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σ model approach to the heterotic string theory*

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ABSTRACT

Relation between the equations of motion for the massless fields in the heterotic string theory, and the conformal invariance of the σ model describing the propagation of the heterotic string in arbitrary background massless fields is discussed. It is emphasized that this σ model contains complete information about the string theory. Finally we discuss the extension of the Hull-Witten proof of local gauge and Lorentz invariance of the σ-model to higher order in α', and the modification of the transformation laws of the antisymmetric tensor field under these symmetries. Presence of anomaly in the naive $N = \frac{1}{2}$ supersymmetry transformation is also pointed out in this context.

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I shall begin my talk by discussing the relation between the fixed point equations of the σ model and the equations of motion for the various massless fields in the heterotic string theory.[1)–3)] I shall work in the light cone gauge and consider a background where the graviton field $g_{ij}(x)$, the antisymmetric tensor field $B_{ij}(x)$, and the gauge field $A_i^M(x)$ acquire vacuum expectation value (vev) only in the eight transverse directions and are independent of the longitudinal coordinates x^0 and x^9. The dilaton field ϕ is taken to be independent of all space-time coordinates, in which case it may be absorbed in various fields and coupling constants[4)] and never appear explicitly in our analysis.* The action for the first quantized heterotic string[5)] in such a background is given by,[1)–3),6)]

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\begin{aligned}
S =& \frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma\Big(g_{ij}(X)\partial_\alpha X^i\partial^\alpha X^j + \varepsilon^{\alpha\beta}B_{ij}(X)\partial_\alpha X^i\partial_\beta X^j + ig_{ij}(X)[\bar\lambda^i\partial\!\!\!/\lambda^j \\
&+ \bar\lambda^i(\Gamma^j_{k\ell}(X) + S^j_{k\ell}(X))\rho^\alpha\lambda^\ell\partial_\alpha X^k] + \bar\psi^s(i\partial\!\!\!/\delta_{st} + A_i^M(X)(T^M)_{st}\rho^\alpha\partial_\alpha X^i)\psi^t \\
&+ \frac{i}{4}F_{ij}^M(X)\bar\psi^s\rho^\alpha(T^M)_{st}\psi^t\bar\lambda^i\rho_\alpha\lambda^j\Big).
\end{aligned}
\tag{1}
$$

where α' is the string tension, X^i are the eight bosonic fields, λ^i are the eight left-handed Majorana-Weyl spinors and ψ^s are the 32 right-handed Majorana-Weyl spinors respectively. We are working in the Neveu-Schwarz-Ramond representation, so that the λ^i's transform in the vector representation of $SO(8)$, whereas the ψ^s's transform in the 32 representation of $SO(32)$ or (16,1)+(1,16) representation of the $SO(16)\otimes SO(16)$ subgroup of $E_8\otimes E_8$. Also here,

$$S_{ijk} = \frac{1}{2}(\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}). \tag{2}$$

$$\Gamma_{ijk} = \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}). \tag{3}$$

and F_{ij}^M is the field strength associated with the vector potential A_i^M. The action

* These constraints on the background fields are needed to ensure that the 0 and 9 directions remain flat as a solution of the classical equations of motion, thus allowing us to choose the light-cone gauge.

(1) has an $N=\frac{1}{2}$ supersymmetry:[‡]

$$\begin{aligned}\delta X^i &= i\epsilon\lambda^i; \quad \delta\lambda^i = -(\partial_\tau - \partial_\sigma)X^i\epsilon.\\ \delta\psi^s &= (-\epsilon\lambda^i A_i^M)(T^M)_{st}\psi^t\end{aligned} \tag{4}$$

