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The Ensemble Projector Monte Carlo Method, Studying the U(1) Lattice Gauge Theory with Fermions in 2 + 1 Dimensions*

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ABSTRACT

The ensemble projector Monte Carlo method is a lattice Monte Carlo method for studying gauge theories with and without Fermions in the Hamiltonian formulation. We study the compact U(1) lattice gauge theory with fermions in $d = 2 + 1$ dimensions. There are matrix elements with positive and negative signs in lattice gauge theories with fermions. As in the case of the Schwinger model, we find the energy expectation values calculated from matrix elements using the two subsets of configurations with total positive and negative score to agree within the statistical errors to each other as well as to the energy expectation value calculated from average scores. We conclude that this method, which was previously only used in $d = 1+1$ dimensions can indeed be applied to lattice gauge theories with fermions in more than one spatial dimensions.

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1. INTRODUCTION

The ability to perform reliable Monte Carlo calculations for lattice gauge theories including fermions is crucial for the further progress of lattice gauge theories. This progress can come either from more powerful computers or from more efficient methods.

The Hamiltonian formulation of lattice gauge theories[1] in $d = 1+1$ dimensions provides the only example for efficient Monte Carlo methods for gauge theories with fermions. So far however, the negative signs, which occur in certain matrix elements in theories involving fermions, have been the stumbling block preventing the extension of these Hamiltonian methods to higher space dimensions.

The application of Hamiltonian methods to lattice gauge theories with fermions started with the local Hamiltonian Monte Carlo method.[2] This method was applied with success to model field theories like the massless[3] and massive[4] Schwinger model, to models with gauge bosons, Higgs bosons and fermions[5] and supersymetric models in $d = 1 + 1$ dimensions.[6]

The projector Monte Carlo method[7] and its extensions using replication[8] and parallel scores[9] demonstrate, that Hamiltonian Monte Carlo methods can be applied in more than one space dimensions to pure lattice gauge theories[8,9] and to lattice theories with fermions[10,11] in $d = 1 + 1$ dimensions.

In[12] we reported the application of the ensemble projector Monte Carlo method to the Schwinger model in a situation where matrix elements and scores have positive and negative signs. It was shown, that the energy expectation values, calculated from matrix elements using the two sub-ensembles of states with positive and negative total scores, as well as calculated from the average scores

were all consistent with each other. They were in addition consistent with the energy expectation values calculated with the local Hamiltonian Monte Carlo method. This latter method displays the correct behaviour in the massless limit, where the expectation values can be calculated analytically. It was also shown in Ref. 13, that the presence of intrinsic negative signs in the matrix elements does not prevent the Hamiltonian Monte Carlo calculations.

Here we use the method of Ref. 12 studying the compact U(1) lattice gauge theory with fermions in $d = 2 + 1$ dimensions. We find, that for these values of the coupling constant, where our method converges in the computer running time used, the different energy expectation values are again consistent. We conclude from this, that the ensemble projector Monte Carlo method can be applied to gauge theories involving fermions in $d = 2 + 1$ dimensions.

In Section 2 we formulate the lattice Hamiltonian of the compact U(1) lattice gauge theory in $d = 2 + 1$ dimensions. In Section 3 we discuss the ensemble projector Monte Carlo method. Section 4 treats the problem, how to use this method for lattice gauge theories including fermions. In Section 5 we apply the method to our lattice Hamiltonian and in Section 6 we present and discuss the results.

2. Hamiltonian Formulation of the U(1) Lattice Gauge Theory with Fermions in $d = 2 + 1$ Dimensions

A Hamiltonian formulation of 3 dimensional lattice QED was already given previously.[14,15] Hamiltonian Monte Carlo calculations were reported from the lattice theory without fermions.[8,9] The lattice Hamiltonian of compact QED has the form

$$H_g = \sum_{\vec{r}} \Bigg\{ \frac{g^2}{2} \left(E^2_{\vec{r},1} + E^2_{\vec{r},2} \right) + \frac{1}{a^2 g^2} \left[1 - \frac{1}{2} \left(a^+_{\vec{r},1} \, a^+_{\vec{r}+\vec{\mu}_x,2} \, a_{\vec{r}} + \vec{\mu}_y, 1 \, a_{\vec{r},2} + h.c. \right) \right] \Bigg\} \tag{1}$$

Here we formulate the Hamiltonian for the $d = 2 + 1$ dimensional compact U(1) gauge theory with fermions.

In $2 + 1$ dimensions, fermions are described by two component Dirac spinors. Free fermions in the continuum are described by the Dirac equation[♯1]

$$\dot{\psi} = \alpha_1 \frac{\partial \psi}{\partial x} + \alpha_2 \frac{\partial \psi}{\partial y} \tag{2}$$

Using single component Susskind fermions[16] $\phi(\vec{r})$ with the following anti-commutation relations

$$\{\phi(\vec{r}_1), \phi(\vec{r}_2)\} = 0; \quad \{\phi(\vec{r}_1)\phi^+(\vec{r}_2)\} = \delta_{\vec{r}_1, \vec{r}_2} \tag{3}$$

♯1 We use the Weyl representation,

$$\alpha_2 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \alpha_2 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \gamma_0 = i\alpha_1\alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

we write a lattice Hamiltonian

$$H_1 = \sum_{\vec{r}} \Big\{ \frac{i}{2a}(\phi^+(\vec{r})\phi(\vec{r}+\vec{\mu}_x) - h.c.) \\ - \frac{(-1)^{x+y}}{2a}(\phi^+(\vec{r})\phi(\vec{r}+\vec{\mu}_y) + h.c.) \Big\} \tag{4}$$

Following the analysis of Susskind[16] it is found, that the Hamiltonian (4) describes in the continuum limit two massless fermions, which we call u and d fermions.

