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CONVEXITY OF THE QUARKONIUM POTENTIAL*

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ABSTRACT

The heavy quark-antiquark potential is shown to be a monotone non-decreasing and convex function of the separation. This property holds independent of the gauge group and the details of the matter sector.

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Some interesting theorems relating the order of energy levels of quarkonium to convexity properties of the heavy quark potential, $V(R)$, were recently derived by Baumgartner, Grosse and Martin.[1] In particular these authors showed that if V is a convex function of R^2, then

$$E_{n,\ell} \leqq E_{n-1,\ell+2} \ , \tag{1}$$

where the energy levels are labelled by the angular momentum ℓ and the number of nodes of the wavefunction n. In this note I would like to point out that convexity of V as a function of R^α, for all $\alpha \geq 1$, is a general property of gauge theories, independent of the choice of gauge group and the details of the matter (light fermion and scalar) sectors.

I should state right at the outset that this paper makes no claim to originality. The convexity of V as a function of R^α means that

$$\frac{d}{d(R^\alpha)} \frac{d}{d(R^\alpha)} V = \frac{R^{2-2\alpha}}{\alpha^2} \frac{d^2V}{dR^2} - \frac{(\alpha-1)}{\alpha^2} R^{1-2\alpha} \frac{dV}{dR} \leqq 0$$

for all $\alpha \geq 1$. This is equivalent to the combined statements:

$$\frac{d^2V}{dR^2} \leqq 0 \tag{2a}$$

$$\frac{dV}{dR} > 0 \ , \tag{2b}$$

i.e. that the quark-antiquark force is everywhere attractive and a monotone non-increasing function of their separation. That this is indeed so is known to physicists who have worked on rigorous aspects of lattice gauge theories,[2] but given the simplicity of its proof, it has been considered unworthy of particular emphasis. It has thus escaped the attention of a wider audience, which I believe it actually deserves since:

(*a*) combined with the result of Baumgartner et al., it elevates the ordering (1) of the levels of charmonium (ψ), bottomium (Y), toponium, etc., to a most general rigorous theorem of quantum field theory, and

(*b*) it provides a subtle and apparently forgotten consistency check for Monte-Carlo simulations of quarkonia.

In this note I will thus present the simple derivation of this result.

Let us start with a pure gauge theory on a hypercubic lattice[3] with sites $\underset{\sim}{s} = (s^1, s^2, s^3, s^4) \in \boldsymbol{Z}^4$. The role of the lattice is only technical, and sets the stage for a rigorous proof; the inclusion of dynamical matter will be discussed later. The fields $U(b)$ are as usual defined on the directed bonds $b = (\underset{\sim}{s}, \underset{\sim}{s}')$ of the lattice, and take values in the gauge group G. A group element can more generally be assigned to every directed path $\omega = (\underset{\sim}{s}_0 \to \underset{\sim}{s}_1 \to \cdots \to \underset{\sim}{s}_f)$ on the lattice

$$U(\omega) = U(\underset{\sim}{s}_0, \underset{\sim}{s}_1)\, U(\underset{\sim}{s}_1, \underset{\sim}{s}_2) \cdots U(\underset{\sim}{s}_{f-1}, \underset{\sim}{s}_f) \ .$$

Traversing the same path in the opposite direction induces hermitean conjugation

$$U(-\omega) = U^+(\omega) \ , \tag{3}$$

with $-\omega = (\underset{\sim}{s}_f \to \underset{\sim}{s}_{f-1} \to \cdots \to \underset{\sim}{s}_1 \to \underset{\sim}{s}_0)$. For the action we take

$$S = \frac{1}{g^2} \sum_{\text{plaquettes p}} \text{Re tr}\, U(p) \tag{4}$$

with the trace in, say, the fundamental representation, but any other one-plaquette action would do. Finally, the heavy quark-antiquark potential can be extracted from the expectation value of long rectangular Wilson loops[3] W, with sides of length T and R (see figure 1a) as

$$V(R) = \lim_{T\to\infty} \left(-\frac{1}{T} \log \langle \mathrm{tr}\, U(W) \rangle + \text{constant} \right) , \tag{5}$$

where

$$\langle \mathrm{tr}\, U(W) \rangle = \int \prod_b [dU(b)]\ e^{-S} \mathrm{tr}\, U(W) \Big/ \int \prod_b [dU(b)]\ e^{-S}$$

and $[dU(b)]$ is the invariant group measure.[♯1] We should emphasize that the potential, so defined, is the vacuum energy in the presence of infinitely heavy external $q\bar{q}$ sources, and does not include spin-dependent interactions.

