Local gauge and Lorentz invariance of the heterotic string theory[*]

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ABSTRACT

The Hull-Witten proof of the local gauge and Lorentz invariance of the σ-model describing the propagation of the heterotic string in arbitrary background field is extended to higher orders in α'. The modification of the transformation laws of the antisymmetric tensor field under these symmetries is discussed. Finally we point out the existence of an anomaly in the naive $N = \frac{1}{2}$ supersymmetry transformation, and show that it is cancelled by the same counterterms which restore local Lorentz and gauge invariance of the σ-model.

It has recently been conjectured[1-4] that the classical equations of motion of the massless fields in string theories may be interpreted as fixed point equations of the appropriate σ-model. As a result we expect the symmetries of the classical equations of motion to be reflected in the σ-model, and vice-versa. For example, in the closed bosonic string theory, and the type II superstring theory, the general coordinate and local Lorentz invariance of the classical equations of motion are consequences of the invariance of the corresponding non-linear σ-models under reparametrization of the internal manifold[5]. The existence of such symmetries, although expected, is hard to prove otherwise. Usually the effective action involving the massless fields is constructed from the scattering amplitude involving the massless fields, and the rule for constructing these scattering amplitudes do not exhibit any general coordinate or local Lorentz invariance.

The heterotic string theory[6] is expected to have general coordinate, local Lorentz and local gauge invariance. One would expect that these symmetries should be manifest in the σ-model which describes the propagation of the string in arbitrary background field. Indeed, these symmetries are present in the classical σ-model action. However, a close look at the model tells us that these symmetries are anomalous. In fact, the presence of these anomalies was shown to be responsible for the appearance of the Chern-Simons terms in the classical equations of motion[1,2], which apparently destroys the local gauge and Lorentz invariance. Ultimately, however, we must recover local gauge and Lorentz invariance of the equations of motion. As was shown by Hull and Witten[7], to one loop order the anomalous variation of the effective σ-model action under these symmetries may be cancelled by redefining the transformation laws of the antisymmetric tensor field. The purpose of this paper is to extend their proof to higher orders in the σ-model perturbation theory, and to derive the exact transformation laws of the antisymmetric tensor field under local Lorentz and gauge transformations.

The light-cone gauge action for the σ-model describing the propagation of

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the heterotic string in arbitrary background field is given by[1-5],

$$
S = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \Big(g_{ij}(X)\partial_\alpha X^i \partial^\alpha X^j + \epsilon^{\alpha\beta} B_{ij}(X)\partial_\alpha X^i \partial_\beta X^j + ig_{ij}(X)[\bar\lambda^i \not\partial \lambda^j
$$
$$
+ \bar\lambda^i(\Gamma^j_{k\ell}(X) + S^j_{k\ell}(X))\rho^\alpha \lambda^\ell \partial_\alpha X^k] + \bar\psi^s(i\not\partial\delta_{st} + A_i^M(X)(T^M)_{st}\rho^\alpha \partial_\alpha X^i)\psi^t
$$
$$
+ \frac{i}{4} F_{ij}^M(X)\bar\psi^s \rho^\alpha (T^M)_{st}\psi^t \bar\lambda^i \rho_\alpha \lambda^j \Big). \tag{1}
$$

where $g_{ij}(x)$, $B_{ij}(x)$ and $A_i^M(x)$ are background graviton, antisymmetric tensor, and gauge fields respectively, all taken to be transverse, and assumed to depend on the transverse coordinates only. The dilaton field is taken to be constant in space-time, so that it may be absorbed in various coupling constants, and does not appear explicitly in the σ-model action. X^i's denote the eight scalar fields, λ^i's are the eight left-handed Majorana-Weyl spinors and ψ^s's are the 32 right-handed Majorana-Weyl spinors. We are working in the Neveu-Schwarz-Ramond representation, so that the λ^i's transform in the vector representation of $SO(8)$, whereas the ψ^s's transform in the 32 representation of $SO(32)$ or $(16,1)+(1,16)$ representation of the $SO(16) \otimes SO(16)$ subgroup of $E_8 \otimes E_8$. Also here,

$$
S_{ijk} = \frac{1}{2}(\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}). \tag{2}
$$

$$
\Gamma_{ijk} = \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}). \tag{3}
$$

and F_{ij}^M is the field strength associated with the vector potential A_i^M. The action (1) has an $N = \frac{1}{2}$ supersymmetry:[*]

$$
\delta X^i = i\epsilon\lambda^i; \quad \delta\lambda^i = -(\partial_\tau - \partial_\sigma)X^i \epsilon.
$$

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[*] The transformation law of ψ given here was not needed in Ref. 1 to prove the supersymmetry of the action (1), since we used the equations of motion of ψ in our proof. If we do not use the equations of motion of the ψ fields we need to use the explicit transformation laws of ψ given here.