Redefining

$$\chi(\vec{r}) = i^x \phi(\vec{r}) \tag{5}$$

we write (4) in the form to be used in the following

$$H_2 = \frac{1}{2a} \sum_{\vec{r}} \Big\{ \left(\chi^+(\vec{r})\chi(\vec{r}+\vec{\mu}_x) + h.c.\right) \\ - (-1)^{x+y} \left(\chi^+(\vec{r})\chi(\vec{r}+\vec{\mu}_y) + h.c.\right) \Big\} \tag{6}$$

We construct a spatial lattice with spacing a and with the fermion operators on the lattice sites. The gauge fields are represented by the following operators sitting on the links between the sites

$$a^+_{\vec{r},i} = U_{\vec{r},i} = \exp(\, iga^{1/2} A_{\vec{r},i}) \tag{7}$$

where $ga^{1/2}A_{\vec{r},i}$ are the angles associated with each link, and

$$E_{\vec{r},i} = \pi_{\vec{r},i}/ga^{1/2} \tag{8}$$

with the commutation relations.

$$\begin{aligned}\left[E_{\vec{r},i},\ U_{\vec{r}',j}\right] &= U_{\vec{r}',i}\ \delta_{\vec{r},\vec{r}'}\ \delta_{ij} \\ \left[E_{\vec{r},i},\ U^{+}_{\vec{r},'j}\right] &= -U^{+}_{\vec{r},'j}\ \delta_{\vec{r},\vec{r}'}\ \delta_{ij}\end{aligned} \tag{9}$$

This suggests to use basic states, for which the electric field operators is diagonal, therefore, we identify as indicated in (7) the operators $U_{\vec{r},i}$ with the creation operators $a^{+}_{\vec{r},i}$.

The total lattice Hamiltonian for QED in $d = 2+1$ dimensions with 2 massless (u and d) fermions becomes

$$\begin{aligned}H = \sum_{\vec{r}} \Bigg\{ & \frac{g^2}{2}\left(E^2_{\vec{r},1} + E^2_{\vec{r},2}\right) \\ & + \frac{1}{a^2 g^2}\left[1 - \frac{1}{2}\left(a^{+}_{\vec{r},1}\ a^{+}_{\vec{r}+\vec{\mu}_x,2}\ a_{\vec{r}+\vec{\mu}_y,1}\ a_{\vec{r},2} + h.c.\right)\right] \\ & + \frac{1}{2a}\left(\chi^{+}(\vec{r}) a^{+}_{\vec{r},1}\ \chi(\vec{r}+\vec{\mu}_x) + h.c.\right) \\ & - \frac{(-1)^{x+y}}{2a}\left(\chi^{+}(\vec{r})\ a^{+}_{\vec{r},2}\ \chi(\vec{r}+\vec{\mu}_y) + h.c.\right)\Bigg\}\end{aligned} \tag{10}$$

3. The Ensemble Projector Monte Carlo Method

The ensemble projector Monte Carlo method in the formulation to be used here is described in Ref. 12.

We start from the lattice Hamiltonian H and lattice eigenstates $\mid \chi\rangle$ and $\mid \phi\rangle$ not orthogonal to the ground state $\mid \psi\rangle$. Using the property of $\exp(-\beta H)$ to be a projection operator to the lowest energy eigenstate, we calculate expectation values of operators as follows:

$$\begin{aligned} e^{-\Delta\beta H} = \langle e^{-\Delta\beta H}\rangle &= \lim_{\beta\to\infty} \frac{\langle\chi|e^{-(\beta+\Delta\beta)H} \mid \phi\rangle}{\langle\chi|e^{-\beta H} \mid \phi\rangle} \\ \langle\psi|Q|\psi\rangle &= \lim_{\beta\to\infty} \frac{\langle\chi|e^{-\beta H}Qe^{-\beta H} \mid \phi\rangle}{\langle\chi|e^{-2\beta H} \mid \phi\rangle} \end{aligned} \tag{11}$$

To calculate the matrix elements appearing in (11) numerically, one splits β into L intervals $\beta = L\cdot\Delta\tau$ and the Hamiltonian H into two or more, in our case three parts $H = \sum H_i$. For sufficiently small $\Delta\tau$, corrections of order $\Delta\tau^2$ and higher became small and, defining $U_k = \exp(-\Delta\tau H_k)$, we approximate the partition function

$$\begin{aligned} Y(\beta) &= \langle\chi|e^{-\beta H}|\phi\rangle = \langle\chi|(U_3,U_2,U_1)^L|\phi\rangle \\ &= \sum_{\substack{i_{3L+1}\,, \\ i_{3L},\ \ldots i_1}} \langle\chi|i_{3L+1}\rangle\langle i_{3L+1}|U_3|i_{3L}\rangle\langle i_{3L}|U_2|i_{3L-1}\rangle \\ &\qquad \langle i_{3L-1}|U_1|i_{3L-2}\rangle \ldots\ldots \langle i_2|U_1|i_1\rangle\langle i_1|\phi\rangle \end{aligned} \tag{12}$$

where the sum runs over complete sets of eigenstates i_M.

The subdivision of the Hamiltonian H is to be done in such a way that all matrix elements $\langle i_i|U_k|i_j\rangle$ are local. Each of these matrix elements can be

represented as a product of a probability P_{ij} and a score S_{ij}

$$\langle i_i|U_k|i_j\rangle = P_{ij}(k) \cdot S_{ij}(k) \tag{13}$$

The P_{ij}, interpreted as probabilities, should be positive definite and normalized to one, and are otherwise arbitrary. The freedom in their choice could be used to construct a Monte Carlo method which converges optimally.

$P_{ij}(k)$ is used as the probability to sample the state $|i_i\rangle$ from a given state $|i_j\rangle$ and $S_{ij}(k)$ gives the weight of the state obtained. As the result of one iteration through all time slices starting from a state $|\ \phi\rangle$ we obtain a sequence of states $|i_1\rangle, |i_2\rangle \ldots |i_{3L+1}\rangle$ with a weight

$$W(i_{3L+1}, i_{3L,} \ldots i_1) = \langle\chi|i_{3L+1}\rangle S_{3L+1,3L} \cdot S_{3L,3L-1} \ldots S_{2,1,} \cdot Sign\langle i_1|\phi\rangle \tag{14}$$

The fluctuations of these weights, which might make the projector Monte Carlo Method rather ineffective, are suppressed in the replication step. We start with an ensemble of say $M \approx 1000$ initial lattice states. After each updating of one state one uses the score $|S_{ij}(k)|$ and some average score S to determine the replication probability $q_{ij} = S_{ij}(k)/S$. If $q_{ij} \leq 1$, a given state is retained in the ensemble with the probability q_{ij}. If $q_{ij} > 1$ we add a number of copies of the state to the ensemble corresponding to the integer part of q_{ij} and use the remainder to determine whether to keep one further copy or not.

