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ASPECTS OF MESON–SKYRMION SCATTERING*

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ABSTRACT

Model-dependent and model-independent results for $\pi N \to \pi N$ and $\pi N \to \pi\Delta$ in the Skyrme model are reviewed. An extension of the formalism to the 3-flavor case is presented.

1. INTRODUCTION

It is by now generally accepted that QCD is the correct theory of the strong interactions. Yet, in searching for meaningful, theory-specific tests of QCD, it is crucial to keep in mind that many properties of hadrons are in fact more *basic* than QCD, depending only on symmetry considerations. One immediately thinks of soft-pion physics, which would still be valid even if QCD itself were overthrown, so long as the observed pattern of chiral symmetry breaking were preserved. What has become clear in the last few months, in the context of skyrmion physics, is that many processes that lie *well beyond* the soft-pion energy range might fall into this category as well. (We'll return to this claim in Sec. 5.)

In this talk, I shall focus on meson-nucleon scattering in the Skyrme model.[♯1] Surprisingly, we shall see that many essential features of the partial-wave scattering amplitudes, most notably the quantum numbers and the relative ordering of the approximately thirty observed nucleon and Δ resonances, follow directly from Skyrme's non-linear sigma model, with no need to appeal to an underlying theory of quarks and gluons (Table I). The 2-flavor results that I shall review (both model-dependent and model-independent) were developed in collaboration with

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♯1 For a variety of approaches to this problem, see Refs. 1-8.

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Table I

Comparison between experiment and Skyrme model predictions for masses of baryon resonances (in MeV)

Channel[a]	Experiment	Fit #1 [b]	% error	Fit #2 [c]	% error
S11	1526	1295	-15	1478	-3
S31	1610	1295	-20	1478	-8
P11	939	939	0	1190	27
P11	1723	1233	-28	1427	-17
P13	1710	1919	12	1982	16
P31	1888	1919	2	1982	5
P33	1232	1436[d]	17	1424[d]	16
P33	1522	1242	-18	1435	-6
P33	1868	1874	0.3	1946	4
D13	1519	1589	5	1715	13
D15	1679	1625	-3	1744	4
D33	1680	1616	-4	1737	3
D35	1901	1607	-15	1730	-9
F15	1684	1723	2	1823	8
F17	2005	1954	-3	2011	0.3
F35	1905	1856[e]	-3	1931[f]	1
F37	1913	1714	-10	1816	-5
G17	2140	2034	-5	2075	-3
G19	2268	2230	-2	2234	-2
G37	2215	2141	-3	2162	-2
G39	2468	2043	-17	2083	-16
H19	2205	2346	6	2327	6
H39	2217	2444	10	2407	9
H311	2416	2346	-3	2327	-4
I111	2577	2631	2	2558	-1
I313	2794	2658	-5	2579	-8
K113	2612	3032	16	2882	10
K315	2990	2943	-2	2810	-6

a) The notation is as follows: The pion partial wave, followed by twice the total isospin and twice the total angular momentum.

b) Fit # 1 – Nucleon mass fixed; $e = 6.29$, $f_\pi = 142$.

c) Fit # 2 – Nucleon mass allowed to vary; $e = 4.79$, $f_\pi = 150$.

d) *Not present* in our lowest-order formalism; this number comes from Eq. 9 of Ref. 14.

e) Average of two peaks at 1732 and 1981 MeV.

f) Average of two peaks at 1831 and 2032 MeV.

Marek Karliner[9] and Mike Peskin[10] at SLAC, and independently by the group at Siegen University[11]–[13]; the reader is directed to these publications for further details. Later, in Sec. 6, I shall discuss how to adapt the formalism to the 3-flavor case.

We shall be examining Lagrangians of the form

$$\mathcal{L} = \frac{f_\pi^2}{16}\,\mathrm{Tr}\,(\partial_\mu U \partial^\mu U^\dagger) + \cdots, \tag{1}$$

where $U \in SU(2)$. The leading term is the usual 2-flavor nonlinear sigma model, while the dots stand for higher-derivative terms whose presence is necessary to yield a stable soliton solution, or "skyrmion,"

$$U_0 = e^{iF(r)\hat{\mathbf{r}}\cdot\vec{\sigma}}\,. \tag{2}$$

Such a "hedgehog" configuration has baryon number (*i.e.*, topological charge) unity and is Skyrme's candidate nucleon.[15] Note that although the skyrmion rotates under isospin $\vec{I}$ (which acts on $\vec{\sigma}$) and angular momentum $\vec{J}$ (which acts on $\hat{\mathbf{r}}$), it is left invariant by the simultaneous action of $\vec{I} + \vec{J} \equiv \vec{K}$. We shall make crucial use of this $\vec{K}$-symmetry a little later on.

The skyrmion lay more or less dormant for nigh on twenty years, until Witten's realization that, in large-N, baryons should indeed emerge as solitons in the field of pions (N = number of colors).[16],[17] With this motivation, we shall examine pion-skyrmion scattering in leading order in the large-N expansion, and gear all our approximations accordingly. The results discussed below are of two types. *Model-dependent* results depend on the detailed form of the Lagrangian $\mathcal{L}$, and correspondingly on the precise shape of the skyrmion profile $F(r)$. *Model-independent* results, in contrast, just depend on the hedgehog structure of the skyrmion, Eq. (2), hence constitute a more-or-less direct test of the $1/N$ expansion.

2. SKETCH OF TWO-FLAVOR FORMALISM

Now, in truth, nucleons (and Δ's) should properly be identified with *rotating* skyrmions, which is done by introducing collective coordinates.[14] However, let us ignore this complication for the moment and concentrate on the simplified problem of how to describe pion scattering from an *unrotated* skyrmion in its canonical "hedgehog" orientation.[♯2] This is naturally accomplished by letting

$$F(r)\hat{\mathbf{r}} \longrightarrow F(r)\hat{\mathbf{r}} + \frac{2}{f_\pi}\vec{\pi}(\vec{x},t) \tag{3}$$

♯2 Throughout this talk, I shall use the expression *physical* scattering to mean that the rotation of the soliton has been properly taken into account through the use of collective coordinates, so that the skyrmion does in fact have the quantum numbers of a nucleon or Δ.

in (2) and expanding $\mathcal{L}$ about the skyrmion in powers of the pion field. We find:

$$\int d^3x\, \mathcal{L} = -M_{\rm skyrmion} + \int d^3x\, \pi^{i*}(x)\hat{L}_{ij}\pi^j(x) + \mathcal{O}(\pi^3/f_\pi), \tag{4}$$

where $\hat{L}_{ij}$ is a self-adjoint 3×3 matrix of differential operators in isospin space. The absence of terms linear in the pion field is of course due to the fact that the skyrmion is a local minimum of the action. In the large-N scheme, $f_\pi \sim \sqrt{N}$, so we will ignore the $\mathcal{O}(\pi^3/f_\pi)$ terms, consistent with our lowest-order approach.