The action (4) enjoys a remarkable property, reflection positivity, which guarantees the existence of a positive-metric Hilbert space, and of a transfer (time-evolution) matrix.[4] Indeed, take any 3-dimensional hyperplane normal to a principal axis of the lattice, for instance the hyperplane $s^1 = 0$, and denote collectively the set of sites, links, plaquettes, etc., that lie above, on or below this hyperplane by L_+, L_0 or L_- respectively. Define a reflection θ on all functionals of bond variables

$$\theta f(U(b)) = f^*(U(\theta b)) \tag{6a}$$

♯1 Strictly speaking, the physical potential, is obtained by taking the appropriate scaling limit $g^2(a) \to 0$ as the lattice spacing a shrinks to zero. Since, however, we will establish convexity for any value of g^2, it will also hold in this limit.

where the reflection on sites, bonds, paths, etc. is, of course:

$$\theta \underset{\sim}{s} = \theta(s^1, s^2, s^3, s^4) = (-s^1, s^2, s^3, s^4)$$

$$\theta b = \theta(\underset{\sim}{s}, \underset{\sim}{s}') = (\theta \underset{\sim}{s}, \theta \underset{\sim}{s}') \tag{6b}$$

$$\theta \omega = \theta(\underset{\sim}{s}_0 \rightarrow \underset{\sim}{s}_1 \rightarrow \cdots \rightarrow \underset{\sim}{s}_f) = (\theta \underset{\sim}{s}_0 \rightarrow \theta \underset{\sim}{s}_1 \rightarrow \cdots \rightarrow \theta \underset{\sim}{s}_f) \ .$$

Then for all functionals f that only depend on the bond variables in $L_+ \cup L_0$ we have

$$\begin{aligned}
\langle f \theta f \rangle = Z^{-1} \int \prod_{b \in L_0} [dU(b)] \exp\left(-\frac{1}{g^2} \sum_{p \in L_0} \operatorname{tr} U(p)\right) \\
\cdot \int \prod_{b \in L_+} [dU(b)]\, f(U(b)) \exp\left(-\frac{1}{g^2} \sum_{p \in L_+} \operatorname{tr} U(p)\right) \\
\cdot \int \prod_{b \in L_-} [dU(b)]\, f^*(U(\theta b)) \exp\left(-\frac{1}{g^2} \sum_{p \in L_-} \operatorname{tr} U(p)\right) \\
= Z^{-1} \int \prod_{b \in L_0} [dU(b)] \exp\left(-\frac{1}{g^2} \sum_{p \in L_0} \operatorname{tr} U(p)\right) \\
\cdot \left| \int \prod_{b \in L_+} [dU(b)]\, f(U(b)) \exp\left(-\frac{1}{g^2} \sum_{p \in L_+} \operatorname{tr} U(p)\right) \right|^2 \geqq 0 \ ,
\end{aligned}$$

which implies the Schwarz-type of inequality

$$\langle f_1 \theta f_2 \rangle^2 \leqq \langle f_1 \theta f_1 \rangle \cdot \langle f_2 \theta f_2 \rangle \ . \tag{7}$$

Using this inequality for a reflection about a hyperplane parallel to the (long) time axis, and normal to the plane of the Wilson loop of figure (1a), and recalling eqs. (3) and (6), we obtain

$$\begin{aligned}\langle \operatorname{tr} U(W)\rangle &= \sum_{i,j} \left\langle U(W_1)_{ij}\theta U(W_2)_{ij}\right\rangle \\ &\leqq \sum_{ij} \left\langle U(W_1)_{ij}\theta U(W_1)_{ij}\right\rangle^{1/2} \left\langle U(W_2)_{ij}\theta U(W_2)_{ij}\right\rangle^{1/2} \\ &\leqq \langle \operatorname{tr}(U(W_1)U(-\theta W_1))\rangle^{1/2} \langle \operatorname{tr}(U(W_2)U(-\theta W_2))\rangle^{1/2} \ .\end{aligned} \tag{8}$$

Here i, j are indices in the fundamental representation of the group G, the paths W_1 and W_2 are defined in figure (1b), and the last step is the conventional Schwarz inequality. From the definition of the heavy-quark potential, Eq. (5), we then deduce immediately for all $0 < r < R$ that

$$V(R) \geqq \tfrac{1}{2} V(R-r) + \tfrac{1}{2} V(R+r) \ ,$$

i.e. V is indeed a convex function of R.

It remains to show that V is also monotone non-decreasing. In view of its convexity, we need only prove this asymptotically, i.e. show that no finite repulsive force can survive at infinite separation. But this follows immediately from the fact, established by Simon and Yaffe,[5] that large Wilson loops can be bounded from above by a perimeter-law decaying exponential, so that the potential is bounded from below by a constant.