There are two independent ways to calculate ground state energy expectation values in the ensemble projector Monte Carlo Method.

(i) The first method consists in measuring energy expectation values by the matrix elements of the Hamiltonian according to (11).

(*ii*) The second method uses the average scores S used in the replication step. S is adjusted in such a way that the number of configurations in the ensemble remains approximately constant from iteration to iteration.

$$E_0 = -\frac{1}{\Delta\tau} \ln S \tag{15}$$

4. Fermions in the Ensemble Projector Monte Carlo

Studying a Hamiltonian lattice gauge theory with Fermions introduces one essential complication into the calculation: There occur matrix elements with positive and negative signs and therefore configurations with positive and negative scores in the ensemble of states. In Ref. 12 we found that the ensemble projector method gives nevertheless correct energy expectation values in the case of the Schwinger model, where the results could be compared with the results obtained before[4] using the local Hamiltonian method.

We proceed as follows:

(*i*) At the beginning of the calculation we define an ensemble of states and perform initial interactions to get the ensemble into thermal equilibrium.

(*ii*) We update all states belonging to the ensemble by one time step using the probabilities P_{ij} derived from the matrix elements.

(*iii*) After the update step follows the replication step, using the score $|S_{ij}|$ derived from the matrix element and the average score S. Some configurations have to be deleted from the ensemble others split into two or more states which will evolve independently in the following time steps. We use the requirement, that the total number of active configurations in the ensemble remains approximately constant to determine the average score S.

(*iv*) In a pure gauge theory calculation without fermions, we could after this step calculate contributions to all the expectation values of interest. Due to the negative signs in the scores we cannot do so here but we store the scores which we need for eventually calculating these expectation values later

$$S_{0,i,j} = \frac{\langle i|0e^{-\Delta\tau H}|j\rangle}{P_{ij}} \tag{16}$$

Likewise, we store the sign of the total score of each configuration of the ensemble, this is the product of the signs of all the single scores encountered during the updating of this state.

(*v*) We repeat steps 2 to 4 for ℓ time steps, advancing β by $\ell \cdot \Delta\tau$. During these successive updatings the sign of the total score of each configuration will change randomly. The fraction of configurations in our ensemble which survives the replication decreases but the total number of configurations remains of course stable. Because of the random nature of the sign changes of the configurations, we expect about equal total numbers of the surviving states with positive and negative total scores. Since each state changes the sign of its total score randomly, we do not expect other differences between the two sub-ensembles with positive and negative total scores.

(*vi*) If our ensemble of states is big enough, and we would decide to select randomly only a certain fraction of the states to calculate expectation values, we would not expect any systematic change of the measured expectation values, except for somewhat bigger statistical errors. In fact, such a procedure was already used in the local Hamiltonian Monte Carlo method, where only the states periodic in time after a total number of L time steps

were considered. (This corresponds to a certain fraction of the states with positive total score.)

If we drop from our ensemble all states with negative total score, we should not expect that this introduces any systematic error into our calculated expectation values. But for states with positive total score it is perfectly legal to perform our ensemble projector Monte Carlo ignoring the signs of the scores at each replication step. At the end, it is only the total score which is important.

In fact, because of the random nature of the sign changes, we could also use only the sub-ensemble with negative total scores and obtain the same expectation values.

This is what we do: We divide the total ensemble of the surviving states into two sub-ensembles with positive and negative total scores. We check, that the number of states in each of these two sub-ensembles is roughly equal (in our calculations these numbers differ by less than 2 - 3 %). Otherwise we would have to continue up-dating the ensemble over more time steps. We calculate the interesting expectation values independently for each of the two sub-ensembles and expect that these expectation values agree within the statistical errors.

(*vii*) Besides the requirement, that the expectation values calculated for the sub-ensembles with positive and negative scores agree, we have a second systematic check of our procedure. We calculate according to (15) the energy expectation values using the average scores used in the replication step. We expect that this third energy expectation value also agrees with the two values already obtained.

5. Application of the Ensemble Projector Monte Carlo Method to the U(1) Lattice Gauge Theory

In order to obtain local matrix elements, we break up the Hamiltonian (10) into three parts. We refer to Fig. 1, where we indicate in the $x-y-$ plane, where the three parts are localized on the spatial lattice. Each part of the Hamiltonian contains the gauge boson contribution corresponding to the four links around the given plaquette. The terms of the Hamiltonian containing the fermionic operators are only in the parts H_1 and H_2. H_3 is a pure bosonic operator.

$$
\begin{aligned}
H_{1,2} = \sum_{\substack{x \text{ odd, even} \\ y \text{ odd, even}}} \Bigg\{ & \frac{g^2}{4} \left(E^2_{\vec{r},1} + E^2_{\vec{r}+\vec{\mu}_x,2} + E^2_{\vec{r}+\vec{\mu}_y,1} + E^2_{\vec{r},2} \right) \\
& + \frac{1}{a^2 g^2} \left[1 - \frac{1}{2} \left(a^+_{\vec{r},1} \, a^+_{\vec{r}+\vec{\mu}_x,2} \, a_{\vec{r}+\vec{\mu}_y,1} \, a_{\vec{r},2} + h.c. \right) \right] \\
& + \frac{1}{2a} \Bigg[\chi^+(\vec{r}) \left(a^+_{\vec{r},1} \, \chi(\vec{r}+\vec{\mu}_x) - a^+_{\vec{r},2} \, \chi(\vec{r}+\vec{\mu}_y) \right) \\
& + \left(\chi^+(\vec{r}+\vec{\mu}_y) a^+_{\vec{r}+\vec{\mu}_y,1} + \chi^+(\vec{r}+\vec{\mu}_x) a^+_{\vec{r}+\vec{\mu}_x,2} \right) \\
& \chi(\vec{r}+\vec{\mu}_x+\vec{\mu}_y) + h.c. \Bigg] \Bigg\}
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
H_3 = \sum_{\substack{x \text{ odd, } y \text{ even} \\ \text{and } x \text{ even, } y \text{ odd}}} \Bigg\{ & \frac{g^2}{4} \left(E^2_{\vec{r},1} + E^2_{\vec{r}+\vec{\mu}_x,2} + E^2_{\vec{r}+\vec{\mu}_y,1} + E^2_{\vec{r},2} \right) \\
& + \frac{1}{a^2 g^2} \left[1 - \frac{1}{2} \left(a^+_{\vec{r},1} a^+_{\vec{r}+\mu_x,2} a_{\vec{r}+\mu_y,1} a_{\vec{\mu}_r,2} + h.c. \right) \right] \Bigg\}
\end{aligned}
\tag{18}
$$