We are thus interested in constructing the tree-level pion propagator in the unrotated skyrmion background. As in any multi-dimensional scattering problem, this task is greatly simplified by judicious use of symmetry. In the familiar case of a particle scattering off a spherical potential, for example, the wavefunction is expanded in terms of the spherical harmonics; this leaves a purely radial problem characterized by a "reduced" S-matrix $\mathsf{s}_L(\omega)$. In the case at hand, traditional pion angular momentum $\vec{L}$ is not conserved, nor for that matter is the pion's isospin, but the peculiar hybrid $\vec{K} = \vec{I}\,(pion) + \vec{L}$ *is*. The appropriate normal modes are the so-called *vector* spherical harmonics

$$\Pi_L^{KK_z}(\Omega) = \begin{pmatrix} \langle L1, K_z-1, 1|KK_z\rangle\; Y_{L,K_z-1}(\Omega) \\ \langle L1K_z0|KK_z\rangle\; Y_{LK_z}(\Omega) \\ \langle L1, K_z+1, -1|KK_z\rangle\; Y_{L,K_z+1}(\Omega) \end{pmatrix} \tag{5}$$

familiar from nuclear physics. The result of the normal-mode expansion is, once again, a purely radial scattering problem, the details of which depend on the exact form of the Lagrangian. This radial problem is characterized by a reduced S-matrix $\mathsf{s}_{KL'L}(\omega)$ ($\equiv \mathsf{s}_{KLL'}(\omega)$) with L and L' labeling the incoming and outgoing pion partial waves. In terms of this reduced S-matrix, the amplitude for a pion with $I_z = n$ in a partial wave $\vec{L}$ to scatter off an unrotated skyrmion (URS) and become a pion of $I_z = m$ in a partial wave $\vec{L}'$ is given by:

$$\begin{aligned} &\mathrm{S}(\pi^n(\vec{L}) + \mathrm{URS} \to \pi^m(\vec{L}') + \mathrm{URS}) = \\ &\qquad \sum_{KK_z} \langle KK_z|L1L_zn\rangle\, \langle L'1L'_zm|KK_z\rangle\, \mathsf{s}_{KL'L}. \end{aligned} \tag{6}$$

So far, we have been considering pions scattering from unrotated skyrmions. As we have seen, neither isospin nor angular momentum—only the sum of the two—is conserved in such a process. A slight generalization is pion scattering from a rotated skyrmion $U_A \equiv AU_0A^{-1}$; the natural result is

$$\mathrm{S}(\pi^n + \mathrm{URS} \to \pi^m + \mathrm{URS}) \longrightarrow \sum_{k,l} A_{mk} \cdot \mathrm{S}(\pi^l + \mathrm{URS} \to \pi^k + \mathrm{URS}) \cdot A^{-1}_{ln}. \quad (7)$$

Now, properly speaking, nucleons and Δ's should be identified with a *superposition* of U_A for all values of the collective coordinate $A \in SU(2)$, weighted by an appropriately constructed wavefunction $\chi(A)$. Pion scattering from nucleons and/or Δ's is described accordingly by the expression

$$\sum_{k,l} \int_{SU(2)} dA\, \chi^*_f(A) A_{mk} \mathrm{S}(\pi^l + \mathrm{URS} \to \pi^k + \mathrm{URS}) A^{-1}_{ln} \chi_i(A). \quad (8)$$

We shall see shortly that this formula, which characterizes "physical" pion-nucleon scattering in the skyrmion picture, properly encompasses the individual conservation of both total isospin and total angular momentum.

The physical picture behind this formula is as follows. In large-N, the rotational velocity of the skyrmion scales like $1/N$, and so can be neglected in lowest order. The approaching pion thus sees a fixed value of the collective coordinate A, which remains essentially constant throughout the interaction. A quantum superposition over all possible values of A weighted by their contribution to the initial and final baryon wavefunctions yields the above formula. Note that this picture breaks down near threshold, where the skyrmion's period of rotation becomes comparable to the interaction time. In this regime, it is Schnitzer's "soft-pion" approach which is appropriate.[18]

It turns out that the integral over A indicated in (8) can be performed in closed form, thanks to the compact expression for the baryon wavefunction[14]

$$\chi(A) = \frac{i}{\pi}\sqrt{\frac{2s+1}{2}}[\epsilon^{(s)} \cdot \mathcal{D}^{(s)}(A)^{-1}]_{s_z,i_z} = \frac{i}{\pi}\sqrt{\frac{2s+1}{2}}[\mathcal{D}^{(s)}(A)^{-1}]_{-s_z,i_z} \cdot (-1)^{s-s_z} \quad (9)$$

with $s = \frac{1}{2}$ for nucleons and $s = \frac{3}{2}$ for deltas. Projecting the initial (final) pion-baryon states onto states of definite *total* isospin $\vec{I}$ ($\vec{I}'$) and angular momentum $\vec{J}$ ($\vec{J}'$) yields, after a little massaging,

$$\mathrm{S}([s\,\vec{I}\,\vec{J}\,L] \to [s'\,\vec{I}'\,\vec{J}'\,L']) = \delta_{II'}\delta_{I_zI'_z}\delta_{JJ'}\delta_{J_zJ'_z} \sum_K P_{LL'ss'IJK} \cdot s_{KL'L} \quad (10a)$$

where we have introduced the "P-symbols"

$$P_{LL'ss'IJK} = (-1)^{s'-s}\sqrt{(2s+1)(2s'+1)}(2K+1) \begin{Bmatrix} K & I & J \\ s' & L' & 1 \end{Bmatrix} \begin{Bmatrix} K & I & J \\ s & L & 1 \end{Bmatrix}. \quad (10b)$$

We make the following observations:
(i) Conservation of isospin and angular momentum is now manifest in the Kronecker δ's. (Note that these emerged from the analysis, and were not simply put in "by hand.")
(ii) The above formula can be shown to satisfy unitary.
(iii) The 6-j symbols embody various triangle inequalities on both the entering and exiting channels (Fig. 1) which one might expect to impose unduly strict selection rules for $\pi N \to \pi N$ and $\pi N \to \pi\Delta$. But in fact, one obtains precisely the same constraints on these processes that would otherwise follow just from conservation of isospin and angular momentum and parity. In particular, there is no "extra" conserved quantum number $\vec{K}(pion)$ for the *physical* πN amplitudes (as opposed to scattering from an unrotated hedgehog).
(iv) The formula is analogous to the Wigner-Eckart theorem, in that a large number of physical amplitudes are expressed in terms of a smaller set of "reduced" amplitudes modulo some group-theoretic coefficients. In particular, all the model-dependence arising from the precise form of the Lagrangian is buried in the reduced S-matrix $s_{KL'L}$; the rest is pure group theory, depending only on the hedgehog structure of the skyrmion.

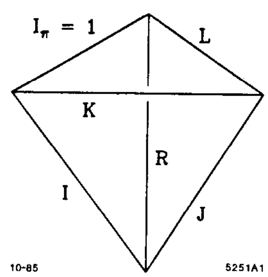

Fig. 1. Triangle inequalities implicit in Eq. (10).

We can take advantage of observation (iv) in one of two ways. On the one hand, one can calculate the $s_{KL'L}$ numerically in the context of a specific model such as Skyrme's, reconstruct the physical amplitudes using (10), and compare the results to experiment; I will discuss the outcome of such a program in Sec. 4. Alternatively, one can find those magical linear combinations of physical amplitudes such that the model-dependent reduced S-matrix elements (*i.e.*, the right-hand side of (10*a*)) cancel out completely. The result is a set of predicted energy-independent linear relations directly between the various *experimental* $\pi N \to \pi N$ and $\pi N \to \pi\Delta$ amplitudes in a given partial wave. These relations depend only on the hedgehog structure of the underlying skyrmion; they therefore serve as model-independent tests of the validity of the skyrmion approach. In addition, they are tests of the legitimacy of our various large-N approximations.

3. MODEL-INDEPENDENT RELATIONS

Let us consider such relations as applied to the elastic case $\pi N \to \pi N$. We can represent the physical πN elastic amplitudes as $\mathbf{S}_{LIJ}$ with $I = \frac{1}{2}, \frac{3}{2}$ and $J = L \pm \frac{1}{2}$. Eq. (10) can then be shown to imply:

$$\mathbf{S}_{L\frac{3}{2},L-\frac{1}{2}} - \frac{L-1}{4L+2} \cdot \mathbf{S}_{L\frac{1}{2},L-\frac{1}{2}} - \frac{3L+3}{4L+2} \cdot \mathbf{S}_{L\frac{1}{2},L+\frac{1}{2}} = 0 \qquad (11a)$$

and

$$\mathbf{S}_{L\frac{3}{2},L+\frac{1}{2}} - \frac{3L}{4L+2} \cdot \mathbf{S}_{L\frac{1}{2},L-\frac{1}{2}} - \frac{L+2}{4L+2} \cdot \mathbf{S}_{L\frac{1}{2},L+\frac{1}{2}} = 0. \qquad (11b)$$