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$$\delta\psi^s = (-\epsilon\lambda^i A_i^M)(T^M)_{st}\psi^t \tag{4}$$

It is more convenient to rewrite the action in terms of the vielbeins e_i^a satisfying $e_i^a e_j^a = g_{ij}$, spin connection ω_i^{ab} constructed from Γ_{ijk}, and the fields $\lambda^a = e_i^a \lambda^i$:

$$\begin{aligned}
S = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \Big(& g_{ij}(X)\partial_\alpha X^i \partial^\alpha X^j + \epsilon^{\alpha\beta}B_{ij}(X)\partial_\alpha X^i \partial_\beta X^j + i[\bar\lambda^a \not\!\partial \lambda^a \\
& + \bar\lambda^a(\omega_k^{ab}(X) - S_k^{ab}(X))\rho^\alpha \lambda^b \partial_\alpha X^k] + \bar\psi^s(i\not\!\partial\delta_{st} + A_i^M(X)(T^M)_{st}\rho^\alpha \partial_\alpha X^i)\psi^t \\
& + \frac{i}{4}F_{ab}^M(X)\bar\psi^s \rho^\alpha (T^M)_{st}\psi^t \bar\lambda^a \rho_\alpha \lambda^b \Big).
\end{aligned} \tag{5}$$

The above action has a local gauge symmetry:

$$A_i^M(X)T^M \to A_i^{M\prime}(X)T^M = U(X)A_i^M(X)T^M U^{-1}(X) + U(X)i\partial_i U^{-1}(X),$$
$$\psi \to \psi' = U(X)\psi, \tag{6}$$

This symmetry, however, is anomalous[8] due to the chiral nature of the fermions ψ. Similar remark holds also for the local Lorentz symmetry:

$$e_i^a \to R^{ab}e_i^b; \qquad \lambda^a \to R^{ab}\lambda^b;$$
$$\omega_i^{ab} \to [R(\omega_i + \partial_i)R^{-1}]^{ab} \tag{7}$$

where R denotes a local SO(8) rotation. The one loop effective action involving only the external bosonic fields transforms under these anomalous symmetries as[†]

$$\delta S^{(1-loop)} = \frac{1}{8\pi}\int d\tau \int d\sigma \; \epsilon^{\alpha\beta}(\partial_\alpha\theta^M A_i^M - \partial_\alpha\theta^{ab}\bar\omega_i^{ab})\partial_\beta X^i \tag{8}$$

where θ^M and θ^{ab} are the infinitesimal gauge and Lorentz transformation param-

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[†] Since we may add any arbitrary local counterterm to the one loop effective action, the expression for the anomaly given in (8) is not unique. We shalll adopt this particular definition of anomaly in order to uniquely define the fermionic loop integrals.

eters respectively, and,

$$\bar{\omega}_i^{ab} = \omega_i^{ab} - S_i^{ab} \tag{9}$$

As was pointed out by Hull and Witten[7], the anomalous variation of the effective action to one loop order may be cancelled by redefining the transformation laws of B_{ij} under local Lorentz and gauge transformations:

$$\delta B_{ij} = -\frac{\alpha'}{4}\left(\partial_{[i}\theta^M A_{j]}^M - \partial_{[i}\theta^{ab}\bar{\omega}_{j]}^{ab}\right) \tag{10}$$

which is identical to the result found by Green and Schwarz[9], except for the replacement of ω by $\bar{\omega}$. This, however, cannot be the end of the story, since this anomalous variation of B_{ij} induces an anomalous variation of S_{ijk} and hence also an anomalous variation of the connection $\bar{\omega}$ which couples to λ. This induces a further variation of the one loop effective action of order α'. A simple way to get rid of this extra variation is to replace S_i^{ab} by H_i^{ab} in the original σ-model lagrangian where H is determined from the equation:

$$H_{ijk} = S_{ijk} + \frac{\alpha'}{8}[\Omega_3(A) - \Omega_3(\omega - H)]_{ijk} \tag{11}$$

where

$$\Omega_3(A)_{ijk} = \frac{1}{2}\left[A_{[i}^M F_{jk]}^M - \frac{2i}{3}A_{[i}^M A_j^N A_{k]}^P Tr(T^M T^N T^P)\right]. \tag{12}$$

and $\Omega_3(\omega - H)$ is given by a similar equation with A replaced by $\omega - H$. If the transformation law of B_{ij} under local Lorentz and gauge transformations is taken to be,

$$\delta B_{ij} = -\frac{\alpha'}{4}\left(\partial_{[i}\theta^M A_{j]}^M - \partial_{[i}\theta^{ab}(\omega_{j]}^{ab} - H_{j]}^{ab})\right) \tag{13}$$

then H_{ijk}, as defined above, is invariant under these symmetries. As a result, the one loop effective action involving the bosonic fields transforms as in Eq.(8) with