We refer to the first and second term of the bosonic Hamiltonian $h_g = h_g^{el} + h_g^{magn}$ as to the electric h_g^{el} and magnetic h_g^{magn} parts. The matrix element of

$\exp(-\Delta\tau h_g)$ is calculated by splitting h_g in the exponent into three parts and keeping only the lowest order contributions.

$$e^{-\Delta\tau h_g} \approx e^{-\frac{\Delta\tau}{2}h_g^{el}}\, e^{-\Delta\tau h_g^{magn}}\, e^{-\frac{\Delta\tau}{2}h_g^{el}} \tag{19}$$

In Fig. 2 we give the bosonic and fermion occupation numbers of the four links and four sites of a plaquette. m_1, m_2, m_3 and m_4 are the bosonic occupation numbers. We obtain for the matrix elements of (19) the expression

$$\begin{aligned}
&\left\langle m_1', m_2', m_3', m_4' \,|\exp(-\Delta\tau h_g)|\, m_1, m_2, m_3, m_4 \right\rangle \\
&= \exp\left[-\frac{\Delta\tau}{2}\,\frac{g^2}{4}(m_1'^2 + m_2'^2 + m_3'^2 + m_4'^2)\right] \exp\left(-\frac{\Delta\tau}{a^2g^2}\right) I_m\left(\frac{\Delta\tau}{a^2g^2}\right) \\
&\quad \cdot \exp\left[-\frac{\Delta\tau}{2}\,\frac{g^2}{4}\,(m_1^2 + m_2^2 + m_3^2 + m_4^2)\right] \cdot \delta_{m_1', m_1 - m}\; \delta_{m_2', m_2 - m} \\
&\quad \cdot \delta_{m_3', m_3 + m}\; \delta_{m_4', m_4 + m}
\end{aligned} \tag{20}$$

where $I_m(\ldots)$ is a modified Bessel function and m is a positive or negative integer, the change of the link occupation numbers. We consider in the Monte Carlo calculation all contributions with $|m| \leq 5$.

The matrix elements for sampling the energy expectation values contain the operators $h_g \exp(-\Delta\tau h_g)$. They can be calculated straightforwardly and are not given here.

The pure bosonic operator H_3 does not change the fermion variables at the sites of the plaquette. We have to consider the fermion operators only when calculating the matrix elements of H_1 and H_2. Using the notation of Fig. 2 we

write the fermionic part of the Hamiltonian (17) as follows

$$h_f = \frac{1}{2a}\left(\chi_1^+ a_1^+ \chi_2 + \chi_2^+ a_1 \chi_1 - \chi_1^+ a_3^+ \chi_3 - \chi_3^+ a_3 \chi_1 \right. \\ \left. + \chi_3^+ a_4^+ \chi_4 + \chi_4^+ a_4 \chi_3 + \chi_2^+ a_2^+ \chi_4 + \chi_4^+ a_2 \chi_2\right) \tag{21}$$

The state vectors containing all the occupation numbers related to one plaquette have the form

$$|i^{\text{Plaquette}}\rangle = (a_1^+)^{m_1}(a_2^+)^{m_2}(a_3^+)^{m_3}(a_4^+)^{m_4} \\ \cdot (\chi_1^+)^{i_1}(\chi_2^+)^{i_2}(\chi_3^+)^{i_3}(\chi_4^+)^{i_4}|0\rangle \tag{22}$$

we calculate the fermionic part of the matrix elements only up to order $\Delta\tau$ and obtain for the h_1 and h_2 matrix elements

$$\left\langle m_1', m_2', m_3', m_4', i_1', i_2', i_3', i_4' \left| e^{-\frac{\Delta\tau}{2} h_g^{el}} \; e^{-\Delta\tau h_f} \; e^{-\Delta\tau h_g^{magn}} \; e^{-\frac{\Delta\tau}{2} h_g^{el}} \right. \right. \\ \left. \left. \cdot |m_1, m_2, m_3, m_4, i_1, i_2, i_3, i_4 \right\rangle \right. \\ = \exp\left[-\frac{\Delta\tau}{2}\,\frac{g^2}{4}\left(m_1^2 + m_2^2 + m_3^2 + m_4^2 + m_1'^2 + m_2'^2 + m_3'^2 + m_4'^2\right)\right] \\ \cdot \exp\left(-\frac{\Delta\tau}{a^2 g^2}\right) I_m\left(\frac{\Delta\tau}{a^2 g^2}\right) \cdot K \\ \cdot \delta_{m_1 - m_1' - m, k_1} \; \delta_{m_2 - m_2' - m, k_2} \\ \cdot \delta_{m_3 - m_3' + m, k_3} \; \delta_{m_4 - m_4' + m, k_4} + O(\Delta\tau^2) \tag{23}$$

The factors K are given in Table 1 for all combinations of fermionic occupation numbers.