These relations are depicted graphically in Fig. 2 for a variety of partial waves L.[♯3] The experimental isospin-$\frac{3}{2}$ amplitudes (indicated by solid lines) have been juxtaposed with the appropriate linear combinations of the experimental isospin-$\frac{1}{2}$ amplitudes in the same partial wave (dotted lines). The general impression is of surprisingly good agreement, especially for F-waves and higher. There is a fairly consistent splitting of ≈ 200 MeV between the isospin-$\frac{1}{2}$ and isospin-$\frac{3}{2}$ curves which can be attributed to a higher order $1/N$ effect (as in the splitting between the N and Δ); such effects have been entirely neglected in our lowest-order treatment. The prediction $P_{13} = P_{31}$ is intriguing in that it relates two out of only four elastic amplitudes that exhibit clear-cut repulsive (*i.e.*, clockwise) motion near threshold. It is an automatic consequence of the fact that both the skyrmion and the P-wave pion have $I = J$.[♯4]

In contrast, the relation $S_{31} = S_{11}$ is obviously grossly violated at low energies, where the amplitudes move in opposite directions around the unitarity circle. The relation is all the more disturbing in light of Weinberg's famous soft-pion calculation of the S-wave scattering lengths,[21] which correctly predicts the observed low-energy behavior for the T-matrix elements: $S_{31} = -2S_{11}$. Here we witness rather dramatically the breakdown of our approximations in the soft-pion regime, where—despite large-N considerations—the period of rotation of the skyrmion can no longer be neglected *vis-à-vis* the time of interaction. In this regime, it is Schnitzer's treatment which is appropriate, and as he has shown, the validity of Weinberg's theorem (as of all soft-pion results) in the context of skyrmion physics is assured.[18] Reassuringly, beyond the soft-pion regime, the relation $S_{31} = S_{11}$ becomes quite reasonably satisfied; note in particular the nice correspondence between the peaks in the real parts of the S_{11} and S_{31} amplitudes at 1.6 GeV. It would clearly be desirable to have a unified formalism that interpolated smoothly between the two regimes.

♯3 The experimental elastic πN amplitudes are drawn from Ref. 19.

♯4 This relation was actually predicted over thirty years ago by Harlow and Jacobsohn[20].

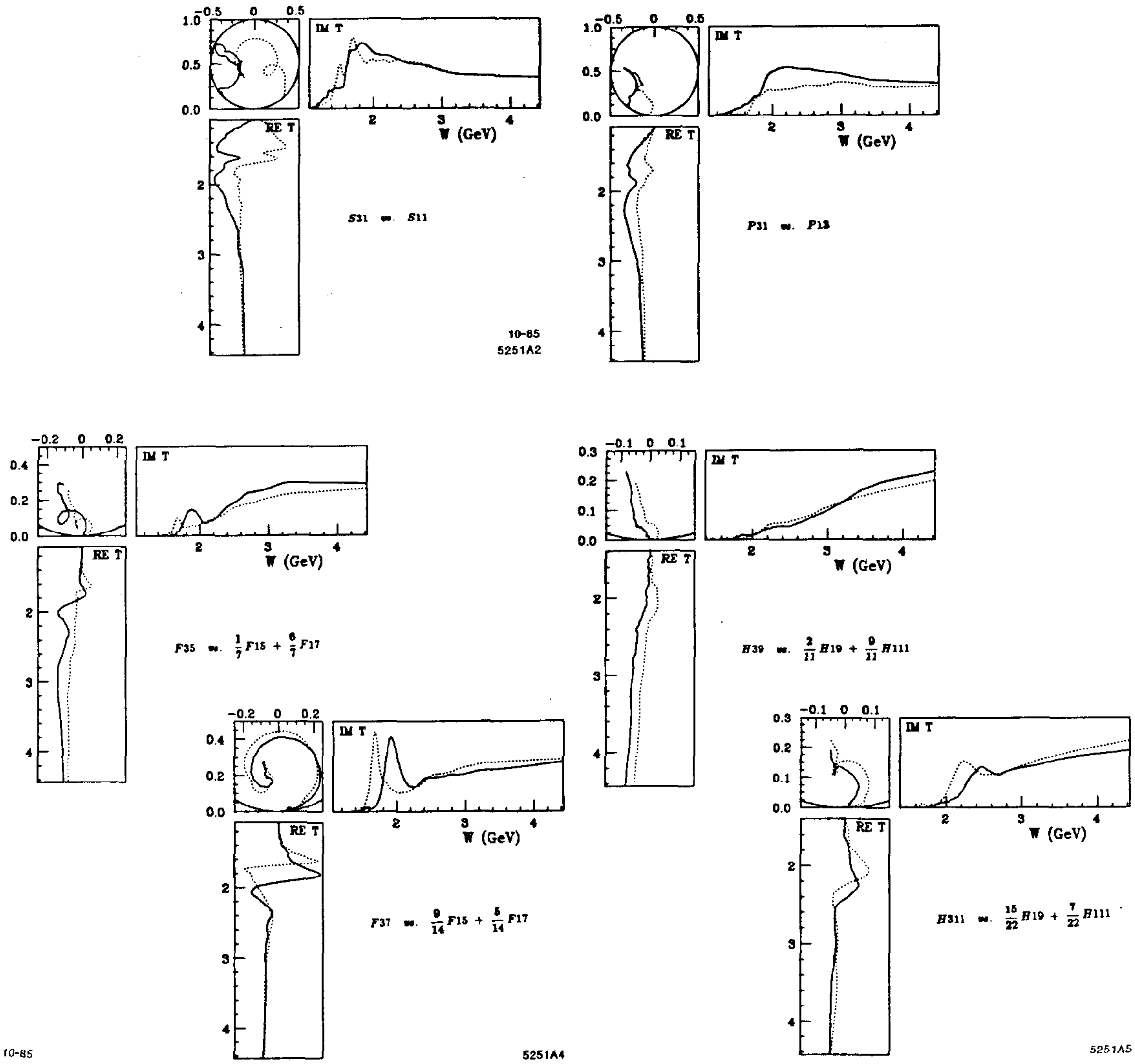

Fig. 2. Comparison of experimental isospin-$\frac{3}{2}$ πN elastic amplitudes (solid lines) with linear combination of isospin-$\frac{1}{2}$ experimental amplitudes (dotted lines).

We can shed additional light on the discrepancy by appeal to the skyrmion's translational and rotational zero-modes. To lowest order in $1/N$, these show up as boundstates in all reduced channels $s_{KL'L}$ with $K = 1$; these boundstates sit precisely at the (unrotated) skyrmion mass. Now, to *next* order in $1/N$, the S-matrix poles corresponding to these bound-states are undoubtedly perturbed in various directions in the complex energy-plane, with possibly drastic effects on the behavior of the $K = 1$ reduced amplitudes near threshold. This will, in turn, cause drastic changes in the low-energy behavior of those *physical* amplitudes which couple to $s_{K=1,L'L}$ via Eq. (10): namely, the S-, P- and D-wave pion-nucleon amplitudes. The moral of the story, then, is that a *zeroth-order* $1/N$ analysis of meson-skyrmion scattering cannot be trusted at low energies in the S-, P- and D-waves. It is reassuring that it is only in these waves that one finds serious violations of the linear relations. We shall return to this point when we consider the specific case of the Skyrme model.

Figure 3 depicts a handful of linear relations obtainable from (10) involving the reaction $\pi N \to \pi\Delta$.[♯5] Although the relative ***signs*** of the amplitudes are always given correctly, the ***sizes*** can occasionally be off by as much as a factor of four, as in Fig. 3(b). Note that Figs. 3(c) and 3(d) actually relate the experimental amplitudes for $\pi N \to \pi N$ to $\pi N \to \pi\Delta$, with reasonable success.

