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An Introduction to the Theory of Strings

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ABSTRACT

These lectures present, from an introductory perspective, some basic aspects of the quantum theory of strings. They treat (1) the kinematics, spectrum, and scattering amplitude of the bosonic string, (2) the spectrum and supersymmetry of Green-Schwarz superstring, and (3) the identification of the underlying gauge invariances of the string theory.

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1. THE BOSONIC STRING

Strings are idealized one-dimensional extended objects. They provide a natural generalization of relativistic point particles, which travel along world lines, in being described by world surfaces stretching through space-time. Strings have more than this simple mathematical attraction, however, because of the possibility, proposed many years ago by Scherk and Schwarz[1] and recently buttressed impressively by Green and Schwarz[2] and by Witten and collaborators[3-5], that strings provide the grand unified theory of everything. Whether or not this claim is true, strings certainly open a rich field of mathematical physics which seems likely to be a very fruitful one over the next few years.

My intent in these lectures is to present an introduction to the theory of strings, covering its main features and extending far enough to reach some of its open problems. To do this, I have adopted the strategy of concentrating on the basic mathematical formalism and steering away from specific phenomenological connections. (The new developments which connect with phenomenology are described in the lecture of David Gross in this volume.) Still, though, the material presented here will give only a taste of the range and depth of the theory. A comprehensive introduction to the theory of strings would require, not three lectures, but thirty. The reader who wishes to study the subject further should go on to consult some of the many excellent, more technical reviews which are available[6-9].

The topics that I will cover are as follows: In this section, I will discuss the basic properties of the simplest string, an unadorned world-sheet immersed in space-time. I will discuss the kinematics and the interactions of this string theory in a unified way, following a viewpoint due to Mandelstam[10]. The second section will discuss properties of a supersymmetric generalization of this object, the superstring of Green and Schwarz[11]. For both of these systems, I will discuss the dynamics using light-cone quantization. This method is, for all its obvious frame-dependence, the route which makes most explicit the physics that these models contain. In the third section, I will discuss some issues which require a more covariant viewpoint, and which bear on the important question of how gauge fields can be built of strings. Whereas the material of the first two lectures is classic, a firm foundation for further development, the material of the last lecture is only a provisional understanding of some deep and still unsolved problems.

Let me begin by writing the simplest action principle for a one-dimensional extended object♯1. The equation of motion for a relativistic point particle imples that the particle sweeps out a geodesic path in space-time:

$$\delta \int ds \Big[\Big(\frac{dx}{ds}\Big)^2 \Big]^{\frac{1}{2}} = 0 . \tag{1}$$

A natural generalization of this idea would be to minimize the invariant area of the 2-dimensional surface swept out by the string's evolution[12]. It is useful to

♯1 For reference, my conventions are:

$$\eta^{\mu\nu} = diag\,(-1, 1, \ldots, 1)\ ; \quad \mu, \nu = 0, 1, \ldots, d-1$$
$$x \cdot y = \vec{x} \cdot \vec{y} - x^+ y^- - x^- y^+ \ ,$$

where $x^\pm$ are light-cone variables:

$$x^\pm = \frac{1}{\sqrt{2}}\,(x^0 \pm x^{d-1}) \ .$$

formulate this idea as the following action principle[13] :

$$S = \frac{-1}{4\pi\alpha'} \int d^2\xi \sqrt{-g}\, g^{ab} \partial_a x^\mu \partial_b x^\mu \ , \tag{2}$$

where $\xi^a = (\tau, \sigma)$ are coordinates on the string surface and $x^\mu(\xi)$, $g_{ab}(\xi)$ are to be varied independently. It is not difficult to see how (2) is connected to the problem of finding geodesic surfaces. Notice first that (2) has (at least classically) three gauge symmetries: 2-dimensional reparametrization invariance and, in addition, a Weyl symmetry:

$$g_{ab} \to e^{\lambda(\xi)} g_{ab} \ , \quad x^\mu \to x^\mu \ . \tag{3}$$

Varying (2) with respect to g^{ab} gives the equation of motion

$$\partial_a x^\mu \partial_b x^\mu = \frac{1}{2} g_{ab} g^{cd} \partial_c x^\mu \partial_d x^\mu \ ; \tag{4}$$

this is equivalent, up to a Weyl motion, to the statement

$$g_{ab} = \partial_a x^\mu \partial_b x^\mu \ , \tag{5}$$

which says that g_{ab} is equal to the induced metric on the 2-dimensional surface. Inserting (5) or (4) into (2) yields the invariant area of the surface

$$\int d^2\xi \sqrt{-g} \ . \tag{6}$$

The equation for x^μ which follows from (2) takes the form of a conservation law

$$\partial_\tau P^\tau{}_\mu + \partial_\sigma P^\sigma{}_\mu \;=\; 0\;, \tag{7}$$

where

$$P^a{}_\mu \;=\; \frac{\delta S}{\delta\,\partial_a x^\mu} \;=\; \frac{-1}{2\pi\alpha'}\sqrt{-g}g^{ab}\,\partial_b x_\mu \tag{8}$$

is the momentum density on the surface. The boundary condition for x^μ should be the condition that no momentum flows out across the boundary

$$P^\sigma{}_\mu = 0\;. \tag{9}$$

This set of equations can be dramatically simplified by a proper choice of gauge. Since g_{ab} is a symmetric 2×2 matrix, it contains only 3 degrees of freedom. The gauge freedom of (2) thus suffices to reduce g_{ab} to the form

$$g_{ab} \;=\; \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\;. \tag{10}$$

This already reduces (7) to the Laplace equation

$$\partial^2\, x_\mu \;=\; 0\;. \tag{11}$$

More simplification can be obtained by a specific choice of the coordinates τ and σ. Set τ equal to the light-cone coordinate x^+, thus insisting that equal-τ slices are made at fixed x^+. Choose σ so that the conserved quantity

$$P^+ = \int d\sigma\, P^{\tau\,+} \tag{12}$$

receives equal contributions from equal increments of σ, *i.e.*, so that $P^{\tau\,+}$ is constant over the surface. This condition implies

$$P^{\sigma +} = 0 \tag{13}$$

for strings with boundaries (open strings), by the use of the boundary condition (9); for strings without boundaries (closed strings), this statement completes the specification of gauge by fixing the origin of σ as a function of τ. Our fixing of τ implies that $\partial_\sigma x^+(\xi) = 0$. Then (13) implies that $g^{\sigma\tau} = 0$. Similarly, the condition that $P^{\tau +}$ is a constant implies, from (8),

$$|g_{\tau\tau}| = |g_{\sigma\sigma}| \, . \tag{14}$$

We are thus already quite close to (10). The last step may be taken by using Weyl invariance to set (14) equal to 1. We have now reached coordinates in which

$$P^\mu_\tau = \frac{1}{2\pi\alpha'}\,\partial_\tau x^\mu \qquad P^\mu_\sigma = \frac{-1}{2\pi\alpha'}\,\partial_\sigma x^\mu \, , \tag{15}$$

and, in particular,

$$P^+_\tau = \frac{1}{2\pi\alpha'} \qquad P^+ = \frac{1}{2\pi\alpha'} \cdot (\text{length of string in } \sigma) \tag{16}$$

so that the total length of the string is given by

$$\Delta\sigma = 2\pi\alpha' P^+ \, . \tag{17}$$

Since we have fixed σ and τ, we must have (at least implicitly) eliminated two coordinate degrees of freedom of the string. In fact, we eliminated $x^+(\xi)$

explicitly, setting it equal to τ. But we have also fixed the value of x^-, since $g_{\sigma\tau} = 0$ implies

$$0 = \partial_\sigma x \cdot \partial_\tau x \;\Rightarrow\; \partial_\sigma x^- = \partial_\sigma \vec{x} \cdot \partial_\tau \vec{x} \ . \tag{18}$$

Therefore x^- is specified as a function of $\vec{x}(\xi)$. The Hamiltonian density generating evolution in τ is

$$P_\tau^- = \frac{1}{2\pi\alpha'}\,\partial_\tau x^- = \frac{1}{2\pi\alpha'}\,[(\partial_\tau \vec{x})^2 + (\partial_\sigma \vec{x})^2]\ . ; \tag{19}$$

thus, the Hamiltonian for the unconstrained degrees of freedom is[14]

$$H = \frac{1}{2\pi\alpha'} \int_0^{2\pi P^+\alpha'} d\sigma\,[(\partial_\tau \vec{x})^2 + (\partial_\sigma \vec{x})^2]\ . \tag{20}$$

This is the Hamiltonian of a set of $(d-2)$ massless 2-dimensional fields. We might study its properties either by diagonalizing it directly or by relating it a functional integral over string configurations. The first of these approaches has been reviewed, for example, in refs. 6 and 7. Let me, then, adopt the second viewpoint here.

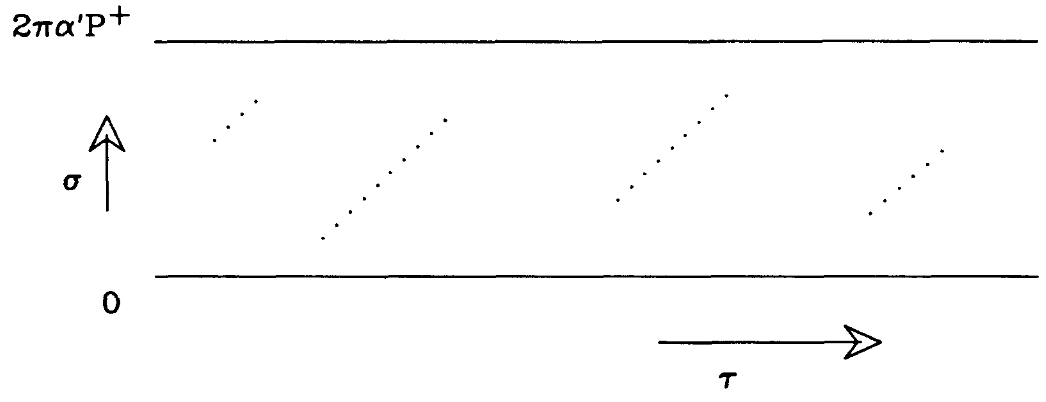
For definiteness in constructing the functional integral, rotate τ to Euclidean space. Then we must study the Euclidean integral

$${}_+\langle 0|0\rangle_- = \int \mathcal{D}x\, e^{-S_E}\ , \tag{21}$$

where

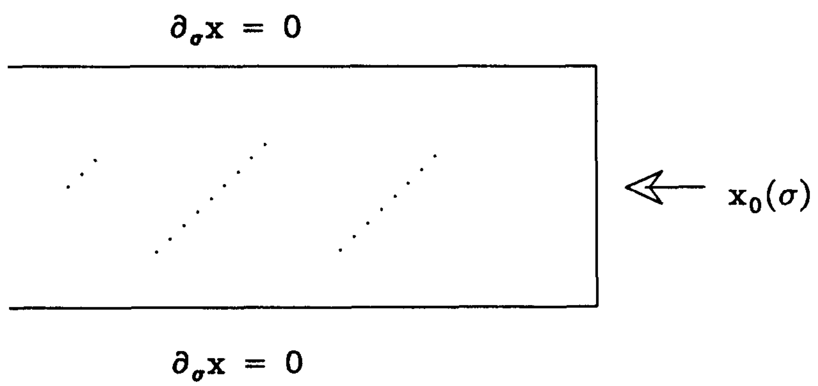
$$S_E = \frac{1}{2\pi\alpha'} \int_0^{2\pi P^+\alpha'} d\sigma\, d\tau [(\partial_\tau \vec{x})^2 + (\partial_\sigma \vec{x})^2] \tag{22}$$

and τ is now Euclidean time($\tau = -i\tau_M$). For the free string, the domain of (τ, σ) is:



To unclutter the notation, choose units of mass so that $2\alpha' = 1$. (For my analysis of a single string, I will also set $P^+ = 1$.) With this formalism, we can readily work out the spectrum of single-string states. We will see that it also allows us to investigate the scattering of strings.

We should first try to find the string ground state. In Euclidean field theory, the ground state wave function is proportional to the amplitude for the system to propagate from Euclidean time $-\infty$ to a fixed configuration $\vec{x}_0(\sigma)$ at $\tau = 0$:



Let us compute this amplitude explicitly. At first glance, this looks like quite a challenge. However, note that the equation for $\vec{x}(\xi)$, the 2-dimensional Laplace equation, is invariant under arbitrary conformal mappings. This property will allow us to solve these equations even on strange-looking domains such as the one considered here. To take advantage of this conformal invariance, let us describe points on the string surface by a complex variable $z = \tau + i\sigma$.

Before confronting the full functional integral, let us consider the classical problem of solving the Laplace equation on the half-strip. The solution is given in terms of the Green's function satisfying

$$-\nabla^2 G(z, z') = \delta^{(2)}(z - z') \tag{23}$$

with the boundary condition (9) (*i.e.*, Neumann boundary conditions) along the edges of the string and Dirichlet boundary conditions at $\tau = 0$. As a step toward constructing this function, let us first construct the Neumann Green's function, G_N, for the full strip: $-\infty < \tau < \infty$. This can be done by mapping the strip onto the upper half plane by the mapping

$$w \;=\; e^z \tag{24}$$

and then using the method of images; one finds

$$G_N(z, z') = -\frac{1}{2\pi} \log\left[(e^z - e^{z'})(e^z - e^{\bar{z}'})\right] \; .$$

For the problem at hand, we need a slightly more complex G, one which also vanishes on the unit circle in the w plane. The desired function is

$$G(z, z') = -\frac{1}{2\pi} \operatorname{Re} \log \frac{(e^z - e^{z'})(e^z - e^{\bar{z}'})}{(e^{-z} - e^{z'})(e^{-z} - e^{\bar{z}'})} \; . \tag{25}$$

The classical solution for $\vec{x}(z)$ which agrees with $\vec{x}_0(\sigma)$ at $\tau = 0$ can be constructed from (25) by

$$x_c(z) = -\oint dz'\, x_0(z')\partial_n' G(z', z) \ , \tag{26}$$

where ∂_n is the derivative in the direction of the outward normal to the boundary.