We split the matrix elements M (20) and (21) in the following way into probabilities P and scores S

$$M = e^{\frac{-\Delta\tau}{a^2g^2}} I_m\left(\frac{\Delta\tau}{a^2g^2}\right) \cdot K \cdot \exp\left\{-\frac{\Delta\tau}{2}\ \frac{g^2}{4}\left[m_1^2+m_2^2+m_3^2+m_4^2+(m_1-m-k_1)^2\right.\right.$$
$$\left.\left.+(m_2-m-k_2)^2+(m_3+m-k_3)^2+(m_4+m-k_4)^2\right]\right\} \tag{24}$$

$$P = A_{\text{Norm.}}\ e^{\frac{-\Delta\tau}{2}g^2m^2}\ e^{\frac{-\Delta\tau}{a^2g^2}}\ I_m\left(\frac{\Delta\tau}{a^2g^2}\right)\cdot|K| \tag{25}$$

$$S = \frac{\text{Sign } K}{A_{\text{Norm.}}} \exp\left\{-\frac{\Delta\tau}{2}\ \frac{g^2}{4}\ \left[2\left(m_1^2+m_2^2+m_3^2+m_4^2\right)+k_1^2+k_2^2+k_3^2+k_4^2\right.\right.$$
$$-2(m_1k_1+m_2k_2+m_3k_3+m_4k_4)-2m(m_1+m_2-m_3-m_4) \tag{26}$$
$$\left.\left.+2m(k_1+k_2-k_3-k_4)\right]\right\}$$

The normalization factors A_{Norm} follow from the requirement to normalize the probabilities

$$\sum P = 1 \tag{27}$$

where the sum runs over all possible transitions. These are the eleven possible values of m which we consider ($|m| \leq 5$) and in addition for each value of m between one and five fermionic transitions, see Table 1 (for instance 1001 $\rightarrow$ 1001, 0101, 0011, 1010, 1100).

Only the matrix elements of h_1 and h_2 containing fermion operators and K-factors of different signs obtain scores of positive and negative signs.

The matrix elements of

$$h_i e^{\Delta\tau h_i} \tag{28}$$

which are needed to calculate energy expectation values can be calculated straight forwardly and are not given here.

6. Results and Discussion

We consider in our Monte Carlo calculation typically an ensemble of 1000 configurations periodic on a spatial lattice of size $n_x \cdot n_y = 6 \times 6$ sites. In most Monte Carlo runs we iterate the ensemble over 600 time steps. The first 200 time steps are used to bring the ensemble into equilibrium. We measure the expectation values always after a $\Delta\beta = \ell\Delta\tau$ advance of 0.2, this are two time steps at $\Delta\tau = 0.1$ and eight time steps at $\Delta\tau = 0.025$. At small values of the coupling constant g, where there seems no interesting g dependence of the expectation values we use a smaller number of iterations, obtaining somewhat larger statistical errors. At larger values of g the expectation values become strongly g-dependent. Since our matrix elements are only correct up to order $\Delta\tau$ we have difficulties to get small systematic errors at large values of the coupling constant g, the Hamiltonian contains terms proportional to $\Delta\tau \cdot g^2$. We use small values of $\Delta\tau$ down to $\Delta\tau = 0.025a$, where a is the spatial lattice spacing. At small values of $\Delta\tau$ another limitation appears. Equilibrium is only reached after a certain amount of $\Delta\beta = \ell \cdot \Delta\tau$ advance. Also the number of configurations with positive and negative scores become only equal after a sufficiently big $\Delta\beta$ advance. Therefore at small $\Delta\tau$, the number of iterations ℓ before measurements become possible grows. This limits the possibilities to measure at large g-values,

since we are only using running times of 10–15 min of an IBM 3081 per value of the coupling constant g.

At each value of the coupling constant g we measure E^+, the energy expectation value obtained from the sub-ensemble with positive total scores, E^-, the energy expectation value obtained from the configurations with negative total score and E^P, the energy expectation value obtained according to (15) from the average scores.

The calculation of E^P proceeds as follows: Each time step $\Delta\tau$ consists in reality of three steps updating the occupation numbers on different plaquettes using the U_1, U_2, and U_3 operators. Because of the non-symmetric division of the Hamiltonian into H_1, H_2 and H_3, the average scores S_i for these three updatings differ. We obtain from the three scores S_i the values of the three subenergies E_1^P, E_2^P and E_3^P

$$E_{1,2}^P = -\frac{4 \log S_{1,2}}{\Delta\tau \, n_x n_y} \tag{29}$$

$$E_3^P = -\frac{2 \log S_3}{\Delta\tau \, n_x n_y} \tag{30}$$

Our average energies are normalized per spatial plaquette or per lattice point. The total energy E^P is simply obtained as

$$E^P = \left(E_1^P + E_2^P + E_3^P\right)/3 \tag{31}$$

Actually, we measure also the subenergies $E_i^{\pm}$ on the three kinds of plaquettes. The comparison of the subenergies E_i obtained with the three different methods is interesting. Only the terms H_1 and H_2 contain fermionic operators. H_3 is a pure bosonic operator, not leading to scores of different signs. The calculation of

the E_3 expectation value is similar to a projector Monte Carlo calculation of the pure gauge theory.[8,9] The calculation is known to converge for $\Delta\tau$ small enough. We consider therefore the differences between three E_3 values E_3^+, E_3^- and E_3^P as an indicator for systematic errors due to the imperfections of our Monte Carlo calculations ($\Delta\tau$ two large, number of time steps two small). Our statistical errors obtained from subdividing the total sample into 5–10 parts, vary between ± 0.02 and ± 0.03. If the E_3 differences are of the order of the statistical errors and the corresponding differences between the three $E_{1,2}$ expectation values are of the same order we consider these residual deviations not as systematic deviations of our method. In fact we find, that we are only able to calculate meaningful expectation values of E_3 or $E_{1,2}$ for coupling constants $g \lesssim 2$.