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$\bar{\omega}$ replaced by $\omega - H$. This, in turn, is cancelled by the variation of B_{ij} given in Eq.(13). Note, however, that if we define a new field,

$$B'_{ij} = B_{ij} + \frac{\alpha'}{4}\omega^{ab}_{[i}H^{ab}_{j]} \tag{14}$$

and S'_{ijk} by Eq.(2) with B replaced by B', then Eq.(11) and (13) may be written as, respectively,

$$H_{ijk} = S'_{ijk} + \frac{\alpha'}{8}[\Omega_3(A) - \Omega_3(\omega)]_{ijk} + \text{covariant terms} \tag{15}$$

and,

$$\delta B'_{ij} = -\frac{\alpha'}{4}\left(\partial_{[i}\theta^M A^M_{j]} - \partial_{[i}\theta^{ab}\omega^{ab}_{j]}\right) \tag{16}$$

which is the standard Green-Schwarz transformation law[9]. This is related to the fact that the part of the right hand side of Eq.(8) (with $\bar{\omega}$ replaced by $\omega - H$) which is proportional to the torsion may be removed by adding a local counterterm to the lagrangian proportional to $\int \varepsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j \omega^{ab}_{[i}H^{ab}_{j]}$.

The replacement of S by H in the action (1) corresponds to the addition of a new term,

$$-\frac{i}{32\pi}\left(\Omega_3(A) - \Omega_3(\omega - H)\right)_{ijk}\bar{\lambda}^i \rho^\alpha \lambda^j \partial_\alpha X^k \tag{17}$$

to the action. This destroys the naive $N = \frac{1}{2}$ supersymmetry given in (4). But before discussing this issue, let us discuss another source of local Lorentz and gauge anomaly. So far, we have considered the one loop effective action involving only the external bosonic lines. Since, however, we have four fermion coupling in our theory, the anomalous contribution to the effective action from a fermion loop will involve external fermion fields as well. This may be analyzed by introducing auxiliary fields Q^{ab}_α, R^{ab}_α, and replacing the four fermion coupling term in (1) by,

$$-\frac{1}{4\pi\alpha'}[Q^{ab}_\alpha F^M_{ab}\bar{\psi}T^M\rho^\alpha\psi + iR^{ab}_\alpha\bar{\lambda}^a\rho^\alpha\lambda^b + 4Q^{ab}_\alpha R^{ab\alpha}] \tag{18}$$

where Q and R are defined to transform covariantly under the local gauge and Lorentz transformations. We may now construct an effective action involving

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the fields X^i, Q_α^{ab} and R_α^{ab} by integrating out the ψ and λ fields. Since the connections coupling to ψ and λ fields contain new terms proportional to Q and R respectively, the variation of this effective action under local gauge and Lorentz transformations now contains new terms given by,

$$-\frac{1}{8\pi}\int d\tau \int d\sigma \; \varepsilon^{\alpha\beta}(\partial_\alpha\theta^M F_{ab}^M Q_\beta^{ab} - \partial_\alpha\theta^{ab} R_\beta^{ab}) \qquad (19)$$

This extra variation may be cancelled by adding new terms to the lagrangian given by,

$$\frac{1}{8\pi}\int d\tau \int d\sigma \; \varepsilon^{\alpha\beta}\partial_\alpha X^i (A_i^M F_{ab}^M Q_\beta^{ab} - \omega_i^{ab} R_\beta^{ab}) \qquad (20)$$

Adding (20) to (18) and eliminating the auxiliary fields by their equations of motion we get the following extra terms in the action besides the four fermion coupling:

$$\frac{1}{32\pi} F_{ab}^M [i A_k^M \partial_\beta X^k \bar{\lambda}^a \rho_\alpha \lambda^b \varepsilon^{\alpha\beta} - \omega_i^{ab} \partial_\beta X^i \bar{\psi} \rho_\alpha T^M \psi \varepsilon^{\alpha\beta} + \frac{\alpha'}{2}\partial_\alpha X^i \partial^\alpha X^k A_k^M \omega_i^{ab}].$$
$$(21)$$