We can now compute the full functional integral by shifting the integration variable

$$x(z) \ = \ x_c(z) \ + \ \delta x(z) \ , \tag{27}$$

where $\delta x(z)$ satisfies Neumann boundary conditions on the horizontal edges and Dirichlet boundary conditions at $\tau = 0$. Since $x_c(z)$ solves the classical equations of motion, the cross term in S_E between $x_c(z)$ and $\delta x(z)$ vanishes. The term involving $(\delta x)^2$ can be integrated over δx, this yields a normalization factor which is independent of the boundary condition $x_0(\sigma)$. The term involving $(x_c)^2$ can be rearranged as follows:

$$\begin{aligned} S_E &= \frac{1}{2\pi}\int d\sigma\, d\tau (\partial_\alpha x_c)^2 + \frac{1}{2\pi}\int d\sigma\, d\tau (\partial_\alpha \delta x)^2 \\ &= \frac{1}{2\pi}\int d\sigma\, d\tau\, x_c(-\partial^2 x_c) + \frac{1}{2\pi}\oint dz\, x_c\, \partial_n x_c \\ &= -\frac{1}{2\pi}\int d\sigma\, d\sigma'\, x_0(\sigma)\left[\frac{\partial}{\partial\tau}\,\frac{\partial}{\partial\tau'}\, G(z,z')\right]_{\tau=\tau'=0} x_0(\sigma') \ . \end{aligned} \tag{28}$$

The kernel is some operator on functions of σ. We might expect that the eigenstates of this operator are the natural Fourier modes of the string:

$$x_0(\sigma) = x_0 + \sum_{n>0} \frac{2}{\sqrt{n}} X_n \cos n\sigma \ . \tag{29}$$

This is, in fact, true, as we may check explicitly:

$$-\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau'}G(\tau+i\sigma,\tau'+i\sigma')\Big|_{\tau=\tau'=0}$$

$$=\frac{1}{2\pi}P\left[\left(\frac{e^{i\sigma}e^{i\sigma'}}{(e^{i\sigma}-e^{i\sigma'})^2}+(\sigma'\to-\sigma')\right)+c.c.\right] , \tag{30}$$

where P denotes the principal value; it is straightforward to integrate this expression with (29) using the integral

$$P\int_0^{2\pi} d\sigma'\,\frac{e^{i\sigma}e^{i\sigma'}}{(e^{i\sigma}-e^{i\sigma'})^2}\,e^{in\sigma'}=P\oint\frac{dy'}{i}\,\frac{e^{i\sigma}}{(e^{i\sigma}-y')^2}\,y'^n=\frac{n}{2}\,2\pi\,e^{in\sigma} . \tag{31}$$

We can then write (28) in the form

$$S_E=\frac{1}{2\pi P^+}\int_0^{\pi P^+}d\sigma\;x_0\mathcal{D}x_0 ,\quad \text{where}\quad \mathcal{D}\cos n\sigma=n\cos n\sigma . \tag{32}$$

This implies that

$$\Psi[x(\sigma)]\;=\;\exp[-S_E]\;=\;\exp\left[-\sum_{n>0}X_n^2\right] \tag{33}$$

which is indeed the wavefunction of the ground state of a system of harmonically oscillating string modes.

We should next investigate the excited states of the string. These states can also be extracted from the functional integral, by recalling that evaluating the Euclidean functional integral over a finite interval of τ is equivalent to applying the operator $e^{-H\tau}$ to a state in Hilbert space. States of given energy are then characterized in the functional integral by factors with fixed exponential dependence on time. We can create such a state from the vacuum by applying an operator at a fixed time to the past of $\tau=0$.

As a first example of this technique, let us try placing the operator $e^{i\vec{k}\cdot\vec{x}(\sigma=0,\tau)}$ under the functional integral at some finite negative τ:

$$\mathcal{A} = \int \mathcal{D}x \; e^{-S_E} \; \exp(i\vec{k}\cdot\vec{x}(\sigma=0,\tau)) \; . \tag{34}$$

Evaluating the integral by the substitution (27), we find our previous result multiplied by the two factors:

$$e^{i\vec{k}\cdot\vec{x}_c(\sigma=0,\tau)} \cdot e^{-\frac{1}{2}k^2\langle\delta x^2\rangle} \tag{35}$$

The second factor is a multiplicative renormalization; ignore it for the moment. The first factor may be written more explicitly as:

$$\begin{aligned}
&\exp\left[i\vec{k}\cdot\int_0^\pi d\sigma' x_0(\sigma')\left[-\frac{\partial}{\partial\tau'}\,G(\tau,\tau'+i\sigma')\right]\right]\\
&\quad=\exp\left[i\vec{k}\cdot\int d\sigma' x_0(\sigma')\,\frac{1}{2\pi}\,\left\{\frac{e^{i\sigma'}}{e^{i\sigma'}-e^{\tau}}+(\sigma'\to-\sigma')\right\}\right]\\
&\quad=\exp\left[i\vec{k}\,\frac{1}{\pi}\int d\sigma' x_0(\sigma')\,\frac{1}{\pi}\,\left[1+e^{-|\tau|}\cos\sigma+e^{-2|\tau|}\cos 2\sigma+\ldots\right]\right]\;.
\end{aligned} \tag{36}$$

As $\tau \to -\infty$, this insertion does nothing more than multiply the amplitude (33) by $e^{i\vec{k}\cdot\vec{x}_0}$; this corresponds to the formation of a state of definite nonzero transverse momentum. But we can also read from this formula the spacing and wave functions of the higher excited states, since the coefficients of exponentials $e^{-\epsilon|\tau|}$ must be states of energy ϵ above the ground state. In general, the excited states appear with integer spacing. If we restore the factors of P^+ and α', we can see that the nth level corresponds to

$$\begin{aligned} \Delta H = \Delta P^- &= \frac{n}{P^+} = \frac{2n}{2P^+ \cdot 2\alpha'} \\ \Rightarrow \quad m^2 &= 2P^+P^- = \frac{n}{\alpha'} \,. \end{aligned} \tag{37}$$

(The contribution of the transverse momentum to P^- is given by the τ dependence of the second factor in (35).)

To fix the first excited state more cleanly, we should modify the operator in (34) so that it creates only states orthogonal to the ground state. A simple choice is

$$\dot{x}(\sigma = 0, \tau)\; e^{i\vec{k}\cdot\vec{x}} \,. \tag{38}$$

Again, the leading contribution comes from saturating with $x_c(\sigma, \tau)$. Since

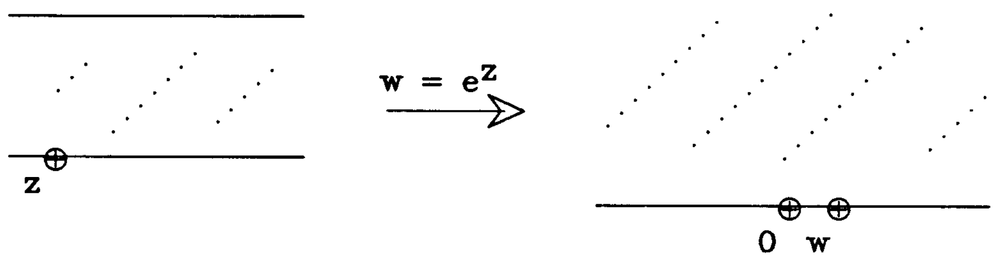
$$\begin{aligned} \dot{x}_c(\sigma = 0, \tau) &= \frac{1}{\pi} \int d\sigma' x_0(\sigma') \left[e^{-|\tau|} \cos\sigma + 2e^{-2|\tau|} \cos 2\sigma + \ldots \right] \\ &\sim \quad e^{-|\tau|} \cdot X_1 \,, \end{aligned} \tag{39}$$

the leading term for large $|\tau|$ created by this operator is

$$e^{-|\tau|} \; \cdot \; X_1^i \exp\left[-\sum_n X_n^2 \right] \,. \tag{40}$$

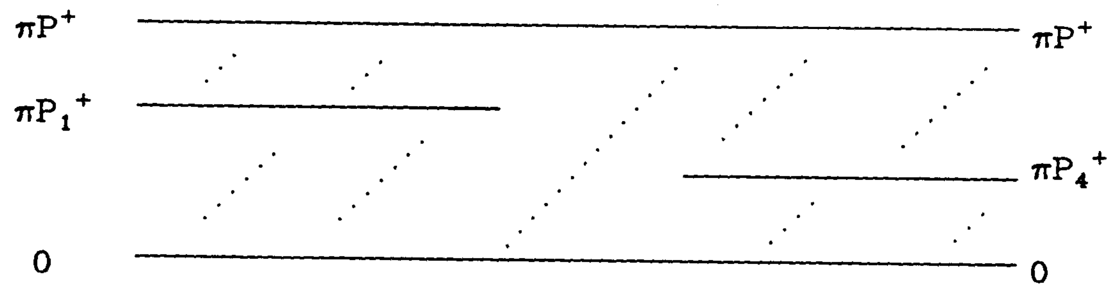
This is the first excited state of the lowest mode of oscillation of the string. Viewed as a relativistic particle, this state has the quantum numbers of a transverse vector. Continuing in this way, one can map out the whole spectrum by associating each state with an appropriate operator. The operators of this class are called *vertex operators.* The spectrum of states can be seen to be precisely the Fock space of string eigenmodes. The nth mass level contains states with spin up to n.

To connect the vertex operators with specific string eigenstates, we needed to apply them at large negative τ in order to given their products time to resolve into a definite state. However, if we have set up the functional formalism in a completely conformally-invariant way, operators positioned near $\tau = -\infty$ are equivalent by conformal mapping to operators positioned at finite distances. For example, under the mapping:



an operator near $\tau = -\infty$ can be viewed as one near $w = 0$. This is a wonderful realization, because it allows us to generalize the analysis we have just done of the single-string spectrum to compute multi-string scattering amplitudes. As long as we can maintain exact conformal invariance, we can convert the operators at the infinite time separation needed to define asymptotic states into operators separated by finite intervals along the boundary of a simple domain.

Let us first try to visualize the process of string-string scattering in the special system of light-cone coordinates which we have constructed. In (17), we have set σ proportional to P^+. Since P^+ is conserved, the interacting string-string system always occupies a strip of definite width in σ. A natural hypothesis for the string-string interaction, first advanced by Mandelstam[10] , is that it is just the process of fusing two strings laid out in σ into a single string, or of splitting one string into two. The process of string-string scattering is then represented by a region in the z plane of the form:



The width of this strip is πP^+, where P^+ is the total in the initial or final state. Mandelstam's hypothesis is equivalent to the statement that the string functional integral evaluated on this domain, with appropriate boundary conditions at $\tau = \pm\infty$, gives

$$\mathbf{T}(AB \to CD) \cdot e^{-\tau P^-} , \tag{41}$$

where $\mathbf{T}$ is the string-string T-matrix, τ is the total elapsed x^+, and the exponential contains the total P^-.

It is not difficult to evaluate the functional integral on this domain using the methods we have developed, as long as we have the freedom to stretch out the domain by an arbitrary conformal mapping. Let us, then, assume that the exact conformal invariance of the classical field theory associated with (20) is maintained in the full evaluation of the functional integral. It is then convenient to transform to the complex variable y defined by

$$z = \sum_j P_j^+ \log(y - Y_i) \qquad \begin{array}{l} P_j > 0 \text{ incoming} \\ P_j < 0 \text{ outgoing} , \end{array} \tag{42}$$

where the Y_j are points on the real axis of the y plane; $j = 1, \ldots, N-1$ if the process involves N strings. The region in the z plane containing the scattering strings (the strip $0 \le \mathrm{Im}\, z \le \pi P^+$) is carried by this mapping into the upper half y-plane. The points Y_j are mapped to $\mathrm{Re}\, y = -\infty$ for incoming momenta, and to $\mathrm{Re}\, y = +\infty$ for outgoing momenta. The imaginary part of z jumps by πP_j^+ as y passes through Y_j. As y passes from Y_j to Y_{j+1}, z comes in from infinity along a line of constant $\mathrm{Im}\, z$, then turns around and goes back out again. Thus, all of the cuts marked in the z plane are mapped onto the real axis in y. The positions of the endpoints of these cuts, which give the times of joining and splitting of strings, are encoded in the values of the Y_j.

Let us now evaluate the functional integral in these coordinates. By conformal invariance, the action is

$$S_E = \frac{1}{\pi} \int d^2Y \, (\partial_\alpha \vec{x})^2 \, ; \tag{43}$$

the propagator for $x(Y)$ is

$$\left\langle x^i(Y) x^j(Y') \right\rangle = \pi G(Y, Y') \delta^{ij} = -\frac{1}{2} \mathrm{Re} \log \left[(Y - Y')(Y - \overline{Y}') \right] . \tag{44}$$

Each asymptotic state can be characterized by placing the corresponding vertex operator at the appropriate Y_j. Then the amplitude for a scattering process involving N string ground states is given by

$$\begin{aligned} &\int dY_2 \ldots dY_{N-2} \left\langle \prod_i \, : \, e^{i\vec{P}_i \cdot \vec{x}(Y_i)} : \right\rangle \\ &= \int dY_2 \cdots dY_{N-2} \prod_{i \neq j} e^{\vec{P}_i \cdot \vec{P}_j (\log |Y_i - Y_j|)} \\ &= \int dY_2 \cdots dY_{N-2} \prod_{i \neq j} |Y_i - Y_j|^{\vec{P}_i \vec{P}_j} \, . \end{aligned} \tag{45}$$

In writing this integral over the Y_j, I have fixed two of the Y_j to definite points: $Y_1 = 0$, $Y_{N-1} = 1$; our whole discussion implicitly keeps $Y_N = \infty$. Fixing these three parameters fixes the three-parameter subgroup of conformal transformations which carries the real axis of the y plane into itself.