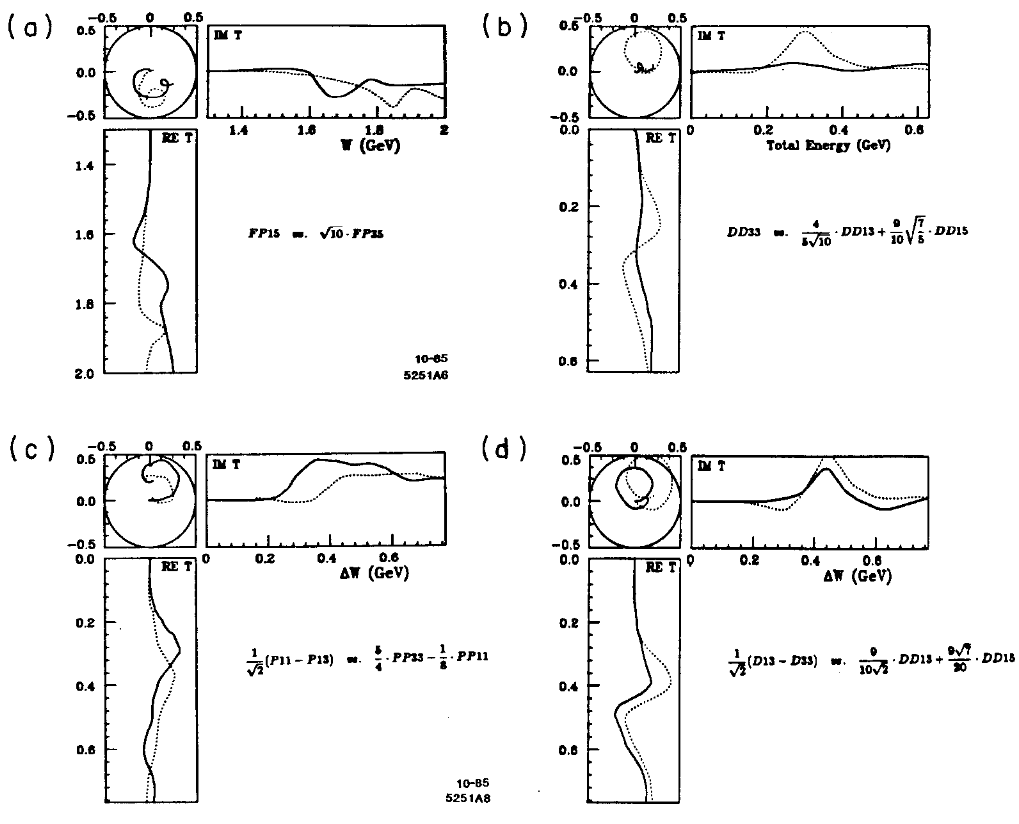

Fig. 3. Test of linear relations for the process $\pi N \to \pi\Delta$.

♯5 The experimental amplitudes for these processes have been drawn from Ref. 22. The two letters labeling each channel stand for L and L', which are not necessarily equal for $\pi N \to \pi\Delta$

4. MODEL-DEPENDENT RESULTS

Returning to Eq. (10), one can apply this formula to a specific model first written down by Skyrme in 1961:[15]

$$\mathcal{L} = \frac{f_\pi^2}{16} Tr \partial_\mu U \partial^\mu U^\dagger + \frac{1}{32e^2} Tr[(\partial_\mu U)U^\dagger, (\partial_\nu U)U^\dagger]^2. \tag{12}$$

The explicit calculation of the reduced S-matrix in the Skyrme model is a lengthy procedure, involving the following steps:
(i) The pion fluctuation field is expanded in vector spherical harmonics, and the action is expanded to second order in the pion field. The integration over solid angles can be carried out explicitly; this leaves a purely radial problem, characterized for each value of K by second-order linear differential equations in r and t, or equivalently, r and ω.
(ii) These differential equations are solved numerically for each (positive) value of ω, by starting with the well-behaved solutions at the origin and integrating out to large r.
(iii) At large r, the skyrmion is negligible, so the theory is one of free pions. Accordingly, the pion field can be fit to a sum of incoming and outgoing spherical bessel functions, and the phase-shifts extracted in the usual manner. This yields the partial-wave reduced S-matrix $s_{KL'L}(\omega)$.
(iv) The physical partial wave pion-nucleon amplitudes in the Skyrme model are then assembled by multiplying the reduced S-matrix elements by the appropriate group theory coefficients, as dictated by Eq. (10).

Figure 4 displays a selection of Skyrme-model phase-shift plots for $\pi N \to \pi N$ juxtaposed with the experimental results. The general impression is of quite good qualitative agreement, particularly for F-waves and beyond. Obviously, the Skyrme model amplitudes for most partial waves take much too big an excursion through the unitarity circle, and hug the rim more closely than their real-life counterparts; this is simply due to the fact that, in our treatment, we haven't allowed for the multitude of inelastic channels which open up at high energies in Nature. In particular, multiple pion production, although formally suppressed by factors of $1/\sqrt{N}$, hence absent from our treatment, becomes the dominant feature of πN scattering at high energies.

Table I (given earlier) presents the resonance mass predictions of the Skyrme model as compared to experiment. These are based on a least-squares fit to the observed spectrum. In Fit #1 (a one-parameter fit), the nucleon mass has been held fixed, while in Fit #2 (a two-parameter fit) it has been treated democratically with the others. In the latter case, a best fit yields for the Skyrme model parameters: $f_\pi = 150$ MeV (*vs.* 186 MeV in Nature) and $e = 4.79$; interestingly, these values yield substantially improved *static* properties of skyrmions as compared with Ref. 14. (Static properties are poorly given by Fit #1.) The average