For a consistent formulation of the string theory in a given background, we require the sigma model described in Eq.(1) to be conformally invariant. This requires all the β-functions of the theory to vanish. Since there are three independent sets of dimension two operators in the theory, namely, $\partial_\alpha X^i \partial^\alpha X^j$, $\varepsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j$, and $\bar{\psi}^s (T^M)_{st}\rho^\alpha \psi^t \partial_\alpha X^i$, that are not related to each other by supersymmetry transformation, we get three different sets of consistency conditions on the background fields. A fourth consistency condition comes from the requirement that the central charge of the Virasoro algebra in this conformally invariant field theory should be the same as in the corresponding free field theory. We have carried out a complete one loop calculation and part of the two loop calculation in this model. The consistency conditions turn out to be,

$$R_{i\ell} + S_{ikm}S_\ell^{km} + \frac{\alpha'}{4}(F_{ki}^M F_\ell^{Mk} - R_{imnp}R_\ell^{mnp}) + \cdots = 0, \tag{5}$$

$$D^i\left(S_{ijk} + \frac{\alpha'}{8}\left[(A_i^M F_{\ell k}^M + A_\ell^M F_{ki}^M + A_k^M F_{i\ell}^M) - \begin{pmatrix} A \to \omega \\ F \to R \end{pmatrix}\right]\right) + \ldots = 0 \tag{6}$$

$$D^k F_{k\ell}^M - f^{MNP} A^{Nk} F_{k\ell}^P - S_\ell^{ij} F_{ij}^M = 0 \tag{7}$$

$$R + \frac{1}{3}S^2 = 0 \tag{8}$$

where ω is the spin connection and f^{MNP} are the structure constants of the group. Here ...denotes terms from two loop contribution of order A^3 and ω^3,

‡ The transformation law of ψ given here was not needed in Ref. 1 to prove supersymmetry of the action (1), since we used the equations of motion of ψ in our proof. If we do not use the equations of motion of the ψ fields we need to use the explicit transformation laws of ψ given here.

as well as terms which vanish when we set the background antisymmetric tensor field and the Ricci tensor to zero. Eq.(7) contains only the complete one loop result, whereas Eq.(8) contains the complete two loop result. These equations turn out to be identical to the equations of motion for the massless fields derived from the Green-Schwarz, Gross-Witten modified Chapline-Manton action[7) :

$$S_{eff} = \int e^{\phi}\sqrt{g}[R + \frac{1}{3}H^2 + \frac{\alpha'}{8}(F^M_{ki}F^{Mki} - R_{imnp}R^{imnp})]d^{10}x \tag{9}$$

where,

$$H_{ijk} = S_{ijk} + \frac{\alpha'}{8}\Omega_3(A)_{ijk} - \frac{\alpha'}{8}\Omega_3(\omega)_{ijk} \tag{10}$$

$$\Omega_3(A)_{ijk} = \frac{1}{2}\Big[A^M_{[i}F^M_{jk]} - \frac{2i}{3}A^M_{[i}A^N_jA^P_{k]}Tr(T^MT^NT^P)\Big]. \tag{11}$$

and $\Omega_3(\omega)$ is obtained by replacing A^M_i by the spin connection ω in Eq.(11). From this we conjecture that there is an exact one to one correspondence between the consistency conditions for the propagation of a string in a given background, and the classical equations of motion for the massless fields derived from the ten dimensional effective action. The higher loop corrections in the σ model will correspond to the higher dimensional operators in the effective action for the massless fields.

I now want to emphasize the following points:

1) If we set the gauge connection to be equal to the spin connection, and set B_{ij} to zero, the action (1) reduces to that of an $N = 1$ supersymmetric non-linear σ model, plus the action for free fermions. Such models are known to have vanishing β- function if the background is Ricci flat and Kahler[8) .

2) The exact one to one correspondence between the classical equations of motion and the equations for the vanishing of the β-function tells us that every solution of the classical equations of motion provide a consistent background for the formulation of the string theory. This includes not only the vacuum solution,

but also various topological and non-topological excitations around the vacuum, e.g. a classical monopole solution.

3) Action (1) is classically invariant under a gauge symmetry transformation,

$$\begin{aligned} A_i^M(X)T^M \rightarrow A_i^{M\prime}(X)T^M =& U(X)A_i^M(X)T^M U^{-1}(X) + U(X)i\partial_i U^{-1}(X), \\ \psi \rightarrow \psi' =& U(X)\psi, \end{aligned} \tag{12}$$

where U(X) is any map from the manifold spanned by the coordinates X^i to the gauge group $E_8 \otimes E_8$ or $SO(32)$. Thus naively one would expect the equations for the vanishing of the β-functions to be invariant under the above symmetry transformation. Eq.(6), however, is not invariant under such symmetry, due to the presence of the gauge non-invariant terms of the form $A^M_{[i}F^M_{\ell k]}$. This is due to the fact that the symmetry (12) is anomalous due to the chiral nature of the fermions ψ^s. This is also responsible for the appearance of the Chern-Simons term in the effective action (9), which is not explicitly gauge invariant. Similar remarks hold also for the local Lorentz transformations.