To extract the T matrix from (45), we must multiply this integral by the factor $exp(\tau P^{-})$. Let us cast this factor into a more convenient form. Using

$$\tau = \mathrm{Re} \sum_{j=1}^{N-1} P_j^{+} \log(y - Y_i) \ , \tag{46}$$

and continuing to consider outgoing momenta as having negative values, we may rewrite

$$\begin{aligned} \tau_{\text{tot}} \Sigma P^{-} &= \sum_{final} \tau_f P_f^{-} - \sum_{initial} \tau_i P_i^{-} \\ &= -\sum_{ij} P_i^{-} P_j^{+} \log |Y_i - Y_j| \ . \end{aligned} \tag{47}$$

Multiplying this into (45), we find

$$T = \int dY_2 \cdots dY_{N-2} \prod_{i \neq j} |Y_i - Y_j|^{P_i P_j} \ . \tag{48}$$

This is the Koba-Nielsen formula[15] for the multistring scattering amplitude.

In my rush to obtain (48), I have, however, overlooked a number of subtleties. First of all, I have consistently dropped the singular $i = j$ terms in the evaluation of the expectation value of vertex operators in (45) and in the evaluation of τ_{tot}. Second, I have ignored the factors of $[\det(-\partial^2)]^{\frac{1}{2}}$ which arise from integrating over the coordinates $x(z)$. Both of thse factors require regularization and hence could break the conformal invariance which I had assumed in setting up this formalism. Fortunately, Mandelstam showed in his original work[10] that conformal invariance is actually maintained, as the result of cancellations among these factors, as long as two conditions are met: First, external momenta must obey

$$-p^2 = m^2 = -\alpha' = -2 \tag{49}$$

This fixes the mass of the string ground state. Second, the number of transverse coordinates must be fixed so that

$$d = 26 . \tag{50}$$

I do not have space to discuss this cancellation in detail. But we can see one piece of it rather easily[16] by returning to the computation of the ground state wave function and trying restore the factors of $[\det(-\partial^2)]^{\frac{1}{2}}$ which we had ignored in the evaluation of (33). Since these determinants count the zero-point energy of the string modes for $(d-2)$ transverse degrees of freedom, the factor we had dropped is

$$(\det[-\partial^2])^{(d-2)/2} = \exp\left[-\tau \cdot (d-2) \cdot \frac{1}{2} \sum_{n=0}^{\infty} n\right] . \tag{51}$$

with suitable regularization, one may write the divergent sum as a Riemann zeta function: $\Sigma n = \zeta(-1) = -1/12$, so we find the factor $\exp[+\tau]$, corresponding to a ground state $P^- = -1$, or a ground state $(\text{mass})^2 = -2$, only if the number of transverse dimensions is 24.

Let me briefly note a few properties of the Koba-Nielsen amplitude. First, let us evalute T for a 4-particle amplitude. We find

$$T = \int_0^1 dY\, Y^{P_1 \cdot P_2}(1-Y)^{P_2 \cdot P_3} = \frac{\Gamma(P_1 \cdot P_2 + 1)\Gamma(P_2 \cdot P_3 + 1)}{\Gamma(P_1 \cdot P_2 + P_2 \cdot P_3 + 2)} . \tag{52}$$

This is the amplitude guessed by Veneziano[17] as a candidate for the T matrix of the strong interactions which was the starting point for the development of string theory. For $P_i^2 = -2$, this amplitude has poles at $s = -2P_1 \cdot P_2 - 4 = 2(n-1)$; these values correspond to the masses of the string modes we found earlier. The same poles, and the same particles, appear as t-channel exchanges. Secondly, let me note that the Koba-Nielsen amplitude can be put into a manifestly conformally-invariant form:

$$T = \frac{1}{\mathcal{V}_\mathcal{G}} \int dY_1 \cdots dY_n \prod_{i \neq j=1}^{N} (Y_i - Yj)^{P_i P_j} , \tag{53}$$

where $\mathcal{V}_\mathcal{G}$ is the volume of the group of conformal transformations which leaves the real y axis fixed.

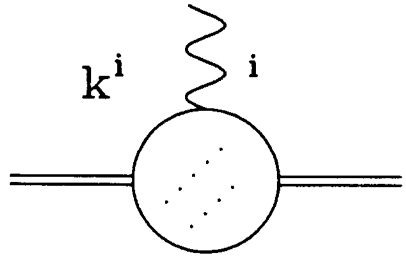
Finally, I would like to discuss the implications of the Koba-Nielsen formula for the properties of the first excited state of the string. The wavefunction (40) indicates that this state is a transverse vector. The longitudinal component of this vector is missing; this is consistent with Lorentz invariance only if this vector has zero mass. But, fortunately, the sum of the zero-point and excitation energies is

$$m^2 = 2(-1+1) = 0 . \tag{54}$$

It is clear, though, that we are still in a dangerous corner: Radiative corrections will drive this mass away from zero unless the result (54) is guaranteed by some underlying principle, such as local gauge invariance. But one can check that the cancellations required by local gauge invariance are working in the string amplitudes. Consider, for example, dotting the vector vertex operator (38) with the (transverse) momentum it introduces:

$$\vec{k}_i \cdot \dot{\vec{x}}(\tau_i) e^{i\vec{k}\cdot\vec{x}} = \frac{d}{d\tau_i} e^{i\vec{k}\cdot\vec{x}} , \tag{55}$$

and inserting this expression into the functional integral for T. Because one integrates over the time of splitting or joining, (55) can be integrated away to $\pm\infty$. Potentially, one may find a contact term when $e^{ik\cdot x}$ is passed through another vertex operator. However, such a term would have the form $|Y - Y_j|^{k\cdot k_j}$, evaluated at $Y = Y_j$. Since this factor is defined by continuation from spacelike momenta, $k \cdot k_j > 0$, the contact terms are *zero*. Thus,



$= 0$

This is the Ward identity for a gauge boson. Neveu and Scherk[18] computed explicitly the T-matrix for the scattering of four vectors and showed that, at low energies $s << \alpha'$, it coincides exactly with the tree graphs of Yang-Mills theory. So it is quite likely that the string theory actually contains a gauge principle which keeps the vectors massless and transverse.

So far in this lecture, I have discussed only open strings. Let me conclude by sketching the generalization to closed strings. This entails generalizing the formalism we have developed to coordinate fields $x(z)$ which propagate on a strip with periodic boundary conditions in σ $(0 \leq \sigma \leq \pi)$. An appropriate Green's function can be found by mapping this domain into the whole complex plane, using the mapping $w = e^{2z}$. One finds, then

$$G = -\frac{1}{2\pi}\log(e^{2z} - e^{2z'}) \ . \tag{56}$$

Because z appears in the form e^{2z}, the spacing of levels in P^- is now *double* the spacing in the open string. There is a second change as well from the open string results; this comes from a constraint which we noted earlier but which becomes important only in this system. The definition of our coordinate system included the condition $\int P^+_\sigma = 0$. This condition was satisfied automatically for open strings but must be imposed as a subsidiary condition for closed strings. If we Fourier decompose

$$x(\sigma) = x + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} X_n e^{in\sigma} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \overline{X}_n e^{-in\sigma} , \tag{57}$$

the ground state of the closed string is:

$$\Phi^{(0)} = \exp\left[-\sum_{n=1}^{\infty} X_n \overline{X}_n \right] , \tag{58}$$

as before. But the natural candidate for the first excited state

$$X_1^i \Phi^{(0)} \tag{59}$$

has net P^+_σ. The first excited state with nonzero P^+_σ has one left-moving and one right-moving excitation

$$X_1^i \overline{X}_1^j \Phi^{(0)} . \tag{60}$$

This state is a transverse spin 2 particle. For consistency with Lorentz invariance, it must be massless; then it must also have the couplings of a graviton. The constraints on the conformal invariance of the closed-string functional integral place the ground state mass at $m^2 = -8$ or $P^- = -4$, so these state do turn out to be massless. The vertex operator for the spin-2 state is given by the natural generalization of (38),

$$\int d\sigma\, \partial_z x^i \partial_{\bar{z}} x^j \ : e^{ik\cdot x} : \ . \tag{61}$$

Dotting (61) with the transverse momentum k^i gives the total derivative

$$\int d\sigma\, i\partial_z \Big(\partial_{\bar{z}} x^j \ : e^{ik\cdot x} : \Big) \ , \tag{62}$$

so this vertex satisfies a Ward identity analogous to that satified by the vector vertex. Indeed, Scherk and Schwarz[1] have verified that, at low energies, these particles have the scattering amplitudes of gravitons.

Thus we find, emerging automatically from the dynamics of quantized strings, gauge bosons and gravitons. We find that the conformal invariance which this dynamics requires restricts the dimensionality of space (unfortunately, to $d = 26$). At this level, still, the theory contains no fermions and, thus, no possibilities for matter. This difficulty, however, is readily resolved by considering a more sophisticated string theory, to which we now turn.

2. THE SUPERSTRING OF GREEN AND SCHWARZ

Having now studied the physics of an unadorned world-sheet propagating in space-time, let us now consider a generalization of that theory which includes fermions and—maybe—everything else of relevance to physics. I will present this theory in the light-cone formulation derived only relatively recently by Green and Schwarz[11] . The full construction is technically rather complex—too complex, unfortunately, to explain in one lecture. I will discuss its general structure, by emphasizing the analogy to the structure of the low-energy limit of the open-string theory, 10-D supersymmetric Yang-Mills theory, an analogy stressed by Green and Schwarz[19] .

In the last section, we discussed a theory whose action was that of d coordinates treated as of Bose fields and coupled invariantly to the geometry of the surface. From our modern perspective, a natural generalization of this theory is given by 2-D *supergravity* interacting with a supermultiplet of matter fields (x^μ, ψ^μ_α). In this construction, ψ_α is a 2-D Majorana fermion; the space-time index μ indexes the matter multiplets and thus stands outside the supersymmetry. This generalization was actually formulated by Neveu, Schwarz, and Ramond[20,21] well before the development of supersymmetry. One can carry out the light-cone quantization of this string as discussed above; one finds a consistent Lorentz-invariant theory—provided that $d = 10$. For the open strings, two possible sets of boundary conditions for the fermions are compatible with the boundary conditions for $x(z)$ given above. One of these sets gives string states which are *fermions*, of which the lowest state is massless; the other gives *bosons*, with a ground-state tachyon and a massless vector as before. However, now a symmetry distinguishes these particles. The theory has a conserved quantity G; the tachyon has $G = -1$, the vector has $G = +1$. Gliozzi, Scherk, and Olive[22] noticed that if one keeps only $G = +1$ bosons and only fermions which are both Majorana and Weyl (as is possible in $d = 10$), the resulting spectrum has no tachyon and is supersymmetric in 10-D space-time, possessing equal number of *fermion* and *boson* degrees of freedom at each mass level.

Green and Schwarz subsequently reformulated the theory by exchanging the 8 transverse ψ^i_α's, which belong to the vector representation of the transverse symmetry group 0(8), for a multiplet S^a_α belonging to the 8-D Weyl fermion representation of 0(8). This is possible because of a magical property of 0(8), that the vector and the two (real and inequivalent) Weyl spinor representations are of the same size and are, in fact, interchangeable by an automorphism of the algebra ("triality"). I will describe this construction for you. To do this, I will proceed as follows: First, I will develop the necessary features of 10-D spinor algebra. Next, I will discuss 10-D supersymmetric Yang-Mills theory and its quantization in the light-cone frame. Finally, I will assemble the ingredients of the string theory in parallel with the analysis of this simpler system.

In order to discuss 10-D fermions and 10-D supersymmetry we need 10-D Dirac matrices. It will be useful to construct these matrices directly in a Majorana representation. I will define the Majorana condition by the rather strong requirement that Dirac spinors may be taken to be real:

$$\xi = \xi^* \; . \tag{63}$$

(A complete discussion of the existence of Majorana representations in various dimensions has been given by van Nieuwenhuizen in ref. 23.) Dirac spinors in d dimensions generally have $2^{d/2}$ components, so ξ will be 32-dimensional. The Dirac equation for ξ reads

$$0 = i\gamma \cdot \partial \, \xi = (i\gamma \cdot \partial \, \xi)^* = -i\gamma^* \cdot \partial \, \xi^* \; ; \tag{64}$$

this is consistent with (63) only if

$$(\gamma^\mu)^* = -\gamma^\mu \; . \tag{65}$$

The anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ then imply

$$(\gamma^0)^T = -\gamma^0 \, , \quad (\gamma^i)^T = \gamma^i \, , \quad \text{so that} \quad \gamma^0 (\gamma^\mu)^T \gamma^0 = -\gamma^\mu \; . \tag{66}$$

Using this relation, we can see that each matrix of the form $\gamma^0 \Gamma^{\mu\nu\lambda\ldots\delta}$, where $\Gamma^{\mu\nu\lambda\ldots\delta} = \gamma^{[\mu}\gamma^\nu\gamma^\lambda \ldots \gamma^{\delta]}$, has a definite symmetry: $(\gamma^0\Gamma^{\mu\ldots\delta})^T = \pm(\gamma^0\Gamma^{\mu\ldots\delta})$.

Now, the matrices $\gamma^0\Gamma^{\mu\nu\lambda\ldots\sigma}$ form a complete set of 32×32 matrices, so we must find 528 symmetric and 496 antisymmetric matrices. It is not hard to enumerate the cases, beginning with

$$(\gamma_0)^T = -\gamma^0 \; , \quad (\gamma^0\gamma^\mu)^T = +\gamma^0\gamma^\mu \; , \quad (\gamma_0\Gamma^{\mu\nu})^T = +\gamma^0\Gamma^{\mu\nu} \; , \tag{67}$$

and, in fact, the counting turns out to work exactly. This counting condition turns out to be a sufficient condition for the existence of a Majorana representation. The corresponding statement works in d =2, 4, 10, 12, 18, 20, ... A similar argument shows that it is consistent to have a representation of the (Euclidean) Clifford algebra $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ in 8 dimensions in which the γ^i are *real symmetric* matrices. Let us label the matrices of this representation as γ_8^i.