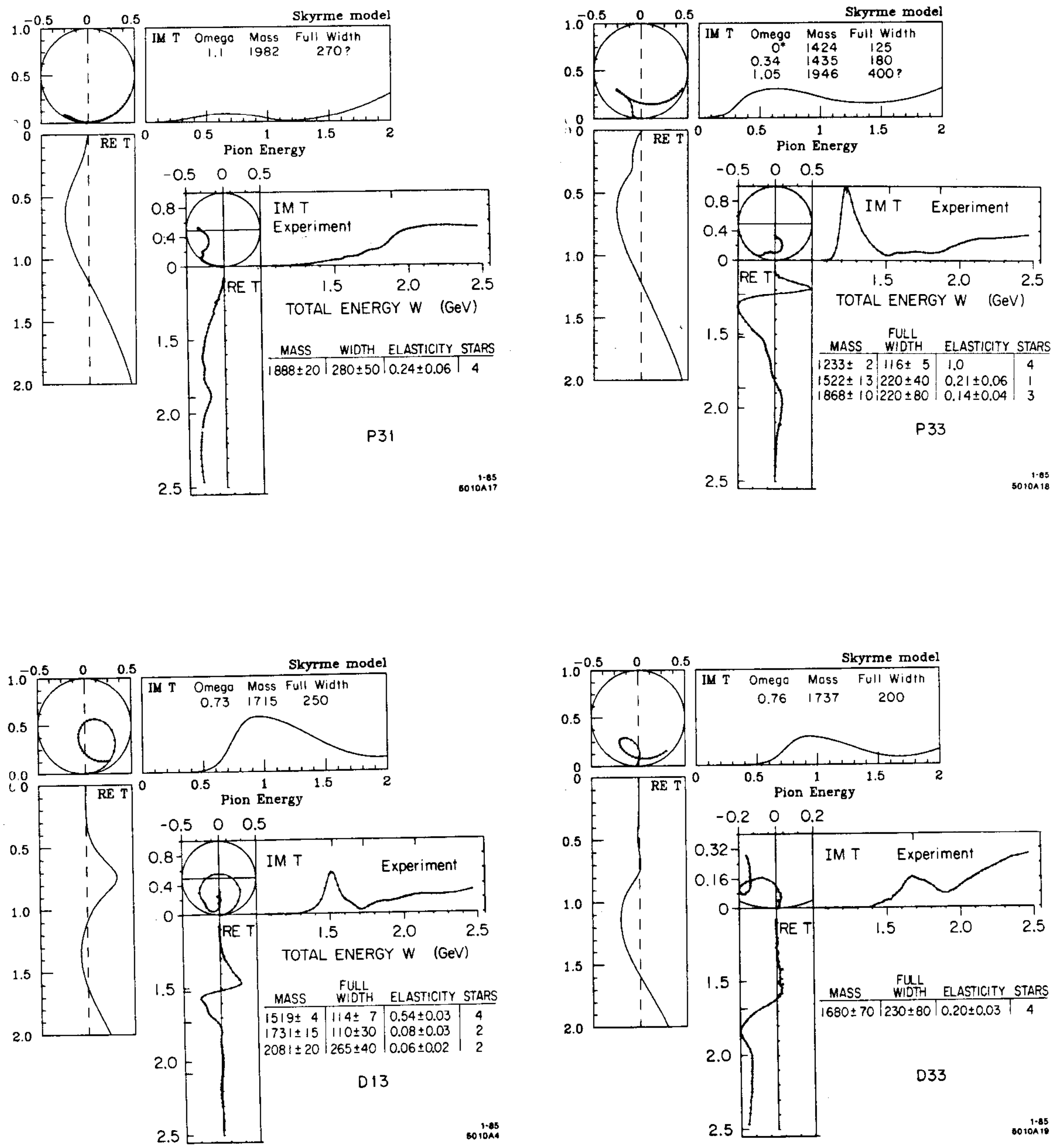

Fig. 4. Experimental elastic πN amplitudes juxtaposed with Skyrme model predictions. Energy is *total* energy in the first case, *pion* energy (in units of $e \cdot f_\pi$) in the second.

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* **The Δ is degenerate with the nucleon to order N^o. The value for the mass comes from Eq. (9) of Ref. 14. The width is also from Ref. 14.**

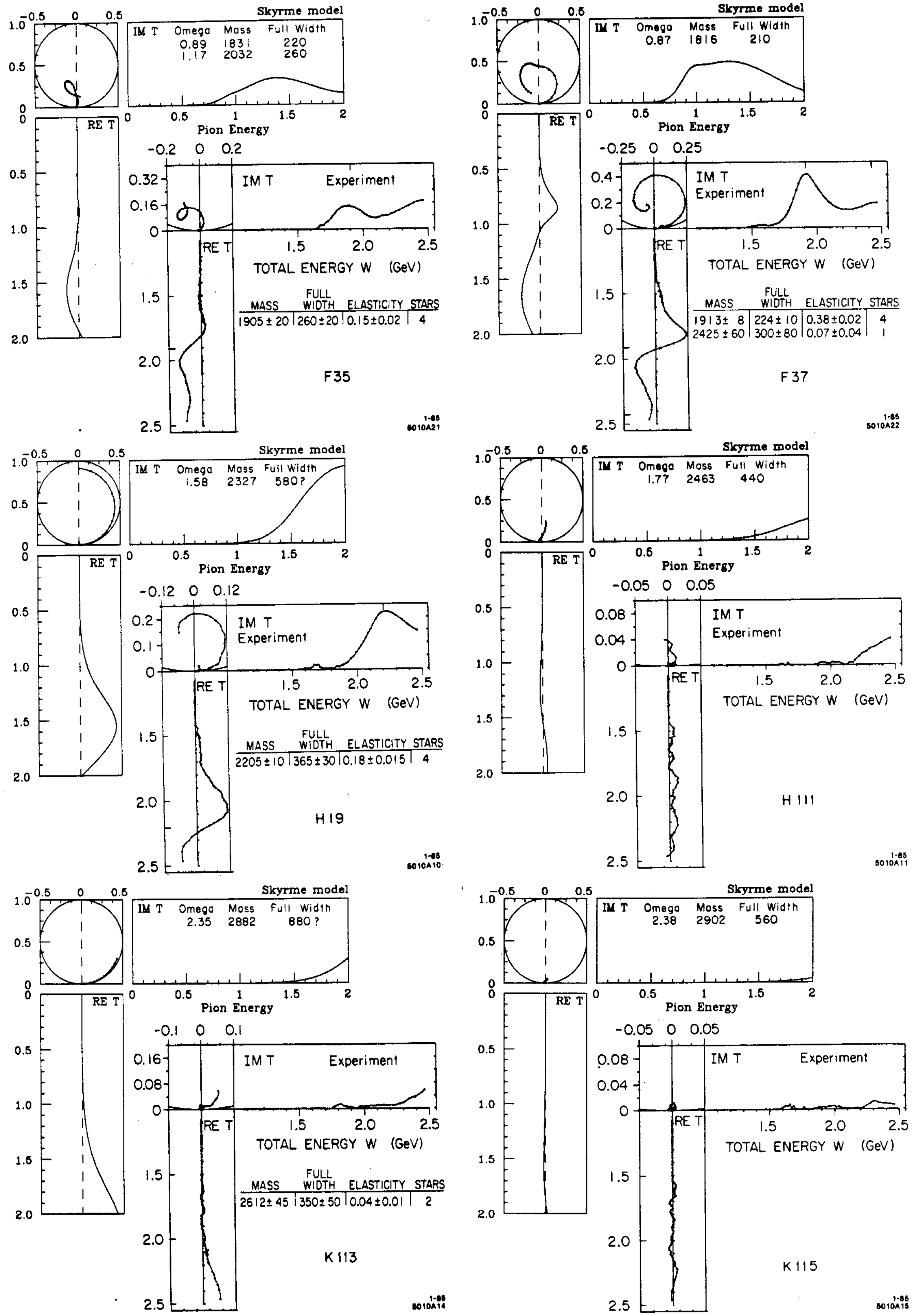

Figure 4 Continued

mass agreement between theory and experiment is 8%, although the nucleon mass is off by 27%. Note that agreement is respectable all the way up to 3 GeV. It is surely surprising that one obtains so much structure from a two-term Lagrangian with no explicit nucleon fields.

In contrast, the P_{33} channel in the Skyrme model is an ostensible disaster. Instead of the grand sweep around the unitarity circle associated with the $\Delta(1232)$, there is threshold *repulsive* behavior followed by two rather feeble resonances at 1435 and 1946 MeV. But the absence of the Δ is only an artifact of our lowest-order treatment. Our pion energies are measured in units of $e \cdot f_\pi$, which $\sim N^0$. But to this order, the Δ is degenerate with the nucleon[14], and therefore would not show up as a resonance. Thus it would actually have been *inconsistent* for us to have seen the Δ in our treatment. This discussion can be recast in terms of the soliton's zero-modes, mentioned previously. Recall that, corresponding to the rotational modes, there is a pole in the S-matrix that, to lowest order, sits right at the skyrmion mass. It is the quantization of the rotational zero-modes which pushes this pole into the physical region, where it manifests itself as the Δ resonance. This quantization—hence the nucleon-Δ mass-splitting—is a $1/N$ effect, and could not appear in our lowest-order treatment. The resonances we *do* already find in this channel are in good correspondence with the observed P_{33} excitations at 1522 and 1868 MeV; unlike the Δ, these resonances have excitation energies $\mathcal{O}(N^0)$.

Overall, the overwhelming majority of observed nucleon and Δ resonances in Nature is accounted for in the Skyrme model. There is good agreement for the values of the masses even in several of the channels where the shapes of the amplitudes are poorly rendered by the model. It should be noted, though, that with the exception of the F- and G-waves, widths in the model are typically 50% too large.

5. THE "BIG-SMALL-SMALL-BIG" PATTERN

An especially noteworthy success of the Skyrme model evident in Fig. 4 is that, despite the overly large motion of the amplitudes as noted earlier, the *relative* sizes of the four independent elastic amplitudes in each partial wave are correctly rendered. Figure 5 illustrates what can be termed the "big-small-small-big" pattern in Nature: the big amplitudes have $\{I = \frac{1}{2}, J = L-\frac{1}{2}\}$ and $\{I = \frac{3}{2}, J = L+\frac{1}{2}\}$ while the small amplitudes have $\{I = \frac{1}{2}, J = L+\frac{1}{2}\}$ and $\{I = \frac{3}{2}, J = L-\frac{1}{2}\}$. (Clearly the D_{35} stands out as the major exception to the rule.) This pattern, which is surely the most striking general feature of the experimental $\pi N \to \pi N$ amplitudes, was noticed a long time ago by Donnachie, Hamilton and Lea, who showed how it might arise by consideration of various particle exchanges.[23] It turns out to have a simple and largely model-independent group-theoretic explanation in the context of skyrmion models, which I shall now explain.