We present the total energy expectation values E^+, E^- and E^P for the coupling constants g considered in Table 2 and Fig. 3. In Table 3 and Fig. 4 we present the expectation values of the sub-energies E_i^+, E_i^- and E_i^P. The results are only presented for such values of g and $\Delta\tau$, where the errors are still tolerable. We find in nearly all cases, and the E^+ and E^- expectation values agree rather well with each other. The biggest deviations occur between E_i^P and $E_i^\pm$. The agreement between the values obtained by the different methods can be interpreted as an indicator that the ensemble projector method gives valid results also in the presence of fermions. Unfortunately, there are no other calculations of the U(1) gauge theory with fermions in $d = 2 + 1$ dimensions available. Therefore, we compare our results only with the lowest order showing coupling estimate per lattice point

$$E_{s.c.} = -\frac{1}{a^4 g^6} + O\left(\frac{1}{a^6 g^8}\right) \tag{32}$$

This is also plotted in Fig. 3. Obviously, in the region, where the strong coupling estimate is best, we are limited by the systematic errors of our calculation. Also the statistical errors are bigger than the very small energy values according to Eq. (32) in the strong coupling region.

The energy expectation value near to $g = 1.5$ shows a sudden jump. This could indicate the existence of a phase transition, but to our knowledge no phase transition is expected in this model.

Concluding, we find the results of this first application of the ensemble projection Monte Carlo method to a model gauge theory with fermions in $d = 2 + 1$ encouraging. We find it worthwhile to continue such studies, further improving the method, extending it to $d = 3 + 1$ and studying the properties of the considered lattice gauge theory, in more detail.

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Table 1. Factors K in the non-vanishing matrix elements (23) of the operator $\exp(-\Delta\tau h_{1,2})$ up to first order in $\Delta\tau$

i_1	i_2	i_3	i_4	$\rightleftarrows$	i_1'	i_2'	i_3'	i_4'	K	k_1	k_2	k_3	k_4
0	0	0	0		0	0	0	0	1	0	0	0	0
1	0	0	0		1	0	0	0	1	0	0	0	0
1	0	0	0		0	1	0	0	$-\Delta\tau/2a$	±1	0	0	0
1	0	0	0		0	0	1	0	$+\Delta\tau/2a$	0	0	±1	0
0	1	0	0		0	1	0	0	1	0	0	0	0
0	1	0	0		0	0	0	1	$-\Delta\tau/2a$	0	±1	0	0
0	0	1	0		0	0	1	0	1	0	0	0	0
0	0	1	0		0	0	0	1	$-\Delta\tau/2a$	0	0	0	±1
0	0	0	1		0	0	0	1	1	0	0	0	0
1	1	0	0		1	1	0	0	1	0	0	0	0
1	1	0	0		0	1	1	0	$-\Delta\tau/2a$	0	0	±1	0
1	1	0	0		1	0	0	1	$-\Delta\tau/2a$	0	±1	0	0
1	0	1	0		1	0	1	0	1	0	0	0	0
1	0	1	0		0	1	1	0	$-\Delta\tau/2a$	±1	0	0	0
1	0	1	0		1	0	0	1	$-\Delta\tau/2a$	0	0	0	±1
1	0	0	1		1	0	0	1	1	0	0	0	0
1	0	0	1		0	1	0	1	$-\Delta\tau/2a$	±1	0	0	0
1	0	0	1		0	0	1	1	$+\Delta\tau/2a$	0	0	±1	0
0	1	1	0		0	1	1	0	1	0	0	0	0
0	1	1	0		0	1	0	1	$-\Delta\tau/2a$	0	0	0	±1
0	1	1	0		0	0	1	1	$+\Delta\tau/2a$	0	±1	0	0
0	1	0	1		0	1	0	1	1	0	0	0	0
0	0	1	1		0	0	1	1	1	0	0	0	0
1	1	1	0		1	1	1	0	1	0	0	0	0
1	1	1	0		1	1	0	1	$-\Delta\tau/2a$	0	0	0	±1
1	1	1	0		1	0	1	1	$-\Delta\tau/2a$	0	±1	0	0
1	1	0	1		1	1	0	1	1	0	0	0	0
1	1	0	1		0	1	1	1	$-\Delta\tau/2a$	0	0	±1	0
1	0	1	1		1	0	1	1	1	0	0	0	0
1	0	1	1		0	1	1	1	$-\Delta\tau/2a$	±1	0	0	0
0	1	1	1		0	1	1	1	1	0	0	0	0
1	1	1	1		1	1	1	1	1	0	0	0	0

Table 2. Energy expectation values calculated at different values of the coupling constant g. The statistical errors of the energy expectation values are between $\Delta E = \pm 0.02$ and ± 0.03. E^+, E^- and E^P are explained in the main text.

$\Delta\tau$	g	$1/g^2$	E^+	E^-	E^P
0.1	0.5	4.00	0.34	0.33	0.29
0.1	$1/\sqrt{3}$	3.00	0.33	0.33	0.29
0.1	$1/\sqrt{2}$	2.00	0.37	0.35	0.33
0.1	$\sqrt{2/3}$	1.5	0.32	0.32	0.28
0.1	1.0	1.00	0.32	0.32	0.29
0.05	$\sqrt{4/3}$	0.75	0.31	0.305	0.28
0.05	1.3	0.5917	0.235	0.235	0.21
0.05	1.5	0.4444	0.205	0.20	0.18
0.025	1.65	0.3673	0.005	0.00	−0.02
0.025	1.80	0.3086	0.01	0.005	−0.01
0.025	$\sqrt{5}$	0.2	−0.01	−0.01	−0.02

Table 3. Energy expectation values E_1, E_2 and E_3 obtained with the three different methods (E_i^+, E_i^- and E_i^P) considered in this paper. Note that H_3 is a purely bosonic operator. Differences between the three values E_3^+, E_3^- and E_3^P indicate the statistical and systematic errors at too large values of $\Delta\tau$ or a too small number of iterations. E_1 and E_2 are influenced by the fermion operators in H_1 and H_2. The statistical errors of the expectation values are of the order of ± 0.02 to 0.03.