Thus by adding terms in the original lagrangian given by (17) and (21), we may recover local Lorentz and gauge invariance of the one loop effective action obtained by integrating out the fermion fields. Furthermore, if we assume the validity of the Adler-Bardeen theorem[10], we may conclude that this result is exact, and that there is no further contribution to the local Lorentz and gauge anomalies.[*]

As was pointed out before, the addition of these new terms seems to destroy the naive $N = \frac{1}{2}$ supersymmetry. This symmetry, however, is anomalous[1,11], since it involves field dependent phase transformations of the chiral fermions ψ^s,

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[*] In order to prove such a theorem, one has to find a gauge invariant regularization prescription for doing higher loop calculations with this effective action. One may be able to achieve this by adding gauge invariant higher derivative terms in the action involving the auxiliary fields Q and R.

and also of the chiral fermions λ^a, since,

$$\delta\lambda^a = e^a_{i,j}(i\epsilon\lambda^j)\lambda^i - e^a_i(\partial_\tau - \partial_\sigma)X^i\epsilon$$

$$\delta\psi^s = (-\epsilon\lambda^i A^M_i)(T^M)_{st}\psi^t \tag{22}$$

In the last of Refs.1 we conjectured that the supersymmetry anomaly may be cancelled by the extra terms (17) and (21) in the lagrangian. In the rest of the paper, we shall verify this conjecture for a specific choice of the background fields, where we set $g_{ij} = \delta_{ij}$ and $B_{ij} = 0$, but keep A^M_i arbitrary. The contribution to the effective action from the ψ-loop is then given by[†]

$$\frac{1}{16\pi}a^M_\alpha\left(g^{\alpha\beta} - (g^{\alpha\alpha'} + \varepsilon^{\alpha\alpha'})\frac{\partial_{\alpha'}\partial_{\beta'}}{\partial^2}(g^{\beta\beta'} + \varepsilon^{\beta\beta'})\right)a^M_\beta + f(a^M_\alpha) \tag{23}$$

where,

$$a^M_\alpha = A^M_i(X)\partial_\alpha X^i + \frac{i}{4}F^M_{ij}\bar{\lambda}^i\rho_\alpha\lambda^j \tag{24}$$

and $f(a^M_\alpha)$ is a gauge invariant function of its argument, involving cubic and higher powers of a^M_α. Since a^M_α couples only to the right handed fermions, $f(a^M_\alpha)$ is a function of $(a^M_\tau - a^M_\sigma)$ only. It can be shown that under a supersymmetry transformation $a^M_\tau - a^M_\sigma$ transforms like a gauge transformation with parameter $i\epsilon A^M_i\lambda^i$. Hence $f(a^M_\alpha)$ is invariant under this transformation. On the other hand, using Eqs.(4) and (24) we may directly evaluate the variation of the first term in (23) under the supersymmetry transformation. Ignoring terms of order α' and higher powers of α', this may be shown to cancel the variation of the terms given in (17) and (21) under the supersymmetry transformation, up to terms proportional to the classical equations of motion of the λ^i and the X^i fields. These terms may be cancelled by redefining the supersymmetry transformation laws of λ^i and X^i.

[†] The first term in Eq.(23) is the two point function and may be calculated directly. Since the gauge variation of this term reproduces the full anomaly given in Eqs.(8) and (19), the rest of the contribution $f(a^M_\alpha)$ must be gauge invariant.

A complete proof of the cancellation of supersymmetry anomaly at one loop order will involve the evaluation of the full one loop effective action. However the cancellation of the one loop supersymmetry anomaly in the presence of background gauge fields is a strong indication that such cancellation indeed occurs even in the presence of arbitrary background fields.

Thus we have shown that the σ-model given in (1), plus the counterterms given in (17) and (21) makes the model invariant under local Lorentz and gauge transformations, with the transformation law of the antisymmetric tensor field given in (13)[or(16)], the covariant torsion H appearing in this equation being given as a solution of Eq.(11). [This equation may be solved iteratively for H]. There is also strong indication that this model retains the $N = \frac{1}{2}$ supersymmetry. The equivalence between Eqs.(11), (13) and (15), (16) also shows us that in Witten's consistency condition[12] $\int Tr(R \wedge R - F \wedge F) = 0$ we may take R as the ordinary curvature or as the generalized curvature including torsion. This is related to the fact that the Pontryagin class of a manifold in invariant under the addition of a globally defined tensor to the connection.

Note added: After completion of this work, we learned about some work by R. Nepomechie[13], which discusses issues similar to that of Ref.7 in a bosonized formulation.

The σ-model approach to the string theory has also been used recently to derive information about the spectrum of massless particles in the theory[14].

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