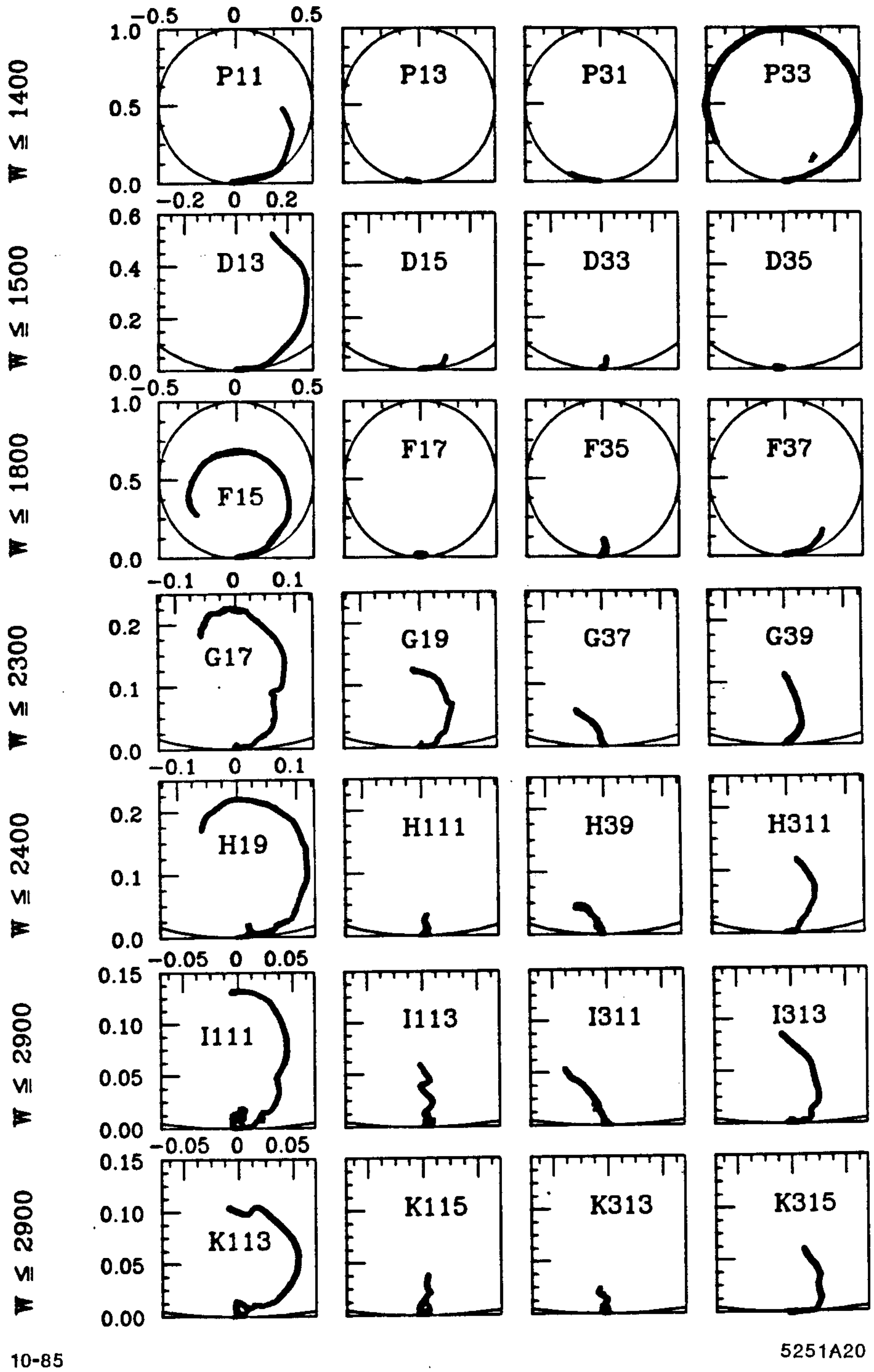

Fig. 5. The "big-small-small-big" pattern exhibited by the experimental elastic πN amplitudes.

Recall that, according to Eq. (10), a πN scattering process in a partial wave L receives contributions from (model-dependent) reduced amplitudes $\mathbf{s}_{KLL}$ with $K = L-1,\ L,\ L+1$. Let us make the plausible dynamical assumption that the deviation from unity of $\mathbf{s}_{L+1,LL}$ can essentially be neglected compared to $\mathbf{s}_{L-1,LL}$ and $\mathbf{s}_{LLL}$; this is, in fact, the case in the Skyrme model. Setting $\mathbf{s}_{L+1,LL} \cong 1$ then yields:[♮6]

$$\mathrm{S}_{L\frac{1}{2},L-\frac{1}{2}} - 1 \cong \frac{2L-1}{3L} \cdot (\mathbf{s}_{L-1,LL} - 1) + \frac{L+1}{3L} \cdot (\mathbf{s}_{LLL} - 1), \tag{13a}$$

$$\mathrm{S}_{L\frac{1}{2},L+\frac{1}{2}} - 1 \cong \frac{L}{3L+3} \cdot (\mathbf{s}_{LLL} - 1), \tag{13b}$$

$$\mathrm{S}_{L\frac{3}{2},L-\frac{1}{2}} - 1 \cong \frac{(2L-1)(L-1)}{6L(2L+1)} \cdot (\mathbf{s}_{L-1,LL} - 1) + \frac{2L-1}{6L} \cdot (\mathbf{s}_{LLL} - 1), \tag{13c}$$

$$\mathrm{S}_{L\frac{3}{2},L+\frac{1}{2}} - 1 \cong \frac{2L-1}{4L+2} \cdot (\mathbf{s}_{L-1,LL} - 1) + \frac{2L+3}{6L+6} \cdot (\mathbf{s}_{LLL} - 1). \tag{13d}$$

The big-small-small-big pattern is simply due to the small coefficients in the middle two equations compared with the outer two! These relations further predict that, of the two "big" amplitudes, the first will be bigger than the last; this, too, is confirmed in Nature, with the exception of the P-waves.

It is important to emphasize that the big-small-small-big pattern will be correctly reproduced, not only by the particular model written down by Skyrme, but in fact by *any* of the large class of skyrmion models whose associated reduced amplitudes satisfy the dynamical condition stated above. As such, the pattern should be considered a *quasi-model-independent* result.

Why is this significant? It is easy to convince oneself that the big-small-small-big pattern—together with the consistent isospin-splitting mentioned earlier —accounts for a good deal of the "fine structure" of the (non-strange) baryon mass table, *viz.*:

-The fact that the only reliable resonances in the H-, I- and K-waves are in the $\mathrm{S}_{L\frac{1}{2},L-\frac{1}{2}}$ and $\mathrm{S}_{L\frac{3}{2},L+\frac{1}{2}}$ channels (*i.e.*, the "big" channels);

-The fact that (ignoring the S-wave) there are 11 and 12 resonances observed in the two "big" channels, but only 5 and 6 in the two "small" channels;

♮6 The notation is the same as for Eq. (11).

-The fact that $\mathbf{S}_{L\frac{1}{2},L-\frac{1}{2}}$ resonances are lighter than their $\mathbf{S}_{L\frac{1}{2},L+\frac{1}{2}}$ partners. (However, there is no such pattern for the isospin-$\frac{3}{2}$ resonances.)

What we have seen is that these features can be *expected*, so long as the "optimal" 2-flavor nonlinear sigma model of Nature falls into the large class of models which satisfy the stated dynamical assumption. This, then, is the justification of the rather bold statement made in the Introduction: that not only the quantum numbers of the baryon resonances, but much of their relative ordering as well, finds a largely *model-independent* explanation in the context of skyrmion physics. To the extent that this is true, these features (like soft-pion results) can be considered more *basic* than the particular underlying gauge theory that gives rise to chiral symmetry breaking.