Since our main interest will be in exploring light-cone dynamics in 10 dimensions, it will be useful to choose a representation for the Dirac matrices in which those matrices corresponding to light-cone directions are especially simple. Thus, writing the 32×32 matrices γ^μ as matrices of 16×16 blocks, choose

$$\gamma^0 = \left(\begin{array}{c|c} & -i \\ \hline i & \end{array}\right) \qquad \gamma^9 = \left(\begin{array}{c|c} & -i \\ \hline -i & \end{array}\right) \tag{68}$$

so that

$$\gamma^+ = \left(\begin{array}{c|c} 0 & -\sqrt{2}i \\ \hline 0 & 0 \end{array}\right) \qquad \gamma^- = \left(\begin{array}{c|c} 0 & 0 \\ \hline \sqrt{2}i & 0 \end{array}\right) \tag{69}$$

$$(\gamma^\pm)^2 = 0 \qquad \{\gamma^+, \gamma^-\} = 2 \; .$$

The other γ's must anticommute with these. A set which satisfies all the necessary requirements is:

$$\gamma^j = \left(\begin{array}{c|c} i\gamma_8^j & 0 \\ \hline 0 & -i\gamma_8^j \end{array}\right) , \tag{70}$$

where the γ_8^i are the real symmetric 16×16 matrices which represent the 8-dimensional Euclidean algebra.

A remarkable feature of the 10-dimensional Dirac algebra, already noted above, is that fact that Majorana fermions may be further restricted by a Weyl condition

$$\gamma^{11}\xi = (\pm 1)\,\xi \; , \quad \text{where } \gamma^{11} \; = \; \gamma^0\gamma^1\cdots\gamma^8\gamma^9 \; . \tag{71}$$

To see how this works, first define

$$\gamma_8^9 = \gamma_8^1\gamma_8^2\ldots\gamma_8^8 \; . \tag{72}$$

This matrix satisfies

$$\gamma_8^9 = (\gamma_8^9)^T = (\gamma_8^9)^{-1} = (\gamma_8^9)^* \; ; \quad (\gamma^9)^2 = 1 \tag{73}$$

We can therefore choose the γ_8^i so that γ_8^9 takes the form

$$\left(\begin{array}{c|c} -\mathbf{1} & 0 \\ \hline 0 & \mathbf{1} \end{array}\right) , \tag{74}$$

where the blocks are now 8×8. In this basis, the γ_8^i take the block form:

$$\left(\begin{array}{c|c} 0 & \gamma_{8(-)}^i \\ \hline \gamma_{8(+)}^i & 0 \end{array}\right) \tag{75}$$

with $\gamma^i_{8(+)} = (\gamma^i_{8(-)})^T$. Using this form for γ^9_8, we can evaluate

$$
\begin{aligned}
\gamma^{11} = \gamma^0\gamma^1\ldots\gamma^8\gamma^9 &= \left(\begin{array}{c|c} & -i \\ \hline i & \end{array}\right)\left(\begin{array}{c|c} \gamma^9_8 & \\ \hline & \gamma^9_8 \end{array}\right)\left(\begin{array}{c|c} & -i \\ \hline -i & \end{array}\right) \\
&= \left(\begin{array}{c|c} -\gamma^9_8 & \\ \hline & \gamma^9_8 \end{array}\right) = diag\;(\mathbf{1}, -\mathbf{1}, -\mathbf{1}, \mathbf{1}) \; .
\end{aligned}
\tag{76}
$$

where $\mathbf{1}$ is the 8×8 identity matrix. γ^{11} is *real*, so the Weyl condition (71) is compatible with the Majorana condition (63). This state of affairs is peculiar to 2, 10, 18, ... dimensions.

It is instructive to see what the supersymmetry algebra looks like in this basis. The standard form of the supersymmetry algebra is

$$
\{Q_\alpha, \overline{Q}_\beta\} = -2\,(\gamma \cdot p)_{\alpha\beta}
\tag{77}
$$

Multiplying by γ^0 and using the explicit forms

$$
\gamma^+\gamma^0 = \sqrt{2}\,\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \quad \gamma^-\gamma^0 = \sqrt{2}\,\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right) \quad \gamma^i\gamma^0 = \left(\begin{array}{c|c} & \gamma^i_8 \\ \hline \gamma^i_8 & \end{array}\right) ,
\tag{78}
$$

we can write out these commutation relations as (8×8 blocks):

$$
\{Q_\alpha, Q^\dagger_\beta\} = \sqrt{2}\begin{pmatrix} 2p^- & 0 & 0 & -\sqrt{2}\vec{\gamma}_{8(-)} \cdot \vec{p} \\ 0 & 2p^- & -\sqrt{2}\vec{\gamma}_{8(+)} \cdot \vec{p} & 0 \\ 0 & -\sqrt{2}\vec{\gamma}_{8(-)} \cdot \vec{p} & 2p^+ & 0 \\ -\sqrt{2}\vec{\gamma}_{8(+)} \cdot \vec{p} & 0 & 0 & 2p^+ \end{pmatrix}
\tag{79}
$$

But it is possible to define a smaller algebra than (79), since, in 10-D, we may take Q to be a Majorana-Weyl spinor. This choice gives the minimal number of supersymmetries ($N = 1$ supersymmetry) in 10 dimensions. The Majorana condition sets $Q^\dagger = Q$. The Weyl condition restricts Q so that, *e.g.*, $\gamma^{11}Q = -Q$. If we apply this condition, we reduce Q to the form

$$Q = 2^{1/4} \begin{pmatrix} 0 \\ Q_- \\ Q_+ \\ 0 \end{pmatrix} \tag{80}$$

where each entry is 8×1. The commutation relations of supersymmetry then take the form:

$$\begin{aligned} \{Q_{+\alpha}, Q_{+\beta}\} &= 2P^+ \delta_{\alpha\beta} \\ \{Q_{+\alpha}, Q_{-\beta}\} &= -\sqrt{2}\,(\vec{\gamma}_{8(-)} \cdot \vec{P})_{\alpha\beta} \\ \{Q_{-\alpha}, Q_{-\beta}\} &= 2P^- \delta_{\alpha\beta} = 2H\delta_{\alpha\beta}\ . \end{aligned} \tag{81}$$

Thus, half of the supersymmetries are relatively trivial, being the square roots of the momentum operator P^+. We may call these *kinematical supersymmetries.* Only the $Q_{-\alpha}$ depend on the nonlinear interactions contained in H.

It would be useful to understand this algebraic structure a bit better. To gain some experience with it, let us make use of it in working out the light-cone dynamics of 10-D supersymmetric Yang-Mills theory. This theory is the simplest supersymmetric theory in 10 dimensions, and it is simple indeed, containing only a vector gauge boson and a Majorana-Weyl gaugino λ, which satisfies $\gamma^{11}\lambda = \lambda$. The action for the theory is

$$\mathcal{L} \ = \ \mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}\bar{\lambda}\, i \not{D}\,\lambda\right]\ . \tag{82}$$

Let us work out the quadratic part of the light-cone Hamiltonian. Begin with the fermion dynamics. Using (69), (70), we can write the free Lagrangian for λ

as

$$\mathcal{L}_2 = \frac{1}{2}\,\lambda^{aT} i\gamma^0\gamma^\mu\partial_\mu\lambda^a$$

$$= \frac{1}{2}\,\lambda^{aT} i\left[\sqrt{2}\,\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right)\,\partial_+ + \sqrt{2}\,\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right)\,\partial_- \right. \tag{83}$$

$$\left. + \left(\begin{array}{c|c} 0 & -\vec{\gamma}_8 \\ \hline -\vec{\gamma}_8 & 0 \end{array}\right)\cdot\vec{\partial}\right]\lambda^a\,.$$

Let us decompose

$$\lambda = \begin{pmatrix} \lambda_- \\ \lambda_+ \end{pmatrix}\;; \tag{84}$$

the Weyl condition on λ induces $\gamma_8^9\lambda_- = -\lambda_-$, $\gamma_8^9\lambda_+ = \lambda_+$. Then

$$\mathcal{L}_2 = \frac{1}{\sqrt{2}}\,\lambda_+^T(i\partial_+)\lambda_+ + \frac{1}{\sqrt{2}}\,\lambda_-^T(i\partial_-)\lambda_- - \frac{1}{2}\,\left[\lambda_-^T i\vec{\gamma}_8\cdot\vec{\partial}\lambda_+ + \lambda_+^T i\vec{\gamma}_8\cdot\vec{\partial}\lambda_-\right]\;, \tag{85}$$

Now recall that the evolution parameter for light-cone dynamics is $\tau = x^+$; x^- and $P^+ = i\partial_-$ are purely kinematical. Fields which appear in the Lagrangian with no derivatives ∂_+ are thus auxiliary fields which must be eliminated to set up the Hamiltonian formalism. λ_- is such a field. Integrating it away reduces (85) to

$$\mathcal{L}_2 = \frac{1}{\sqrt{2}}\,\lambda_+^T i\partial_+\lambda_+ - \frac{1}{2\sqrt{2}}\,\lambda_+^T\,\frac{(i\vec{\gamma}_8\cdot\vec{\partial})^2}{i\partial_-}\,\lambda_+\,. \tag{86}$$

Let us rename $\lambda = 2^{-1/4}\,\lambda_+$. (Hopefully, this object, which is a Majorana-Weyl spinor of $O(8)$, will not be confused with the original λ.) Then $\mathcal{L}_2$ falls into the final form

$$\mathcal{L}_2 = \lambda^T i\partial_+\lambda - \frac{1}{2}\,\lambda^T\,\frac{(-\partial^2)}{P^+}\,\lambda \tag{87}$$

Turn now to the bosonic half of the theory. In a light-cone reduction, the gauge boson Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{4}\,(F^a_{ij})^2 + F_{+i}F_{-i} + \frac{1}{2}\,F_{+-}F_{+-}\ . \tag{88}$$

Choose the gauge $A_- = 0$; this insures that $iD_- = i\partial_- = P^+$ remains simple in the presence of interactions. Then

$$\mathcal{L}_2 = -\frac{1}{2}\left[A_i(-\partial^2)A_i - (\partial_i A_i)^2\right] + [\partial_+ A_i - \partial_i A_+]\partial_- A_i + \frac{1}{2}\,(\partial_- A_+)^2\ . \tag{89}$$

In this Lagrangian, A_+ is an auxilliary field. Integrating it away, we find

$$\begin{aligned}\mathcal{L}_2 &= -\frac{1}{2}\,A_i(-\partial^2)A_i + \frac{1}{2}\,(\partial_i A_i)^2 + \partial_+ A_i\partial_- A_i \\ &\qquad - \frac{1}{2}\,(\partial_-\partial_i A_i)\,\frac{1}{-\partial_-^2}\,(\partial_-\partial_j A_j) \\ &= \partial_- A_i\,\partial_+ A_i - \frac{1}{2}\,A_i(-\partial^2)A_i\ .\end{aligned} \tag{90}$$

We have now reduced (82) to the expressions (87), (90), which involve only the dynamical fields λ, A_i. Note that these fields include equal numbers of fermions and bosons: After the Majorana and Weyl conditions on λ, each multiplet contains 8 real components. The Lagrangians (87), (90) lead to the commutation relations

$$\begin{aligned}[\lambda_\alpha(x), \lambda_\beta(y)]_+ &= \frac{1}{2}\,\delta^{(9)}(x-y)\,\left(\frac{1+\gamma^9_8}{2}\right)_{\alpha\beta} \\ [A^i(x), \partial_- A^i(y)]_- &= \frac{i}{2}\,\delta^{(9)}(x-y)\,\delta^{ij}\end{aligned} \tag{91}$$

and the light-cone Hamiltonian

$$H_2 = \int d^9x\left\{\frac{1}{2}\,A_i\vec{P}^2 A_i + \frac{1}{2}\,\lambda^T\,\frac{\vec{P}^2}{P^+}\,\lambda\right\} \tag{92}$$

To make this look more symmetrical, let us define $\Lambda = \frac{1}{\sqrt{P^+}}\,\lambda$. Then the equations defining the light-cone dynamics take the form

$$
\begin{aligned}
[\Lambda_\alpha(x), \Lambda_\beta(y)] &= \frac{1}{2P^+}\, \delta^{(9)}(x-y)\, \delta_{\alpha\beta} \\
[A^i(x), A^j(y)] &= \frac{1}{2P^+}\, \delta^{(9)}(x-y)\, \delta^{ij} \\
H_2 &= \int d^9x \left\{ \frac{1}{2}\, A_i \vec{P}^2 A_i + \frac{1}{2}\, \Lambda^T \vec{P}^2 \Lambda \right\} \ .
\end{aligned} \tag{93}
$$

The manifest fermion-boson symmetry of (93) is a direct reflection of the kinematical part of the supersymmetry algebra (81). The kinematical supersymmetry charges $Q_{+\alpha}$ take the explicit form:

$$
Q_{+\alpha} = \int d^9x\ 2i\ (A^i \gamma_8^i (P^+)^{3/2} \Lambda)_\alpha \ . \tag{94}
$$

These $Q_{+\alpha}$ satisfy the commutation relations

$$
\begin{aligned}
[Q_{+\alpha}, A^j] &= -i\sqrt{P^+}\ \gamma^i_{\alpha\beta}\ \Lambda_\beta \ , \\
[Q_{+\alpha}, \Lambda_\beta] &= -i\ \sqrt{P^+}\, \gamma^i_{\alpha\beta} A^i \ ,
\end{aligned} \tag{95}
$$

so they are indeed the square roots of P^+. For your reference, the dynamical supersymmetry generators for the free theory, which square to H_2, are given by

$$
Q_{-\alpha} = \int d^9x (-\gamma^i \gamma^j A^j \sqrt{2P^+}\, \partial^i \Lambda)_\alpha \ . \tag{96}
$$

The reduction we have seen here of a space-time vector to its propagating transverse components is quite analogous to the reduction of the string coordinate $x^\mu(\xi)$ which we saw at the beginning of the previous section. It is then not unreasonable that a 10-D string theory, formulated in the light-cone frame, in which the transverse coordinates $x^i(z)$ are supplemented by an $O(8)$ Majorana-Weyl fermion might possess space-time supersymmetry. Indeed, Green and Schwarz succeeded in constructing a manifestly supersymmetric string theory by working precisely along this line. We are now ready to review their construction.