$\Delta\tau$	g	E_1^+	E_1^-	E_1^P	E_2^+	E_2^-	E_2^P	E_3^+	E_3^-	E_3^P
0.1	0.5	−0.02	−0.01	−0.05	+0.01	+0.01	−0.07	1.02	1.01	0.98
0.1	$1/\sqrt{3}$	−0.01	+0.01	−0.05	+0.03	−0.03	−0.07	1.01	1.02	0.98
0.1	$1/\sqrt{2}$	+0.03	0.00	−0.02	+0.02	0.00	−0.03	1.05	1.05	1.02
0.1	$\sqrt{2/3}$	−0.01	−0.02	−0.07	−0.05	−0.03	−0.07	1.01	1.00	0.97
0.1	1.0	−0.02	−0.04	−0.07	+0.01	0.00	−0.06	1.01	1.01	0.98
0.05	$\sqrt{4/3}$	−0.06	−0.07	−0.08	−0.05	−0.06	−0.08	1.00	1.01	1.00
0.05	1.3	−.11	−.11	−.15	−.10	−.105	−.145	0.93	0.97	0.93
0.05	1.5	−0.11	−0.12	−0.16	−0.19	−0.17	−0.21	0.89	0.89	0.90
0.025	1.65	−.33	−.33	−.37	−.38	−.38	−.45	.15	.12	.74
0.025	1.80	−0.35	−0.37	−0.40	−0.33	−0.33	−0.39	.72	.71	.75
0.025	$\sqrt{5}$	−0.36	−0.36	−0.45	−0.35	−0.34	−0.43	.69	.69	.75

FIGURE CAPTIONS

Fig. 1. Break up of the Hamiltonian H into the three parts H_1, H_2 and H_3.

Fig. 2. Boson and fermion occupation numbers on the four links and four sites of a spatial plaquette.

Fig. 3. Energy expectation values E^+, E^- and E^P as function of the square of the inverse coupling conduct $1/g^2$. The line gives the lowest order strong coupling estimate, see Eq. (32).

Fig. 4. Expectation values of the sub-energies $E_1^{\pm}$, E_1^P, $E_3^{\pm}$ and E_3^P as function of the square of the inverse coupling constant $1/g^2$. The $E_2^{\pm}$ and E_2^P values are in general near to the E_1 values and omitted in the plot.

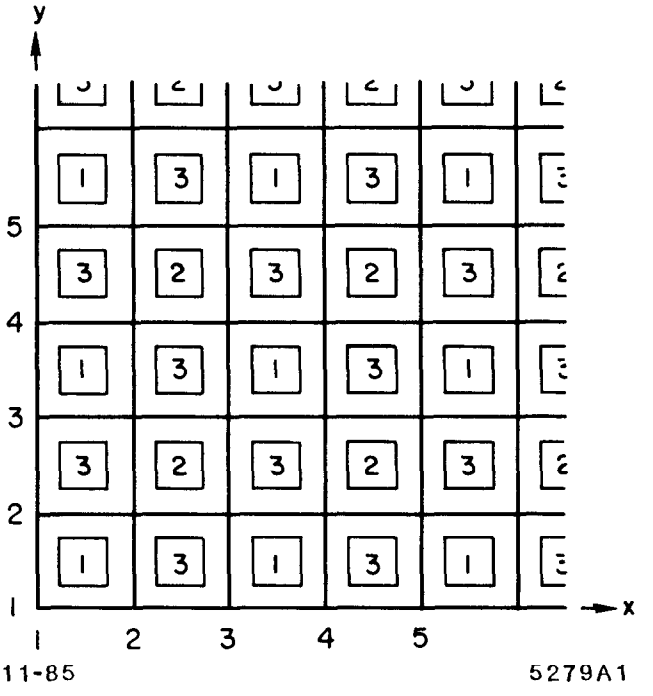


Fig. 1

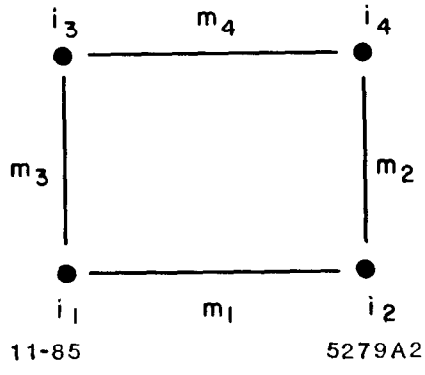
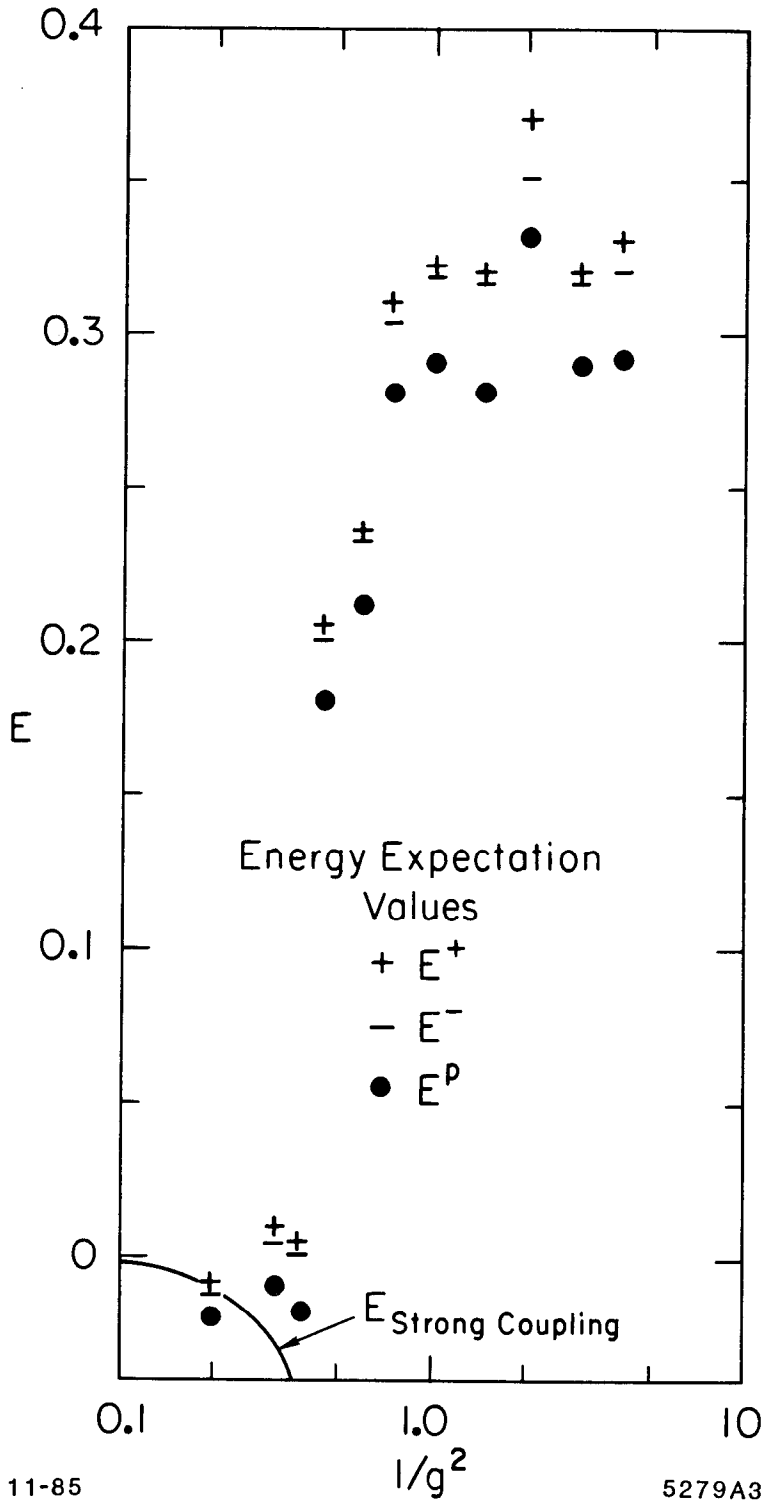