6. EXTENSION TO THREE FLAVORS

I would now like to discuss how to extend the scattering formalism to the case of three light flavors, restricting ourselves to the idealized case of exact $SU(3)_{flavor}$. The unrotated skyrmion can be embedded in an $SU(2)$ subgroup as follows:[17],[24]

$$U_0 = \exp\{iF(r)\sum_{i=1}^{3}\hat{\mathbf{r}}^i\lambda^i\} \tag{14}$$

with λ^a , $a = 1, ..., 8$, the Gell-Mann matrices. We identify the isospin subgroup as the group generated by $\frac{1}{2}\lambda^i$ with $i = 1, 2, 3$. Then the skyrmion is invariant under $\vec{K} = \vec{I} + \vec{J}$ as in the 2-flavor case. In addition, it is invariant under hypercharge Y, which is proportional to λ^8. Another novel feature of models with three or more flavors is that the Lagrangian, Eq. (1), must be augmented by the so-called Wess-Zumino term, which reproduces the correct flavor-current anomalies.[25],[17]

The general procedure outlined earlier for reducing a 3-dimensional scattering problem to a purely radial one is to expand the meson field in a complete set of eigenstates of the symmetries of the potential. In the case of scattering from an unrotated 3-flavor skyrmion, these are the meson states of definite $(\mathbf{K}^2, K_z, Y)$. Accordingly, the fluctuations about the skyrmion, which we identify with the octet of pseudoscalar mesons, decompose into the following non-interacting sectors:

(a) There are the vector spherical harmonics familiar from the 2-flavor analysis, Eq. (5), where the first, second and third components stand for fluctuations in the π^+, π^0 and π^- directions, respectively. These are definite states of $\mathbf{K}^2$ and K_z by construction, and have $Y = 0$. Since L is not conserved, $\Pi_{K-1}^{KK_z}$ and $\Pi_{K+1}^{KK_z}$ mix. By parity, neither of these mixes with $\Pi_K^{KK_z}$. We will call the associated reduced S-matrices $s_{KL'L}^{pion}$ with $K = L-1, L, L+1$. For the 3-flavor Skyrme model, these are in fact exactly the same as for the 2-flavor case; in particular, the Wess-Zumino term does not contribute.

(b) Analogously, one can define "spinor spherical harmonics"

$$\mathcal{Y}_L^{KK_z}(\Omega) = \begin{pmatrix} \langle L\frac{1}{2}, K_z - \frac{1}{2}, \frac{1}{2}|KK_z\rangle\, Y_{L,K_z-\frac{1}{2}}(\Omega) \\ \langle L\frac{1}{2}, K_z + \frac{1}{2}, -\frac{1}{2}|KK_z\rangle\, Y_{L,K_z+\frac{1}{2}}(\Omega) \end{pmatrix}, \tag{15}$$

where the first and second components stand for fluctuations in the K^+ and K^0 directions, respectively. These have $Y = 1$. By parity, $\mathcal{Y}_{K-\frac{1}{2}}^{KK_z}$ does not mix with $\mathcal{Y}_{K+\frac{1}{2}}^{KK_z}$, so that $L = L'$. The associated reduced S-matrices will be called $\mathbf{s}_{KL}^{kaon}$ with $K = L \pm \frac{1}{2}$. The Wess-Zumino term contributes a term linear in time derivatives to the differential equations that determine the meson field (from which the phase-shifts are extracted).[26]

(c) Same as above, with the first and second component now standing for $-\overline{K}^0$ and K^-. These fluctuations have $Y = -1$. The corresponding reduced S-matrices, which we shall call $\mathbf{s}_{KL}^{k\text{-}bar}$, are extracted from precisely the same differential equations as $\mathbf{s}_{KL}^{kaon}$, except that the Wess-Zumino term contributes with the opposite sign.[26] In the absence of the Wess-Zumino term, we would have $\mathbf{s}_{KL}^{k\text{-}bar}(\omega) \equiv \mathbf{s}_{KL}^{kaon}(\omega)$.

(d) Finally, fluctuations in the η direction, expanded in the usual spherical harmonics. These have $\overrightarrow{K} = \overrightarrow{L}$ and $Y = 0$. The corresponding reduced S-matrix is $\mathbf{s}_L^{eta}$. It does not receive any contribution from the Wess-Zumino term.

This decomposition allows us to characterize the S-matrix for the process $\phi^a(\overrightarrow{L}) + \text{URS} \to \phi^b(\overrightarrow{L}') + \text{URS}$, with ϕ the meson field and URS standing for "unrotated skyrmion" as before. Here a and b are $SU(3)$ flavor indices labeling the octet; each is short for the triplet of indices (i, i_z, Y). When a and b are of the form $(1, i_z, 0)$, Eq. (6) is valid. When a and b are both of the form $(\frac{1}{2}, i_z, +1)$ or $(\frac{1}{2}, i_z, -1)$, the relevant formula is

$$\delta_{LL'} \sum_{KK_z} \left\langle KK_z \middle| L\frac{1}{2}L_z, i_z \right\rangle \left\langle L'\frac{1}{2}L'_z, i'_z \middle| KK_z \right\rangle \mathbf{s}_{KL} \tag{16}$$

where the reduced amplitude stands for $\mathbf{s}_{KL}^{kaon}$ or $\mathbf{s}_{KL}^{k\text{-}bar}$, respectively. And when $a = b = (0,0,0)$, the S-matrix is just

$$\delta_{LL'}\delta_{L_zL'_z}\mathbf{s}_L^{eta}. \tag{17}$$

These formulas together form the 3-flavor analog of Eq. (6). *Physical* meson-baryon scattering, that is, scattering with the collective coordinate structure of the skyrmion properly taken into account, is of course now described by the analog of Eq. (8), with the integral now ranging over the group $SU(3)$. For this we require the correct $SU(3)$ generalization of the $SU(2)$ baryon wavefunctions, Eq. (9), which is given by:

$$\chi(A) = \frac{i}{\pi}\sqrt{\frac{\dim R}{2}}[\mathcal{D}^{(R)}(A)^{-1}]_{a,b} \cdot (-1)^{s-s_z} \tag{18}$$

where $a = (s, -s_z, 1)$, $b = (i, i_z, Y)$, and R denotes the representation of the baryon.[♮7]

As in the 2-flavor case, it is particularly illuminating to combine the initial (final) pseudoscalar octet with the baryon octet/decuplet to form a definite $SU(3)$ state in a representation R_{tot} (R'_{tot}). This is accomplished with the help of an $SU(3)$ Clebsch-Gordan coefficient[♮8]

$$\langle R_1 i_1 i_{z1} Y_1; R_2 i_2 i_{z2} Y_2 | R_{\text{tot}} \gamma I_{\text{tot}} I_{z\text{tot}} Y_{\text{tot}} \rangle$$

which can be factored conveniently into the product

$$\langle i_1 i_2 i_{z1} i_{z2} | I_{\text{tot}} I_{z\text{tot}} \rangle \cdot \left(\begin{array}{cc|c} R_1 & R_2 & R_{\text{tot}}\gamma \\ i_1 Y_1 & i_2 Y_2 & I_{\text{tot}}\ Y_{\text{tot}} \end{array} \right)$$

of an $SU(2)$ Clebsch with a so-called isoscalar factor.[27] The projection onto states of definite total angular momentum $\vec{J}$ ($\vec{J}'$) is of course accomplished with ordinary $SU(2)$ Clebsches as before.

The same manipulations as in the 2-flavor case then allow us to express the physical scattering of a pseudoscalar-octet meson in a partial wave L (L') from a spin-s (spin-s') baryon in an $SU(3)$ representation R (R'), where the initial (final) meson-baryon system is projected onto a state of definite total angular momentum $\vec{J}$ ($\vec{J}'$) and total flavor as described by the quantum-numbers $\{R_{\text{tot}}, \gamma, I_{\text{tot}}, I_{z\text{tot}},$

♮7 These differ somewhat from the wavefunctions given in Ref. 24, which have nonstandard transformation properties under **I** and **J**. The fact that the "left-handed hypercharge" is unity is a nontrivial quantization condition arising from consideration of the Wess-Zumino term.[24] Our normalization in (9) and (18) is such that $\int_{SU(2)} dA = 2\pi^2 = \int_{SU(3)} dA$.

