances. Since the frequency of oscillation in the longitudinal degree of freedom is typically quite small compared to the transverse degrees of freedom, this coupling is usually treated separately. A review of this subject in Ref. 3 appears in these proceedings.

To conclude, let us emphasize that this paper has almost exclusively concentrated on integrable or nearly-integrable motion. Although the transition to chaotic behavior is extremely important in the design of particle accelerators, we have only briefly mentioned the overlap criterion and the residue criterion for determining the breakdown of invariant surfaces. We hope that this will encourage the reader to consult Refs. 9 and 15 and the other relevant papers in these proceedings.

ACKNOWLEDGEMENTS

I would like to thank John Jowett, Phil Morton and Bill Weng for useful discussions, Tor Raubenheimer for his help with figures and computations, and finally Robert Warnock for carefully reading this paper and for many useful suggestions.

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