4) Next I want to point out that the criterion that the fixed point equations of the σ-model are derivable from an action is a very strong constraint on the σ-model itself, and is probably true only for those σ-models which represent the propagation of strings in background fields. For example, in the σ-model approach, the $S^{ij}_\ell F^M_{ij}$ term in Eq.(7) appears from one loop fermion self-energy graphs, whereas the $D^i(A^M_{[i}F^M_{jk]})$ term in Eq.(6) appears from a two loop graph, one of whose internal loop is a fermion loop with anomalous contribution. When we derive these equations from the effective action (9), both these terms come from the variation of the $S_{ijk}\Omega_3(A)^{ijk}$ term in (9). Thus the criterion for the fixed point equations to be derivable from an effective action relates a non-anomalous one-loop contribution to the β-function to an anomalous two loop contribution. In particular, if we construct a new σ-model by adding 32 left-handed fermions which couple to the gauge field $A_i^M(X)$ in the same way as the 32 right-handed fermions ψ^s, the gauge symmetry (12) ceases to be anomalous,

and the $D^i(A^M_{[i}F^M_{jk]})$ term must disappear from Eq.(6). The $S^{ij}_{\ell}F^M_{ij}$ term, coming from the one-loop fermion self-energy contribution, knows nothing about the addition of the new fermions, and continues to be present in Eq.(7). Thus in the new σ-model constructed this way the fixed point equations are no longer derivable from an action.

5) The action (9) contains cubic as well as quartic and higher order terms in the massless fields, occuring due to the interchange of heavy intermediate states. This indicates that the calculation of the σ-model β-functions to all orders in the perturbation theory reproduces the full effective action for the massless fields, obtained by summing all the tree graphs with massless fields as external lines and massive fields as internal lines. (This is also equivalent to constructing the effective action by eliminating the massive fields by their classical equations of motion). In string theory, knowing this effective action we may calculate the scattering amplitude involving arbitrary external massless and massive states, by using the factorization properties of the amplitudes. Hence the σ-model described by the action (1) contains complete information about the heterotic string theory,* although we have not coupled the string to the massive fields explicitly.

Now I want to show how the result that the fixed point equations of the σ-model are identical to the classical equations of motion for the massless fields may be used to derive non-trivial information about the string effective action. We shall show, for example, that the Lorentz Chern-Simons term in the effective action must have as its argument the generalized spin connection which includes torsion.† Among other things, this will imply that the consistency condition[4)] $\int F\wedge F - \int R\wedge R = 0$ must be replaced by $\int F\wedge F - \int \tilde{R}\wedge\tilde{R} = 0$ in the presence of torsion, where $\tilde{R}$ is the generalized curvature. The simplest way to see why

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⋆ This statement is not completely correct, since (1) does not contain the most general massless background fields. This difficulty may be avoided by working in the conformal gauge as in Ref.2.

† This investigation was inspired by a queation asked by A. Strominger during the talk.

this should be so is to write the part of the action (1) quadratic in λ as,

$$i\bar{\lambda}^a[\partial\!\!\!/\delta_{ab} + \rho^\alpha(\omega_i^{ab} - S_i^{ab})\partial_\alpha X^i]\lambda^b \equiv i\bar{\lambda}^a[\partial\!\!\!/\delta_{ab} + \rho^\alpha\bar{\omega}_i^{ab}\partial_\alpha X^i]\lambda^b \tag{13}$$

where a and b are tangent space indices, ω is the ordinary spin connection, and $\bar{\omega}$ is the generalized spin connection. [The only other term in the action involving λ is the four-fermion coupling, but we may ignore it completely in a two loop calculation of the β-function]. During the calculation of the β-function using the background field method,[9] we may treat $\bar{\omega}$ as a new parameter, independent of g_{ij} and B_{ij}. The connections $\bar{\omega}_i^{ab}$ and A_i^M then appear in the σ-model lagrangian exactly in the same way except for the fact that $\bar{\omega}$ couples to the left handed fermions λ, whereas A_i^M couples to the right handed fermions ψ^s. The presence of the Chern-Simons term involving A_i^M in the fixed point equations then automatically implies the presence of a similar Chern-Simons term with $\bar{\omega}$ as its argument.*