♮8 Here γ is a largely redundant index whose only real purpose is to distinguish between degenerate representations that can occur in the product of two $SU(3)$ representations, as for example the 8_{sym} and 8_{antisym} in 8×8.[27] It is *not* in general conserved, as is clear if one considers the non-vanishing coupling $\text{Tr}(\{\overline{B}, \Phi\}[B, \Phi])$ of the baryon and meson octets.

$Y_{\rm tot}\}$ $(\{R'_{\rm tot}, \gamma', I'_{\rm tot}, I'_{z{\rm tot}}, Y'_{\rm tot}\})$. The desired expression reads:

$$
\begin{aligned}
&\mathbf{S}([LsRR_{\rm tot}\gamma I_{\rm tot}I_{z{\rm tot}}Y_{\rm tot}\vec{J}\,] \to [L's'R'R'_{\rm tot}\gamma' I'_{\rm tot}I'_{z{\rm tot}}Y'_{\rm tot}\vec{J}\,'\,]) \quad = \\
&\delta_{R_{\rm tot}R'_{\rm tot}}\,\delta_{I_{\rm tot}I'_{\rm tot}}\,\delta_{I_{z{\rm tot}}I'_{z{\rm tot}}}\,\delta_{Y_{\rm tot}Y'_{\rm tot}}\,\delta_{JJ'}\,\delta_{J_zJ'_z} \quad \times \\
&\Bigg\{\delta_{LL'}\,\delta_{ss'}\,\frac{\sqrt{\dim R\cdot\dim R'}}{\dim R_{\rm tot}}\begin{pmatrix} R_{\rm tot}\gamma' \\ s1 \end{pmatrix}\Bigg|\begin{matrix} R' & 8 \\ s1 & 00\end{pmatrix}\begin{pmatrix} R & 8 \\ s1 & 00\end{matrix}\Bigg|\begin{pmatrix} R_{\rm tot}\gamma \\ s1\end{pmatrix}\,\mathbf{s}^{eta}_{L} + \\
&\sum_{I}\sqrt{\frac{\dim R\cdot\dim R'}{(2s+1)(2s'+1)}}\,\frac{2I+1}{\dim R_{\rm tot}}\Bigg[\delta_{LL'}\begin{pmatrix} R_{\rm tot}\gamma' \\ I0\end{pmatrix}\Bigg|\begin{matrix} R' & 8 \\ s'1 & \frac{1}{2},-1\end{pmatrix} \quad \times \\
&\begin{pmatrix} R & 8 \\ s1 & \frac{1}{2},-1\end{matrix}\Bigg|\begin{pmatrix} R_{\rm tot}\gamma \\ I0\end{pmatrix}\sum_{K=L\pm\frac{1}{2}}\widetilde{P}_{LLss'IJK}\cdot\mathbf{s}^{k\text{-}bar}_{KL} \quad + \\
&\delta_{LL'}\begin{pmatrix} R_{\rm tot}\gamma' \\ I2\end{pmatrix}\Bigg|\begin{matrix} R' & 8 \\ s'1 & \frac{1}{2}1\end{pmatrix}\begin{pmatrix} R & 8 \\ s1 & \frac{1}{2}1\end{matrix}\Bigg|\begin{pmatrix} R_{\rm tot}\gamma \\ I2\end{pmatrix}\sum_{K=L\pm\frac{1}{2}}\widetilde{P}_{LLss'IJK}\cdot\mathbf{s}^{kaon}_{KL} \quad + \\
&\begin{pmatrix} R_{\rm tot}\gamma' \\ I1\end{pmatrix}\Bigg|\begin{matrix} R' & 8 \\ s'1 & 1,0\end{pmatrix}\begin{pmatrix} R & 8 \\ s1 & 1,0\end{matrix}\Bigg|\begin{pmatrix} R_{\rm tot}\gamma \\ I1\end{pmatrix}\sum_{K=L,L\pm1}P_{LL'ss'IJK}\cdot\mathbf{s}^{pion}_{KL'L}\Bigg]\Bigg\}\,.
\end{aligned}
\tag{19}
$$

This is the 3-flavor analog of (10). The $\widetilde{P}$-symbols are defined exactly like the P-symbols, Eq. (10b), with the single exception that the 1's in the 6-j symbols are to be replaced by $\frac{1}{2}$'s (reflecting the fact that kaons have isospin $\frac{1}{2}$). Only half-integral values of the index I contribute to the coefficient of $\mathbf{s}^{pion}$, while only integral values contribute for $\mathbf{s}^{kaon}$ and $\mathbf{s}^{k\text{-}bar}$. The long string of Kronecker δ's in the first line of Eq. (19) expresses the reassuring fact that total angular momentum and $SU(3)_{flavor}$ are conserved in the scattering process.

An interesting consequence of this formula is that, even if we restrict the initial and final states to pions, nucleons and deltas, one must nevertheless take into account contributions from reduced amplitudes with nonzero strangeness. One can therefore ask: Will the big-small-small-big pattern survive the extension to (unbroken) $SU(3)$? Following the reasoning of Sec. 5, let us make the dynamical assumption that, for each partial wave L, the deviations from unity of the reduced amplitudes $\mathbf{s}^{pion}_{L+1,LL}$, $\mathbf{s}^{eta}_{L}$, $\mathbf{s}^{kaon}_{L+\frac{1}{2},L}$ and $\mathbf{s}^{k\text{-}bar}_{L+\frac{1}{2},L}$ are essentially negligible compared to the other four. This is in fact numerically the case in the $SU(3)$ Skyrme model.[28] Setting these amplitudes equal to unity, one indeed finds the same pattern of bigger and smaller coefficients as in the 2-flavor case (Eq. 13), albeit with somewhat different numbers.

As in the 2-flavor case, Eq. (19) points the way to two possible avenues of inquiry. On the one hand,[28] one can calculate the reduced amplitudes in a specific model, such as the 3-flavor Skyrme model. On the other hand, one can extract from Eq. (19) a host of model-independent linear relations between scattering amplitudes in different representations of $SU(3)$ and different channels of total angular momentum, albeit in the same partial wave. Work along both lines is in progress. I shall close by considering how inclusion of a third light flavor modifies the linear relations for $\pi N \to \pi N$ as given by Eq. (11).

Let us recall the 2-flavor situation for a moment. For each partial wave L, there are four independent elastic πN amplitudes, corresponding to $I = \frac{1}{2}$ or $\frac{3}{2}$ and $J = L \pm \frac{1}{2}$. By Eq. (10), these are expressed as linear combinations of the three reduced amplitudes $\mathbf{s}_{KLL}$ with $K = L-1, L, L+1$. One therefore expects one nontrivial linear relation between physical amplitudes for each L. In fact, there are two, which enabled us in Eq. (11) to solve for the $I = \frac{3}{2}$ amplitudes as linear combinations of the $I = \frac{1}{2}$ amplitudes. In the 3-flavor case, in contrast, the same four physical πN amplitudes are expressed as a superposition of *eight* reduced S-matrix elements. As a result, there is no longer any exact linear relation between physical amplitudes.

It should therefore come as a surprise that (11a) and (11b) are nevertheless *almost* satisfied in the 3-flavor case. Specifically, the right-hand sides, instead of vanishing, are proportional to the differences of the reduced amplitudes $\mathbf{s}^{kaon}_{L\pm\frac{1}{2},L} - \mathbf{s}^{k\text{-}bar}_{L\pm\frac{1}{2},L}$. Recall that these differences would vanish identically were it not for the presence in the 3-flavor Lagrangian of the Wess-Zumino term, and can therefore be expected to be small.[♯9] But it is certainly an intriguing possibility that the Wess-Zumino term, which is forced on us only when we consider the existence of a third light flavor, might partially account for discrepancies in purely 2-flavor predictions.

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♯9 This is numerically the case in the 3-flavor Skyrme model.[28]

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