The final form of the light-cone string action derived in the previous section, eq. (22), is

$$S_X = -\frac{1}{4\pi\alpha'}\int d^2z\,(\partial_\alpha x^i\partial^\alpha x^i)\;; \qquad i=1,\ldots,8 \tag{97}$$

We must now add to this an action for fermions propagating on the string surface. Let us introduce a two multiplets S^A, $A = 1,2$, of light-cone-Majorana-Weyl fermions. The conditions on S^A are, more explicitly,

$$\gamma^+S^A = 0\;, \qquad (S^A)^* = S^A\;, \qquad \gamma^{11}S^A = +S^A\;. \tag{98}$$

The first condition on S is the opposite of the condition we imposed on λ; the reason for this will become apparent a bit later. Each S^A is formally a 32-component object; however, it is reduced to 8 real components by the conditions (98). In particular, the light-cone condition effects the reduction:

$$S^\dagger\gamma^0\gamma^-S = \sqrt{2}\,S^\dagger\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0\end{array}\right)S \tag{99}$$

Let us represent the 2-dimensional Dirac matrices on the string surface as follows:

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0\end{pmatrix}\;. \tag{100}$$

Note that these matrices also have been constructed in a Majorana representation which allows a further Weyl reduction, since $\rho^3 = \rho^0\rho^1$ is real diagonal. Since

$$\rho \cdot \partial = -i \left(\begin{array}{c|c} 0 & \partial_\tau - \partial_\sigma \\ \hline -\partial_\tau - \partial_\sigma & 0 \end{array} \right) , \tag{101}$$

the Dirac equation on the string surface is solved by spinors whose upper ($\rho^3 = +1$) component is a function only of $(\tau - \sigma)$, a right-moving wave, and whose lower ($\rho^3 = -1$) component is a left-moving wave.

We can merge these ingredients by considering the index A of S^A as a 2-dimensional spinor index. Then a natural theory of free fermions constrained as in (98) is given by

$$\begin{aligned} S_S &= \frac{i}{\sqrt{2}\, 2\pi} \int S^+ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right) \rho^0 \rho . \partial S \\ &= \frac{i}{4\pi} \int \overline{S} \gamma^- \rho \cdot \partial S \ , \end{aligned} \tag{102}$$

where $\overline{S} = S\gamma^0\rho^0$. The boundary condition on S (for open strings) should be that the momentum flux of the fermions $J_\sigma = \overline{S}\gamma^-\rho^\sigma S$ must be zero on the boundary. This gives the condition

$$S^+\gamma^-\rho^\sigma S = \sqrt{2}\ S^+ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right) \left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array} \right) \ S = 0 \tag{103}$$

We can satisfy this condition naturally by insisting that $S^1(\sigma) = S^2(\sigma)$ at both boundaries of the string: $\sigma = 0, \pi$.

As we did for x^i, we should Fourier-expand S^A consistently with the boundary conditions:

$$x^i(\sigma) = x_0 + \sum_{n>0} 2\, X_n \cos n\sigma / P^+$$

$$S^{\alpha A} = \sum_{n=-\infty}^{\infty} S_n^\alpha \begin{pmatrix} e^{in\sigma/P^+} \\ e^{-in\sigma/P^+} \end{pmatrix} . \tag{104}$$

We have used the fact that the Dirac eigenfunctions will be right-moving (left-moving) for $\rho^3 = +1$ (-1). As we showed explicitly for x, the modes of S with $n \neq 0$ correspond to excitations of the string of higher P^-, i.e. of higher mass. In fact, the S system has the same steps as for the bosonic string:

$$m^2 \;=\; 2P^+P^- \;=\; 2n \quad \text{or} \quad n/\alpha' . \tag{105}$$

For the bosonic string, the $n = 0$ mode costs little energy to populate, so we wrote arbitrary functions of x_0:

$$\Psi_k[x] = e^{i\vec{k}\cdot\vec{x}_0}\Psi^{(0)} , \tag{106}$$

which are naturally interpreted as states of arbitrary transverse momentum. The dynamics of the $n = 0$ mode of S is, however, more intricate. The equal-time anticommutation relations for $S^{\alpha A}(\sigma)$ lead to anticommutation relations for the Fourier components S_0^α of the form

$$\begin{aligned} \{S_0^\alpha, S_0^\beta\} &= 2\left[\gamma^+\gamma^-\left(\frac{1+\gamma^{11}}{2}\right)\right]^{\alpha\beta} \\ &= 2\delta^{\alpha\beta} \quad \text{on the constrained subspace} . \end{aligned} \tag{107}$$

The S_0^α are actually a set of 8-D γ matrices! The states on which the S_0^α act form a collection of string modes with $P^- = 0$ which may be identified with the massless particle states of the sting theory.

As a step toward identifying this space, it is worth thinking for a moment about the spinor representations of $O(8)$. Dirac spinors in 8 dimensions are 16-component objects, which may be reduced by Weyl conditions to an 8-dimensional $\gamma_8^9 = +1$ spinor (which I will call the $\mathbf{8}_s$) and an 8-dimensional $\gamma_8^9 = -1$ spinor (the $\mathbf{8}'_s$). The matrices γ_8^i anticommute with γ_8^9, so they flip the γ_8^9 eigenvalue. The matrices γ_8^i then map from the $\mathbf{8}_s$ into the $\mathbf{8}'_s$ and vice versa; the block matrices $\gamma_{8(\pm)}^i$ defined in (75) are essentially Clebsch-Gordan coefficients for coupling the $\mathbf{8}_s$, the $\mathbf{8}'_s$ and the vector representation $\mathbf{8}_v$. The light-cone condition we have placed on S, eq. (98), implies that S belongs to the $\mathbf{8}'_s$; we can then represent S_0^α as a block of γ_8^i viewed as a matrix linking the $\mathbf{8}_s$ and the $\mathbf{8}_v$. Since $O(8)$ allows an automorphism which interchanges $\mathbf{8}_v$, $\mathbf{8}_s$, and $\mathbf{8}'_s$, the Dirac matrices viewed in this way still have their standard anticommutation relations. The S_0^α then act on a multiplet of states containing an $\mathbf{8}_v$ and an $\mathbf{8}_s$; these states may be identified with the states created by A^i and Λ in our discussion of 10-dimensional Yang-Mills theory on the light-cone. Note that, because we have found a purely transverse vector and a chiral spinor, the string theory can be Lorentz-invariant only if there is no zero-point contribution to P^-. Fortunately, any such contribution will cancel if supersymmetry on the string surface is maintained.

It is worth exploring the action of S_0^α on the zero-mass states somewhat more explicitly. To do this, it is useful to know how to split the product $S_0^\alpha S_0^\beta$ into its pieces symmetric and antisymmetric under $\alpha \leftrightarrow \beta$. This is accomplished by the following identity:

$$S_0^\alpha S_0^\beta = (\delta^{\alpha\beta}) + \frac{1}{4}\,(\Gamma^{ij})^{\alpha\beta} R^{ij} \tag{108}$$

where

$$R^{ij} = \frac{1}{8}\left(S_0 \Gamma^{ij} S_0\right) \qquad \Gamma^{ij} = \gamma_8^{[i} \gamma_8^{j]} \ . \tag{109}$$

The symmetric part follows from the anticommutation relations for S_0^α. The antisymmetric part arises as follows: The only Γ matrices built of γ_8^i which are antisymmetric in their indices and commute with γ_8^9 are Γ^{ij} and $\Gamma^{ijk\ell mn}$. But $\Gamma^{ijk\ell mn} = \epsilon^{ijk\ell mnpq}\gamma_8^9\Gamma^{pq}$, which reduces to Γ^{pq} on $\gamma_8^9 = +1$ states. The normalization can be fixed by tracing with some Γ^{ij}.

Now we can represent the action of the S_0^α on the appropriate Hilbert space. Let $|i\rangle$ denote a transverse vector state and $|\beta\rangle$ a spinor state. Then

$$S_0^\alpha \,|i\rangle = \gamma^i_{\alpha\beta}\,|\beta\rangle \ , \quad S_0^\alpha\,|\beta\rangle = \gamma^i_{\alpha\beta}\,|i\rangle \tag{110}$$

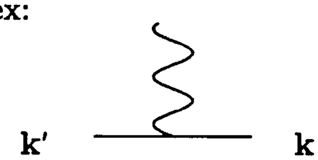
where $\gamma^i_{\alpha\beta}$ is an element of the real symmetric γ_8^i. These relations are inverse to one another, in the sense that

$$\{S_0^a, S_0^b\}\,|i\rangle = 2\delta^{ab}\,|i\rangle \ , \tag{111}$$

as required. Notice that these relations are *exactly* what we found before for the action of the kinematical supersymmetry operator $Q_{+\alpha}$ on the states created by the light-cone dynamical fields of supersymetric Yang-Mills theory, Λ and A^i. supersymmetric Yang-Mills theory. One can check that R^{ij} acts as a helicity operator:

$$\begin{aligned} R^{ij}\,|k\rangle &= \delta^{ik}\,|j\rangle - \delta^{ik}\,|i\rangle \equiv (\Sigma^{ij})_{k\ell}\,|\ell\rangle \\ R^{ij}\,|\alpha\rangle &= \left(\frac{1}{2}\Gamma^{ij}\right)_{\alpha\beta}|\beta\rangle \equiv (\Sigma^{ij})_{ab}\,|b\rangle \ . \end{aligned} \tag{112}$$

Given the states, we should next construct the vertex operators. But because the correspondence to the light-cone description of supersymmetric Yang-Mills theory is so close, we can almost guess the right form by looking at the interactions in that theory. This analysis is carried out in detail in ref. 19; I will discuss here only the simplest part of the result. Let us first recall the correspondence between local field vertices and string vertex operators for the case of the bosonic string. The local field of a charged scalar boson couples to a massless vector via the vertex:



$$= \quad (\mathrm{k} + \mathrm{k}')^i \quad \epsilon^i$$

The form of this vertex is reflected directly in the structure of the corresponding string vertex operator

$$\epsilon^i \, \dot{x}^i \, e^{ip\cdot x} \ , \tag{113}$$

since $\dot{x}^i$ is the momentum density at the boundary of the string. The couplings of a vector to local fields of higher spin contain also a spin term; the expression given above for scalars is modified to

$$[(k + k')^i + 2\Sigma^{ij} P_j] \ . \tag{114}$$

We might then expect that the vertex operator for an external transverse, $p^+ = 0$, gauge boson in the Green-Schwarz superstring is:

$$V_B \;=\; \epsilon^i [\dot{x}^i(\sigma = 0, \tau) + R^{ij}(\sigma = 0, \tau) P_j] \, e^{iP\cdot x} \ , \tag{115}$$

where

$$R^{ij}(\sigma, \tau) \;=\; \frac{1}{8} S \Gamma^{ij} S(\sigma, \tau) \ . \tag{116}$$

Crossing the inserted boson in (114) with an external fermion suggests the following form for the vertex function corresponding to a massless, $p^+ = 0$, fermion state:

$$V_F = \overline{u}[\dot{x}^i\gamma^i S - \frac{1}{3}R^{ij}p^i\gamma^j S]\, e^{ip\cdot x} \ . \tag{117}$$

Green and Schwarz have shown, first, that the single-string theory we have discussed is Lorentz invariant and fully supersymmetric, and, second, that these vertex operators are carried into one another by supersymmetry and lead to Lorentz-covariant scattering amplitudes.

To complete this discussion of the structure of the superstring, we should now turn our attention to the closed strings. I will restrict my discussion to a description of the zero-mass states of the closed string. The analysis of these states follows straightforwardly from our previous discussion. The fields on the string surface obey the same local equations as for the open string but receive a three-fold our previous discussion. The fields on the string surface obey the same local equations as for the open string but receive a three-fold modification from the new boundary conditions. First, the boundary condition (103) which fixed $S^{\alpha 1} = S^{\alpha 2}$ disappears, so that we now have two independent 2-dimensional spinor fields, each of which will have its own multiplet of associated zero-energy states. Second, as for the bosonic string, the condition $\int P_\sigma^+$ now become nontrivial; in particular, a string state containing a zero-mass left-moving state must also contain a zero-mass right-moving state. Finally, since no constraint now links S^1 and S^2, we may choose these to have the same or opposite chirality under γ^{11}.

The zero-mass states of the closed superstring can thus be written as follows, using the notation $|a,b\rangle$ to denote a state which is the direct product of the state $|a\rangle$ of the S_0^1 and the state $|b\rangle$ of the S_0^2:

$$|i,j\rangle \ , \quad |\alpha,i\rangle \ , \quad |i,\alpha\rangle \ , \quad |\alpha,\beta\rangle \ . \tag{118}$$

$|i,j\rangle$ is a general transverse tensor; this can be decomposed into a symmetric tensor h_{ij}, an antisymmetric tensor a_{ij}, and a trace ϕ. $|\alpha,i\rangle$ contains a spin-$\frac{3}{2}$ particle $\psi_{\alpha i}$, plus a trace part $\chi_\alpha = \gamma^i_{\alpha\beta}\,|\beta,i\rangle$. $|i,\alpha\rangle$ gives a second set of fermions ψ_i, χ. $|\alpha,\beta\rangle$ gives a set of bosons whose content depends on the relative chirality of S^1 and S^2.