Fig. 2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0.3
0.2
0.1
0
E
Energy Expectation
Values
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− E^-
● E^p
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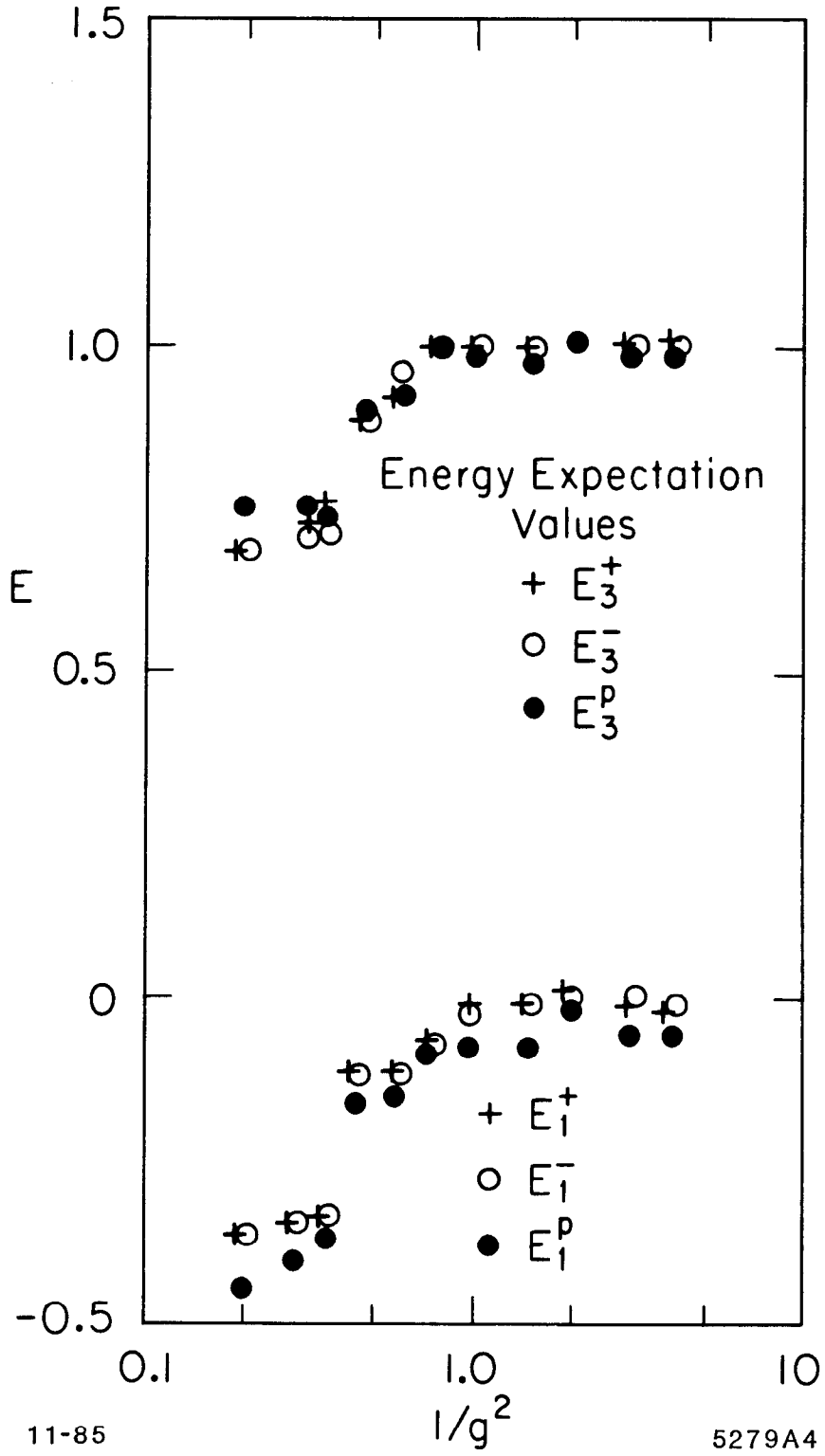
1.0
10
$1/g^2$
11-85
5279A3

Fig. 3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0
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E
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10
1/g²
Energy Expectation Values
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○ E₃⁻
● E₃ᵖ
+ E₁⁺
○ E₁⁻
● E₁ᵖ
11-85
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Fig. 4