Let us finally discuss the inclusion of dynamical matter. Adding light fermions will not destroy convexity, even though it drastically modifies the shape of the potential (in particular one looses heavy-quark confinement due to screening). The reason is that a gauge theory with light fermions is still reflection positive,[♯2] and reflection positivity was the only ingredient in our proof of convexity. The same is true in the presence of (Higgs) scalars $\phi(\underset{\sim}{s})$, except that the definition of the heavy quark potential, Eq. (5), should now be modified to take into account the direct Yukawa couplings of the scalars to the external sources:

$$V(R) = \lim_{T\to\infty} \left(-\frac{1}{T} \log \left\langle \operatorname{tr} U(W) \cdot \exp\left(\lambda \sum_{\underset{\sim}{s} \in W} \phi(\underset{\sim}{s}) \right) \right\rangle + \text{constant} \right) ,$$

where λ is the Yukawa coupling constant and the summation runs over all lattice sites on the Wilson loop W. The proof of convexity then goes through as before.

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♮2 Provided one uses only nearest-neighbour fermion interactions.[4] . By universality this is, presumably, not an essential restriction.

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FIGURE CAPTIONS

1. (a) The large Wilson loop W, with sides of length T and R. The dotted line is its intersection with the reflection-hyperplane.

   (b) The paths W_1 and W_2, going from A to B, used in inequality (8), and their reflections. Note that W is the combination of W_1 and $-\theta W_2$.

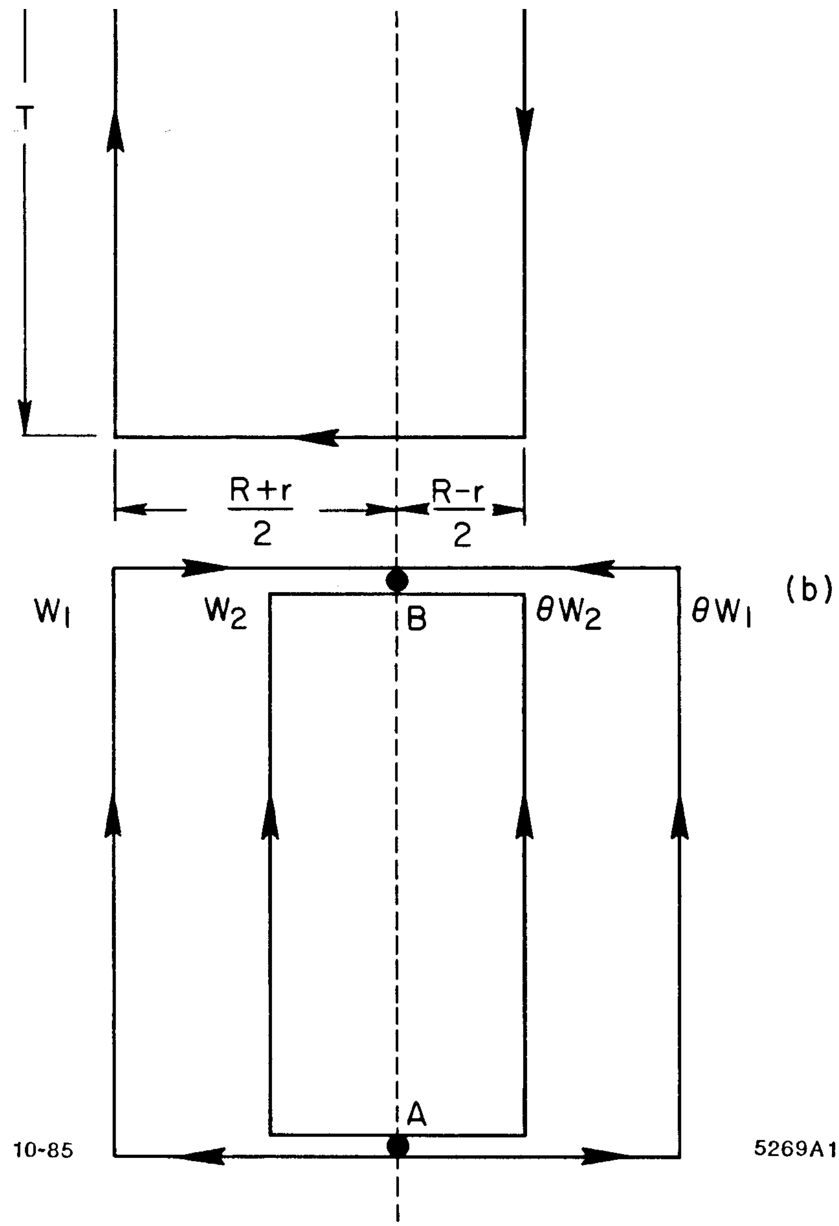

Fig. 1