Let us first consider the case

$$\gamma^{11}S^1 \;=\; +S^1\;, \quad \gamma^{11}S^2 \;=\; -S^2\;. \tag{119}$$

This is called the type IIA superstring. We can finish determining its particle content by noting that the states $|\alpha,\beta\rangle$ fill the reducible representation $8_s \times 8'_s$ of $O(8)$. This representation contains a vector, since the invariant γ^i maps 8_s to $8'_s$, and a rank-3 antisymmetric tensor. Thus, the complete content of the massless sector is:

$$h_{ij} \quad a_{ij} \quad \phi \quad \begin{matrix} \psi_{i(+)} & \chi_{(+)} \\ \psi_{i(-)} & \chi_{(-)} \end{matrix} \quad v^i \quad f^{[ijk]}\;, \tag{120}$$

where, for the spinors, $(\pm)$ denotes the eigenvalue of γ^9_8. As for the bosonic string, the symmetric tensor h_{ij} may be interpreted as the graviton. Similarly, the spin-$\frac{3}{2}$ fermions $\psi_{i(\pm)}$ may be interpreted as gravitinos. This theory, in fact, does contain two sets of supersymmetry charges, with $\gamma^{11}Q \;=\; \pm Q$. The massless content of this theory is exactly what one would obtain by dimensionally reducing to 10 dimensions an 11-dimensional theory of with the field content

$$H_{ij} \quad \Psi^a_i \quad F^{ijk}\;, \tag{121}$$

which is precisely the content of the 11-dimensional supergravity theory. (The correct field content of 11-dimensional supergravity was actually discovered in this way[24] .)

If we take both S^1 and S^2 to be $\gamma^{11} = +1$ fields, $|\alpha, \beta\rangle$ belongs to $8_s \times 8_s$. This contains a scalar, since the 8_s is real, and antisymmetric tensors of even rank. There are still two gravitinos (so that the theory still has $N = 2$ supersymmetry), but now both gravitinos have the same chirality and cannot be obtained as components of a higher-dimensional object. This theory is called the type IIB superstring. Its full massless content is

$$h_{ij} \quad a_{ij} \quad \phi \quad \begin{matrix} \psi_i^a & \chi \\ \psi'^a_i & \chi' \end{matrix} \quad \eta \quad b_{[ij]} \quad c_{[ijk\ell]} \ , \tag{122}$$

where $c_{[ijk\ell]}$ is self-dual.

To obtain a theory with $N = 1$ supersymetry, we can reduce the spectrum of states of the type IIB theory by identifying states with reversed orientation

$$|\Psi[x(\sigma), S(\sigma)]\rangle = \big|\Psi[x(\pi P^+ - \sigma), S(\pi P^+ - \sigma)]\big\rangle \ . \tag{123}$$

Then we obtain only the following states at the zero mass level:

$$|ij\rangle + |ji\rangle \quad |ia\rangle + |ai\rangle \quad |ab\rangle - |ba\rangle \ . \tag{124}$$

This theory, the type I closed superstring, thus has as its massless states

$$h_{ij} \quad \phi \qquad \psi_i^a \quad \chi \qquad b_{[ij]} \ , \tag{125}$$

which is the content of the $N = 1$ supergravity in 10 dimensions. Unlike the previous closed string theories, this one can be consistently coupled to the open string theory, which is necesssarily only $N = 1$ supersymmetric.

Both the type I and type IIB theories have *chiral* massless spectra. This means that they are not necessarily free of chiral anomalies in the matrix element of gauge bosons and gravitons. But here some miracles occur. Alvarez-Gaume and Witten[25] have shown that the supergravity theory with the content of the massless states of the IIB theory is free of gravitational anomalies. Green and Schwarz[2] have computed the gauge anomaly explicitly in the type I theory with open strings and have shown that the anomaly cancels when the theory is embellished with an 0(32) gauge group. They have argued that the gravitational anomalies also cancel in this particular theory. The same cancellations should occur in a string theory with $N = 1$ supersymmetry and the gauge group $E_8 \times E_8$. Such a theory (the *heterotic string*) has recently been constructed by Gross, Harvey, Martinec, and Rohm[26] . Since E_8 and its subgroups have long been recognized as natural grand unification groups[27,28] , this latter theory is a promising candidate for a theory containing all of the fundamental particles and forces.

3. GAUGE PARTICLES FROM STRINGS

In our explicit analysis of the properties of bosonic and super strings, we found zero-mass transverse vectors and tensors which could naturally be identified with gauge bosons and gravitons. These bosons obey the correct on-shell Ward identities. But, clearly, there is much more here to be understood. One might pose the basic question in two different ways. On the one hand, one might focus on the fact that the equations of gravity and Yang-Mills theory have a geometrical structure. How does this structure arise from string geometry? On the other hand, one might consider the appearance of vectors and gravitons as reflecting the presence of exact gauge invariances of the string theory, viewed as a field theory in the embedding space-time. What are these gauge invariances? The answers to these questions are not especially well understood; indeed, they are central issues of the current research into string theories. In the past year, however, a certain amount of progress has been made in unraveling these issues, and I would like to indicate what has been learned. One would expect that the two questions I have posed have the same answer. But I will only be able to give you two quite different answers, each somewhat incomplete. Their completion—and connection—I must leave to you.

The first pieces of insight which I will describe comes from generalizing the theory of strings embedded in flat space, which is what we have discussed up to now, to strings embedded in a curved background space-time. This theory has been worked out both for bosonic strings and for superstrings by Fradkin and Tseytlin[29], Sen[30], and Callan, Martinec, Perry, and Friedan[31]. I will restrict myself here to illustrating their analysis for the case of the bosonic string.

The key to their argument is conformal invariance, the symmetry we used in a very strong way in our discussion of the bosonic string to unveil its structure and to compute its scattering amplitudes from finite-time information. Even in a flat-space background, there is an anomaly which can spoil the conservation of conformal generators; this anomaly cancells only in $d = 26$. We have noted this point in our earlier discussion; let us now present it formally in way that will generalize. Return, then, to the action (2):

$$S = \frac{-1}{4\pi\alpha'} \int d^2\xi \sqrt{-g}\, g^{ab} \partial_a x^\mu \partial_b x^\mu \ . \tag{126}$$

In our earlier discussion, we used that fact that the Weyl symmetry

$$g_{ab} \to e^{\lambda(\xi)} g_a b \quad . \tag{127}$$

is a gauge invariance of (126). However, since (127) has the form of a scale transformation, we should expect that, when the theory is quantized, this symmetry will be spoiled by the regularization. Polyakov[32] discovered that there is a subtlety in this computation: One must consider the effect of the regulator both on the d coordinate fields $x^\mu(\xi)$ and on the gauge-fixing determinant which appears when one fixes g_{ab} to the conformal gauge. Considering both effects, Polyakov found

$$\left\langle \frac{\delta S_{\rm eff}}{\delta \lambda} \right\rangle = \langle T^a_a \rangle = \frac{d-26}{48\pi^2} \left[-\frac{1}{\alpha'}\, \partial^2 \sigma \right] = \frac{d-26}{48\pi^2} \sqrt{g}\, R^{(2)} \ , \tag{128}$$

where T^a_a is the trace of the energy momentum tensor on the string surface, and $R^{(2)}$ is the curvature of the string 2-geometry. This anomaly disappears in $d = 26$, leading to the exact conformal invariance which we assumed in our earlier analysis.

It is natural to expect that, for strings in a curved background geometry, the problem of anomalies becomes even more severe. Let us now analyze that situation. To begin, we need a generalization of the action (126) to a arbitrary background metric; a reasonable choice is

$$S = \frac{1}{4\pi\alpha'} \int d^2 z\, \sqrt{g}\, g^{ab} G_{\mu\nu}(x)\, \partial_a x^\mu \partial_b x^\mu \ . \tag{129}$$

If we fix the conformal gauge, $g_{ab} = \delta_{ab}$, this becomes a two-dimensional nonlinear sigma model with variables which live on the space-time manifold with metric

$G_{\mu\nu}$. But that means that we are in trouble. Consider, for example, the case in which the background manifold is a sphere. Then (129) contains an additional parameter, the radius of the sphere, which acts as a coupling constant. This coupling constant is well known to have a nontrivial β function; thus, in this case, the model (129) is not conformally invariant. It is useful to cast this problem into a more general setting. Friedan, in his thesis[33], studied the renormalization of a nonlinear σ model with variables on a general manifold, considering the space of metrics $G_{\mu\nu}$ as a generalized space of coupling constants. He then computed the β-functional for $G_{\mu\nu}(x)$ which determines the scale-dependence of the action. He found that this functional takes a geometrical form; to one loop,

$$\left\langle \frac{\delta S}{\delta \sigma} \right\rangle = R_{\mu\nu}(x) \cdot (\sqrt{g}\, g^{ab} \partial_a x^\mu \partial_b x^\nu) + (\text{Polyakov piece}) \ , \tag{130}$$

where $R_{\mu\nu}$ is the curvature computed from $G_{\mu\nu}$. Thus, the more general theory (129) is conformally invariant if $d = 26$ and

$$R_{\mu\nu} = 0 \ . \tag{131}$$

Note that this is equivalent to $R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R = 0$, Einstein's equation for a background metric in empty space. This suggests that, in more general circumstances, the requirement of conformal invariance implies the field equations of the background geometry.

Let us check this hypothesis for the most general set of massless background fields available in the bosonic closed-string theory. The vertex operator for massless states is:

$$\eta_{ij} \partial_z x^i \partial_{\bar{z}} x^j e^{ik\cdot x} \ . \tag{132}$$

This is the 1-particle matrix element of a term in the Lagrangian of the 2-

dimensional field theory which would be of the form

$$\delta S = \int \delta t_{ij}(x)\, \partial_z x^i\, \partial_{\bar{z}} x^j \ . \tag{133}$$

Separating t into symmetric and antisymmetric parts, and making (133) generally covariant in 2 dimensions, this equation becomes

$$\delta S = \int d^2z[\delta H_{\mu\nu}\sqrt{g}\, g^{ab}\partial_a x^\mu \partial_b x^\nu + \delta B_{\mu\nu}\epsilon^{ab}\partial_a x^\mu \partial_b x^\nu] \tag{134}$$

Note that, at the classical level, $T^a_a = \sqrt{g}\, g^{ab}\partial_a x^\mu \partial_b x^\nu$ is traceless; thus, the trace of H does not couple into (134) at this level. Polyakov's calculation suggests that the scalar closed-string state, the trace of t_{ij}, couples for the first time via a counterterm of the form[29] :

$$\int d^2z\sqrt{g}\, R^{(2)}\Phi(x) \ . \tag{135}$$

Combining all of these insights, we can write the complete coupling of the string to massless background fields as

$$\begin{aligned} S \ = \ \frac{1}{4\pi\alpha'}\int d^2z\Big\{&\sqrt{g}\, g^{ab}G_{\mu\nu}(x)\partial_a x^\mu \partial_a x^\nu + \epsilon^{ab}B_{\mu\nu}(x)\partial_a x^\mu \partial_b x^\nu \\ &+ \frac{\alpha'}{4}\sqrt{g}\, R^{(2)}\Phi(x)\Big\} \end{aligned} \tag{136}$$

We must now compute $\left\langle\frac{\delta S}{\delta\sigma}\right\rangle$. In principle, the coefficient of each term in (136) can be changed; however, by dimensional analysis, no new terms can arise. Thus,

$$\left\langle\frac{\delta S}{\delta\sigma}\right\rangle = \beta^G_{\mu\nu}(\sqrt{g}\, g^{ab}\partial_a x^\mu \partial_b x^\nu) + \beta^B_{\mu\nu}\epsilon^{ab}\partial_a x^\mu \partial_b x^\nu + \beta^\Phi\, \frac{1}{4}\sqrt{g}\, R^{(2)} \tag{137}$$

Following ref. 31, let us discuss the leading contributions to each of the coefficients in this equation in the natural perturbation theory of the nonlinear sigma model. This is an expansion in powers of the curvature of the associated manifold; the dimensionless expansion parameter is $k^2 \times \alpha'$, where k is a characteristic momentum of the background field. For the evolution of $G_{\mu\nu}(x)$, one finds

$$\beta^G_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} H^{\lambda\sigma}_{\mu} H_{\nu\lambda\sigma} + 2\nabla_\mu \nabla_\nu \Phi \ , \tag{138}$$

where $H_{\nu\lambda\sigma}$ is the field strength for $B_{\mu\nu}$:

$$H_{\mu\nu\lambda} = 3\nabla_{[\mu} B_{\nu\lambda]} \ . \tag{139}$$

The first term in (138) is Freidan's result quoted above. To understand the second term, notice that the coupling of $B_{\mu\nu}$ in (136) has the form of a Wess-Zumino term added to the 2-dimensional action, and recall Witten's result[34] that the β function of nonlinear sigma model on a group space has a zero for a particular nonzero value of the coefficient of the Wess-Zumino term. The renormalization of the Wess-Zumino term itself is given by

$$\beta^B_{\mu\nu} = \nabla_\lambda H^\lambda{}_{\mu\nu} - 2\nabla_\lambda \Phi \, H^\lambda{}_{\mu\nu} \ . \tag{140}$$

If we set Φ to a constant, this vanishes when $H_{\lambda\mu\nu}$ is covariantly constant. Finally, the next correction to the result (128) is

$$\beta^\Phi = \frac{1}{\alpha'} \left(\frac{d-26}{48\pi^2} \right) + \frac{1}{16\pi^2} \left\{ 4(\nabla\Phi)^2 - 4\nabla^2\Phi - R + \frac{1}{12} H^2 \right\} \ . \tag{141}$$

The second term of β^ϕ can be shown to be independent of x by using $\beta^G_{\mu\nu} = 0$, $\beta^B_{\mu\nu} = 0$, and the Bianchi identities. To recover the conformal invariance of the string, we must insist that all three of the β^A are zero. If we also insist that d remains equal to 26[♮2],

$$\beta^B_{\mu\nu} = 0 \quad \Rightarrow \quad \nabla_\lambda e^{-2\Phi} H^\lambda_{\mu\nu} = 0 \ ,$$

$$\beta^\Phi_{\mu\nu} = 0 \quad \Rightarrow \quad e^{-2\Phi}(R - \frac{1}{12} H^2 + 4(\nabla\Phi)^2) - 4\nabla_\mu(e^{-2\Phi}\nabla^\mu\Phi) = 0 \ ,$$

$$\beta^G_{\mu\nu} + 8\pi^2\beta^\Phi G_{\mu\nu} = 0 \Rightarrow (R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R) = T_{\mu\nu} \ , \tag{142}$$

where $T_{\mu\nu}$ has exactly the form that allows all three equations to follow from the effective action

$$\mathbf{S} = \int d^\alpha x \sqrt{G}\, e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 - \frac{1}{12} H^2 \right) \ . \tag{143}$$

This is a geometrically invariant action principle. It is, as well, covariant under shifts of the scalar field $\Phi(x)$:

$$\Phi \to \Phi + \Lambda \quad \Rightarrow \quad \mathbf{S} \to e^{-2\Lambda}\mathbf{S} \ . \tag{144}$$

$\Phi(x)$ is usually called the dilaton field; (144) suggests that a shift of this field changes the coupling constant of the effective action. If we now take the step of identifying the background fields in (136) with their associated string particles, we can consider this argument to be a derivation of the Einstein-Hilbert action for gravity. The authors of refs. 29–31 have derived a coupled Einstein-Yang-Mills system by extending this construction to the heterotic string.

♮2 The analogous statement for the superstring is required to maintain supersymmetry.

The power of this background-field technique seems very puzzling, however, when one realizes that this technique yields only consistency conditions for string dynamics rather than the actual equations of motion of the string theory. But what are these more fundamental equations? To explore this question, we must go back to the beginning and set off again in a different direction. A particular aspect of this problem which I will explore is the form of the underlying gauge invariance which gives rise to gauge bosons and gravitons as string modes. As yet, no formulation of the equations of string theory is known which is fully gauge-invariant and describes the theory completely. However, following a very beautiful covariant-gauge second quantization of the string displayed by Siegel[35], several authors[36,37] identified the gauge invariances of the linearized string theory and constructed gauge-invariant free-string actions. Let me now explain briefly what form these invariances take.

We might begin by trying to write a second-quantized string action with as much reparametrization invariance as possible. To do this, let us consider again the Fourier decomposition of a single open string in an orthonormal (but not necessarily light-cone) gauge:

$$x^\mu(\sigma) = x^\mu + \sum_{n>0} \frac{2}{\sqrt{n}} X_n^\mu \cos n\sigma \qquad 0 \le \sigma \le \pi \ . \tag{145}$$

The conjugate momenta to $x(\sigma)$ are given by

$$p^\mu(\sigma) = \frac{1}{\pi} \left(p^\mu + \sum_{n>0} \sqrt{n}\, P_n^\mu \cos n\sigma \right) \ . \tag{146}$$

Define α_n^μ by

$$X_n = \frac{i}{2\sqrt{n}} \left(\alpha_n - \alpha_{-n}\right) \ , \quad P_n = \frac{1}{\sqrt{n}} \left(\alpha_n + \alpha_{-n}\right) \ , \quad p = \alpha_0 \ ; \tag{147}$$

then the α's are raising and lowering operators for string eigenmodes:

$$[\alpha_n^\mu, \alpha_m^\nu] = n\ \delta(n+m)\eta^{\mu\nu} \ , \tag{148}$$

and $p(\sigma)$ and $x'(\sigma)$ are simple functions of the α's:

$$\begin{aligned} p^\mu(\sigma) &= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \alpha_n^\mu \cos n\sigma \\ \frac{d}{d\sigma}\, x^\mu(\sigma) = x'^\mu(\sigma) &= \sum_n \alpha_n^\mu\, i \sin n\sigma \ . \end{aligned} \tag{149}$$

It is useful to combine

$$\pi p \pm x' = \sum_{n=-\infty}^{\infty} \alpha_n e^{\mp in\sigma} \ . \tag{150}$$

Then we can write

$$\frac{1}{2}\, (\pi p \pm x')^2 = \sum_{n=-\infty}^{\infty} L_n e^{\mp in\sigma} \ , \tag{151}$$

where I have defined

$$L_n = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\ \alpha_{n+m}^\mu \alpha_{-m}^\mu\ : \ . \tag{152}$$

The L_n are called the Virasoro operators[38] . To understand their role, recast

$$\frac{1}{2}\, (\pi P \pm x')^2 = \frac{1}{2}\, \left[(\pi P)^2 + (x')^2\right] \pm \pi\left[P \cdot x'\right] \equiv \pi(\mathcal{H} \pm \mathcal{P}) \ ,$$

where $\mathcal{H}$ and $\mathcal{P}$ are Hamiltonian and momentum densities on the two-dimensional surface. Thus, the L_n are the generators of τ and σ reparametrizations. The operator L_0 contains the excitation counting operator:

$$L_0 = \frac{p^2}{2} + \sum_{n>0} \alpha_{-n} \cdot \alpha_n \; ; \tag{153}$$

thus,

$$2(L_0 - 1) = p^2 + 2\left(\sum_{n>0} \alpha_{-n}\alpha_n - 1\right) = (p^2 + \mathcal{M}^2) \; , \tag{154}$$

where $\mathcal{M}$ is mass operator for string eigenstates (with eigenvalues $2(n-1)$). This means that the operator $2(L_0 - 1)$ is a reasonable first guess for the free string Lagrangian. The other L_n's generate non-constant conformal transformation of the (σ, τ) space. A fully conformally invariant wavefunction would satisfy:

$$\delta \left|\Phi\right\rangle = i \sum_{n=-\infty}^{\infty} b_n L_{-n} \left|\Phi\right\rangle = 0 \; , \tag{155}$$

with $b_n = b^*_{-n}$.

Let us now attempt to construct a field theory of strings with the L_n as symmetry generators. In the standard fashion by which one constructs a second-quantized field theory from the quantum mechanics of a single particle, we promote the single-string wavefunction $\left|\Phi\right\rangle$ to a classical string field $\Phi[x(\sigma)]$. This field is a functional of the string coordinates. We seek a free-field action of the form

$$S \; = \; -\frac{1}{2} \int \mathcal{D}X(\sigma) \; \Phi[x(\sigma)] \; K\Phi[x(\sigma)] \; \equiv \; -\frac{1}{2} \left(\Phi \; , K\Phi\right) \; , \tag{156}$$

where K is a kinetic-energy operator to be determined. The condition that (155) is a symmetry of this classical field theory takes the form:

$$\delta S = i \sum_{n=-\infty}^{\infty} b_n (\Phi, [K, L_{-n}]\Phi) = 0 \; . \tag{157}$$

It is not at all difficult to construct a K which commutes with L_0, but the problem of finding a K which commutes with all of the L's is more subtle. It was solved, as a problem in mathematics, by Feigin and Fuks in their abstract study of the algebra of the Virasoro operators[39]. Let us now discuss their construction.

We begin by defining a subspace of the full space of functionals $\Phi[x(\sigma)]$ which I will call the subspace of level 0 states. These are the states which satisfy the condition

$$L_n \Phi_0 = 0 , \quad \text{for } n > 0 . \tag{158}$$

These states are often called simply ***physical states.*** On this subspace, the conformal motion (155) can be rewritten

$$\delta \Phi_0 = i \sum_{n \geq 0} b_n L_{-n} \Phi_0 . \tag{159}$$

If Φ is restricted to level 0, and we use the modified transformation law (159), the variation of S becomes

$$\delta(\Phi_0, K\Phi_0) \; = \; ib_0(\Phi_0, [K, L_0]\Phi_0) \; + \; i \sum_{n>0} b_n(\Phi_0, KL_{-n}\Phi_0) + c.c. \tag{160}$$

This variation vanishes if K commutes with L_0 and if

$$K\,L_{-n} \; = \; 0, \qquad n > 0 . \tag{161}$$

This second condition is equivalent to the condition that K contains a projector onto the subspace of level 0 states. To explain this point, let me make a small digression.

The operators L_n obey a slightly nontrivial algebra, the Virasoro algebra

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{d}{12} n(n^2-1)\, \delta(n+m) \ . \tag{162}$$

The homogeneous term reflects the algebra of conformal transformations. The c-number term (or, central charge) is a quantum-mechanical anomaly which makes it inconsistent to demand that a state be fully conformally invariant, in the sense that it is annihilated by all of the L_n. This term may be easily obtained by regulating the definition (152):

$$L_n = \lim_{\Lambda \to \infty} \frac{1}{2} \sum_{m=-\Lambda}^{\Lambda} \ : \alpha_{n+m} \alpha_{-m} \ : ; \tag{163}$$

after commuting two of the operators (163), one finds, first, that the result needs to be normal-ordered and, secondly, that the new c-number terms which arise from normal-ordering can be grouped together only after a shift of the summation variable, so that they cancel incompletely. The appearance of this term is a direct reflection of the conformal anomaly alluded to in our discussion of the bosonic string; the precise connection has been explained by Friedan[40] .

The Virasoro algebra has an infinite number of generators. However, since

$$L_0 \, L_{-n} \ = \ L_{-n} \, (L_0 + n) \, , \tag{164}$$

L_{-n} rasies the mass level of the string by n units and so a finite number of generators suffice to describe the string at finite levels of excitation. We can thus build up the representations of the Virasoro algebra level by level in the following way: Start from the states at level 0, which satisfy $L_n \Phi_0 = 0$ $(n > 0)$.

Define the states at level 1 as those formed by applying L_{-1} to a level 0 state. Define the states at level n as the states formed by applying to level 0 states any product of L_{-m} operators whose indices sum to n; for example, the states at level 2 are formed by applying L_{-2} and L_{-1}^2 to level 0 states. The tower of states constructed from a particular Φ_0 is called a Verma module. States at different levels of a Verma module are orthogonal; for example,

$$(\Phi_1, \Phi_0) = (L_{-1}\Phi_0', \Phi_0) = (\Phi_0', L_1\Phi_0) = 0. \tag{165}$$

Except at discrete values of the parameter $\alpha_0^2 = p^2$ (none of which occur in the Euclidean region beyond the string ground state mass: $p^2 > 2$), the decomposition into levels is an orthogonal decomposition of the space of functionals $\Phi[x(\sigma)]$. Thus, since $L_{-n}\Phi$ is at least of level n, it must be annihilated by the projector onto level 0.

This digression has actually given us the information we need to construct a K_R which commutes with all of the L_{-n}. Such an operator must necessarily take the same value on all states in a Verma module. Thus, define K_R conveniently on the level 0 states—following (154), we should set $K_R = 2(L_0 - 1)$ there—and define K_R on the rest of the Verma module to be equal to its value on the level 0 state from which the module is generated.

The analysis leading to this conclusion gives as a byproduct another possible expression for K, the simple form

$$K \;=\; 2(L_0 - 1)\cdot P\ , \tag{166}$$

where P is the projector onto level 0. This action does not have the full reparametrization symmetry (155), though it does preserve the subset (159). Its advantage is that it possess an additional, enormous group of gauge symmetries, symmetries which are, in fact, local on the space of strings. Since P annihilates L_{-n}, any motion of the form

$$\delta\Phi[x(\sigma)] \;=\; L_{-n}\Psi_n[x(\sigma)] \tag{167}$$

is a symmetry of (156) if this expression is chosen for K. A transformation of the form of (167) includes many more symmetries than a global or a local gauge transformation, since the gauge parameter is local on the next higher space. It is appropriate to call this a *chordal* gauge transformation.

To understand more precisely the content of the symmetry (167), it is useful to expand Φ and each Ψ_n in eigenstates of the string mass operator. Let $\Phi^{(0)}$ be the string ground state, defined, as before, by the condition $\alpha_n\Phi^{(0)} = 0$ $(n > 0)$. Note that $\Phi^{(0)}$ does not include x, the zero mode of $x(\sigma)$. We may then expand

$$\Phi[x(\sigma)] = [\phi(x) - i\,A_\mu(x)\alpha^\mu_{-1} - \frac{1}{2}\,h_{\mu\nu}(x)\,\alpha^\mu_{-1}\alpha^\nu_{-1} - i\,V_\mu(x)\alpha^\mu_{-2} + \ldots]\,\Phi^{(0)}\,; \tag{168}$$

the dependence on x resides in the coefficient functions, which become local fields of increasing spin. Acting on an expression of this structure with $L_{-1} = \alpha_0\cdot\alpha_{-1} + \alpha_{-2}\cdot\alpha_1 + \ldots = p\cdot\alpha_{-1} + \alpha_{-2}\cdot\alpha_1 + \ldots$, one finds

$$L_{-1}\Psi_1 = [-i\partial^\mu\phi_1\alpha_{-1} - \partial^\mu A_1^\nu\alpha^\mu_{-1}\alpha^\nu_{-1} - iA_1^\mu\alpha^\mu_{-2} + \ldots]\Phi^{(0)}\,. \tag{169}$$

Similarly, the gauge motion involving L_{-2} is given by

$$L_{-2}\Psi_2 = [-i\partial^\mu\phi_2\alpha_{-2} - \frac{1}{2}\,\phi_2\alpha_{-1}\cdot\alpha_{-1} + \ldots]\Phi^{(0)}\,. \tag{170}$$

From these equations, one can read off the gauge transformation laws for the fields at the lowest three mass levels: The ground state field is gauge-invariant:

$$\delta\phi(x) \;=\; 0\,. \tag{171}$$

The vector field at the first mass level transforms as

$$\delta A_\mu(x) \;=\; \partial_\mu\phi_1(x)\;, \tag{172}$$

which is just the linearized tranformation law of a gauge field. The fields at the next level have their own gauge invariance:

$$\delta h_{\mu\nu} \;=\; 2\partial_{\{\mu}A_{1\nu\}} - \eta_{\mu\nu}\phi_2\,, \quad \delta V_\mu \;=\; A_{1\mu} + \partial_\mu\phi_2 \tag{173}$$

The choice (166) for the string kinetic energy operator thus leads, at the level of free field theory, to exactly the gauge symmetry we had orginally sought, plus enormously more gauge symmetry than we might have suspected.

Let us, then, adopt the choice (166) and work out the action (156) for the lowest mass levels. Up to the first excited level, one can easily see that P takes the form

$$P = 1 - L_{-1}\frac{1}{2L_0}L_1 + \cdots\;, \tag{174}$$

since this choice reduces to 1 on level 0 states and annihilates level 1 states. Then, using (164),

$$K = 2(L_0-1)P = 2(L_0-1) - L_{-1}L_1 + \cdots \tag{175}$$

Apply this K to Φ given by (168) and examine the result mass level by mass level. At the lowest level,

$$K\Phi|_0 = (p^2 + m^2)\phi \ , \tag{176}$$

where $m^2 = -2$. At the first excited level

$$K\Phi|_1 = (p^2\eta^{\mu\nu} - p^\mu p^\nu)A_\nu \ ; \tag{177}$$

this is equivalent to

$$-\frac{1}{2}(\Phi, K\Phi)|_1 = \int -\frac{1}{4}(F_{\mu\nu})^2 \ , \tag{178}$$

the result required by gauge-invariance. On high mass levels, one finds higher-spin gauge invariant theories. These theories are actually nonlocal in the present formulation, but they can be made local by introducing Stueckelberg compensating fields. The simplest example of this phenomenon arises at the massless level of the closed string theory; let us, then, turn to that theory.

We saw in our earlier discussion that closed strings have a doubled spectrum of normal modes, corresponding to left- and right-moving waves. Excitations of these modes are created and destroyed by two commuting sets of operators $\{\alpha_n\}$, $\{\overline{\alpha}_n\}$, each of which obeys the algebra (148). Applying (152), to each set, we can form two sets of operators $\{L_n\}$ and $\{\overline{L}_n\}$ such that the two sets are mutually commuting and the elements of each set have the commutation relations of the Virasoro algebra. After a Euclidean continuation $(\tau \mp \sigma) \to i(\tau \pm i\sigma)$, the L_n generate mappings of the z plane which are analytic functions of z, and the $\overline{L}_n$ generate mappings which are analytic functions of $\overline{z}$. The two algebras share their zero mode:

$$\alpha_0^\mu = \overline{\alpha}_0^\mu = \frac{1}{2}p^\mu \tag{179}$$

but the two types of excitations contribute independently to the mass of the string state. The generalization of (154) to this system is then

$$(p^2 + \mathcal{M}^2) = 4\big[(L_0 - 1) + (\overline{L}_0 - 1)\big] \ . \tag{180}$$

The generalization of our gauge-invariant action is constructed by adding to this expression the projector onto level 0 in both of the commuting reparametrization algebras:

$$S \ = \ -\frac{1}{2}(\Phi|4\big[(L_0 - 1) + (\overline{L}_0 - 1)\big]P\overline{P}|\Phi) \ . \tag{181}$$

The chordal gauge symmetry of this Lagrangian is

$$\delta\Phi \ = \ L_{-n}\Psi_n \ + \ \overline{L}_{-n}\overline{\Psi}_n \tag{182}$$

Specializing to the zero-mass states with total $P^+ = 0$, this reads:

$$\begin{aligned} &\delta\Big[t^{\mu\nu}\alpha^\mu_{-1}\overline{\alpha}^\nu_{-1}\Big]\Phi^{(0)} \\ &\quad = \ L_{-1}\Big[\xi^\mu\overline{\alpha}^\mu_{-1}\Big]\Phi^{(0)} \ + \ \overline{L}_{-1}\Big[\overline{\xi}^\mu\alpha^\mu_{-1}\Big]\Phi^{(0)} \ , \end{aligned} \tag{183}$$

or, more simply,

$$\delta t^{\mu\nu} \ = \ \partial^\mu\xi^\nu \ + \ \partial^\nu\overline{\xi}^\mu \ . \tag{184}$$

To understand the content of this transformation, note that $t^{\mu\nu}$ is a tensor of general symmetry which can be decomposed into its symmetric and antisymmetric parts. Label the antisymmetric part of this tensor as $b^{\mu\nu}$; this field has the transformation law

$$\delta b^{\mu\nu} = \partial^{[\mu}(\xi - \overline{\xi})^{\nu]} \,. \tag{185}$$

This is the natural gauge transformation of a 2-form field; it leaves invariant the field strength

$$H_{\mu\nu\lambda} = \partial_{[\mu} b_{\nu\lambda]} \,. \tag{186}$$

The symmetric part, $h^{\mu\nu}$, tranforms according to

$$\delta h^{\mu\nu} = \partial^{\{\mu}(\xi + \overline{\xi})^{\nu\}} \,. \tag{187}$$

If we identify the spin-2 field h with the linearized gravitational field, (187) is just a linearized general coordinate transformation.

It is straightforward to work out the actions for these fields by evaluating (181) using the formula

$$P\overline{P} = \left[1 - L_{-1}\frac{1}{2L_0}L_1\right]\left[1 - \bar{L}_{-1}\frac{1}{2\bar{L}_0}\bar{L}_1\right] + \ldots \tag{188}$$

The term in S quadratic in $b_{\mu\nu}$ turns out to be just

$$\int \left(-\frac{1}{12}\,(H_{\mu\nu\lambda})^2\right) \,. \tag{189}$$

The term quadratic in $h_{\mu\nu}$ is slightly more tangled. It is convenient to add and subtract a term to bring it into the form:

$$
\begin{aligned}
S_h = \frac{1}{8}\int d^dx \Bigg[h_{\mu\nu}\Big\{ &-\partial^2\Big(\eta^{\mu\lambda}-\frac{\partial^\mu\partial^\lambda}{\partial^2}\Big)\Big(\eta^{\nu\sigma}-\frac{\partial^\nu\partial^\sigma}{\partial^2}\Big)+(\lambda\leftrightarrow\sigma) \\
&-2(-\partial^2)\Big(\eta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\partial^2}\Big)\Big(\eta^{\lambda\sigma}-\frac{\partial^\lambda\partial^\sigma}{\partial^2}\Big)\Big\} h_{\lambda\sigma} \\
+2h_{\mu\nu}\Big\{\Big(\eta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\partial^2}\Big)&(-\partial^2)\Big(\eta^{\lambda\sigma}-\frac{\partial^\lambda\partial^\sigma}{\partial^2}\Big)\Big\} h_{\lambda\sigma}\Bigg].
\end{aligned}
\tag{190}
$$

The first two lines of this expression may be recognized as the quadratic term in the expansion of the Einstein-Hilbert action

$$\int \sqrt{-g}R \tag{191}$$

obtained by replacing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The last line can be written, using

$$R = \partial^\mu\partial^\nu h_{\mu\nu} - \partial^2 h^\mu_\mu + \dots, \tag{192}$$

as a nonlocal interaction of curvatures:

$$\int R\Big(\frac{1}{-\partial^2}\Big)R\ . \tag{193}$$

This interaction can be made local by introducing an additional scalar field $\varphi(x)$; the expression

$$\int \big((\partial_\mu\varphi)^2 - 2\varphi R\big) \tag{194}$$

reduces to (193)when φ is eliminated using its equation of motion.

We have now obtained a linearized action involving exactly the field content expected at the massless level of the closed string—an antisymmetric tensor field, a graviton, and a scalar dilaton. The gauge invariances of this model, though as yet present only at the linearized level, follow natural from a principle based in string geometry. As a bonus, the final action we have obtained, incorporating (194), is precisely a linearization of the effective action (143) obtained by the background field method.

To complete this discussion, let me add two notes. First, recall our observations here that the projected action is nonlocal, that this nonlocality can be removed by adding to the model additional fields, and that these new fields are necessary to complete the particle content of the theory. This turns out to the generic situation at higher mass levels. Complete local actions for the bosonic string have recently been constructed by several groups [37,41–43] . Second, the whole discussion I have given here can be extended to the superstring; there, linearized local supersymmetry also appears among the chordal gauge motions.

Presumably, the two partial answers we have found to the problem of formulating the gauge invariance of string theories connect to one another and to some more geometrical foundation. At the moment, no one knows what that connection is. I expect, though, that we will soon see progress on this question and soon deepen our understanding of the space of strings and the nature of field theories on this space. I hope that this first look at the theory of strings might have provided a useful step toward the imposing territory we have yet to explore.

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REFERENCES

1. Scherk J. and Schwarz, J. H., *Nucl. Phys.* _B81_, 118 (1974).
2. Green, M. B. and Schwarz, J. H., *Phys. Lett.* _149B_, 117 (1984), *Nucl. Phys.* _B255_, 93 (1985).
3. Witten, E., *Phys. Lett.* _156B_, 55 (1984).
4. Dine, M., Rohm, R., Seiberg, N., and Witten, E., *Phys. Lett.* _156B_, 55 (1985).
5. Candelas, P., Horowitz, G., Strominger, A., and Witten, E., *Nucl. Phys.* _B258_, 46 (1985).
6. Mandelstam, S., *Phys. Repts.* _13C_, 259 (1974).
7. Scherk, J., *Rev. Mod. Phys.* _47_, 123 (1975).
8. Schwarz, J. H., *Phys. Repts.* _89_, 223 (1982).
9. *Proceedings of the Workshop on Unified String Theories (Santa Barbara, 1985)*, Green, M. B. and Gross, D. J., eds. (World Scientific, 1985).
10. Mandelstam, S., *Nucl. Phys.* _B64_, 205 (1973).
11. Green, M. B. and Schwarz, J. H., *Nucl. Phys.* _B181_, 502 (1981), *Phys. Lett.* _109B_, 444 (1982).
12. Nambu, Y., lectures at the Copenhagen Symposium, 1970.
13. Brink, L., Di Vecchia, P., and Howe, P., *Phys. Lett.* _65B_, 471 (1976).
14. Goddard, P., Goldstone, J., Rebbi, C., and Thorn, C. B., *Nucl. Phys.* _B56_, 109 (1973).
15. Koba, Z., and Nielsen, H. B., *Nucl. Phys.* _B10_, 633 (1969).
16. Brink, L. and Nielsen, H. B., *Phys. Lett.* _45B_, 332 (1973).

17. Veneziano, G., *Nuov. Cim.* _57A_, 190 (1968).

18. Neveu, A. and Scherk, J., *Nucl. Phys.* _B36_, 155 (1972).

19. Green, M. B., and Schwarz, J. H., *Nucl. Phys.* _B218_, 43 (1983).

20. Ramond, P., *Phys. Rev.* _D3_, 2415 (1971).

21. Neveu, A. and Schwarz, J. H., *Nucl. Phys.* _B31_, 86 (1971).

22. Gliozzi, F., Scherk, J., and Olive, D.,*Phys. Lett.* _65B_, 282 (1976), *Nucl. Phys.* _B22_, 253 (1977).

23. van Nieuwenhuizen, in *Relativity, Groups, and Topology II (Les Houches, 1983)*, DeWitt, B. S. and Stora, R., eds. (North-Holland, 1984).

24. Cremmer, E., Julia, B., and Scherk, J., *Phys. Lett.* _76B_, *409 (1978).*

25. *Alvarez-Gaumé, L. and Witten, E., Nucl. Phys.* _B234_, 269 (1984).

26. Gross, D. J., Harvey, J. A., Martinec, E. J., and Rohm, R.,*Phys. Rev. Lett.* _54_, 502 (1985), *Nucl. Phys.* _B 256_, 253 (1985), and to appear.

27. Gürsey, F., Ramond, P., and Sikivie, P., *Phys. Lett.* _60B_, 177 (1976).

28. Bars, I. and Günaydin, M., *Phys. Rev. Lett.* _45_, 859 (1980).

29. Fradkin, E. S. and Tseytlin, A. A., *Phys. Lett.* _158B_, 316 (1985), _160B_, 69 (1985).

30. Sen, A., *Phys. Rev.* _D32_, 2102 (1985), *Phys. Rev. Lett.* _55_, 1846 (1985).

31. Callan, C. G., Martinec, E. J., Perry, M., and Friedan, D., Princeton preprint PRINT-85-0734 (1985).

32. Polyakov, A. M., *Phys. Lett.* _103B_, 207 (1981)

33. Friedan, D., *Phys. Rev. Lett.* _45_, *1057 (1980), Ann. Phys. (N.Y.)* _163_, 318 (1985).

34. Witten, E., *Comm. Math. Phys.* 92, 455 (1984).

35. Siegel, W., *Phys. Lett.* 149B 157, 162 (1984), 151B 391, 396 (1985).

36. Kaku, M., *Nucl. Phys.* B, to appear.

37. Banks, T. and Peskin, M. E., *Nucl. Phys.* B, to appear.

38. Virasoro, M., *Phys. Rev.* D1, 2933 (1970).

39. Feìgin, B. L. and Fuks, D. B., *Dokl. Akad. Nauk USSR* 269, no. 5 (1983) [*Sov. Math. Dokl.* 27, 465 (1983)].

40. Friedan, D., in *Recent Advances in Field Theory and Statistical Mechanics (Les Houches, 1982)*, J.-B. Zuber and R. Stora, eds. (North-Holland, 1984).

41. Siegel, W. and Zwiebach, B., Berkeley preprint UCB-PTH-85/30 (1985).

42. Itoh, K., Kugo, T., Kunitomo, H., and Ooguri, H., Kyoto preprint KUNS-800 HE(TH) 85/04 (1985).

43. Neveu, A. and West, P. C., CERN preprint TH.4200/85 (1985); Neveu, A., Nicolai, H., and West, P. C., CERN preprint TH.4233/85 (1985); Neveu, A., Schwarz, J., and West, P. C., CERN preprint TH.4248/85 (1985).