This result leads us naturally to ask whether the torsion S_i^{ab} appearing in the expression for $\bar{\omega}_i^{ab}$ should be replaced by the covariant torsion H_i^{ab} defined in Eq.(10) when we calculate the higher order terms in the β-function. The answer to this question is connected intimately as to how the local gauge and Lorentz invariance is restored in higher orders in the world sheet perturbation theory. Under local gauge and Lorentz transformation of the background fields, the one loop effective action involving the bosonic fields transforms as[10] ,

$$\delta S^{(1-loop)} = \frac{1}{8\pi}\int d\tau \int d\sigma\ \varepsilon^{\alpha\beta}(\partial_\alpha\theta^M A_i^M - \partial_\alpha\theta^{ab}\bar{\omega}_i^{ab})\partial_\beta X^i \tag{14}$$

where θ^M and θ^{ab} are the gauge and Lorentz transformation parameters respectively. As was pointed out by Hull and Witten[11] , the anomalous variation of

* This can also be seen from the analysis of Ref. 11.

the effective action to one loop order may be cancelled by redefining the transformation laws of B_{ij} under local Lorentz and gauge transformations:

$$\delta B_{ij} = \frac{\alpha'}{4}(\theta^M \partial_{[i} A^M_{j]} - \theta^{ab}\partial_{[i}\bar{\omega}^{ab}_{j]}) \tag{15}$$

This anomalous variation of B_{ij}, however, induces an anomalous variation of S_{ijk} and hence also an anomalous variation of $\bar{\omega}$, which induces a further variation of the action in order α'^2. A simple way to get rid of this problem is to replace S^{ab}_i by H^{ab}_i in the original σ-model lagrangian, since H transforms covariantly under local gauge and Lorentz transformation. The new transformation law of B_{ij} is then given by Eq.(15) with $\bar{\omega}$ replaced by $\omega - H$, and the covariant torsion H_{ijk} is now determined from the equation,

$$H_{ijk} = \partial_{[i} B_{jk]} + \frac{\alpha'}{8}[\Omega_3(A) - \Omega_3(\omega - H)]_{ijk} \tag{16}$$

which can be solved iteratively for H.

In order to restore local gauge and Lorentz invariance in higher order in α', we must also take care of the fact that the presence of the four fermion coupling in (1) gives rise to new contribution to local Lorentz and gauge anomaly other than those discussed in Ref. 11. This may be analyzed by introducing auxiliary fields S^{ab}_α, R^{ab}_α, and replacing the four fermion coupling term in (1) by,

$$-\frac{1}{4\pi\alpha'}[S^{ab}_\alpha F^M_{ab}\bar{\psi}T^M\rho^\alpha\psi + iR^{ab}_\alpha\bar{\lambda}^a\rho^\alpha\lambda^b + 4S^{ab}_\alpha R^{ab\alpha}] \tag{17}$$

where S and R are defined to transform covariantly under the local gauge and Lorentz transformation. We may now construct an effective action involving the fields X^i, S^{ab}_α and R^{ab}_α by integrating out the ψ and λ fields. Since the connections coupling to ψ and λ fields contain new terms proportional to S and R respectively, the variation of this effective action under local gauge and Lorentz transformation now contains new terms given by,

$$-\frac{1}{8\pi}\int d\tau \int d\sigma\ \varepsilon^{\alpha\beta}(\partial_\alpha \theta^M F^M_{ab} S^{ab}_\beta - i\partial_\alpha \theta^{ab} R^{ab}_\beta) \tag{18}$$

besides those given in Eq. (14). This extra variation may be cancelled by adding new terms to the lagrangian given by,

$$\frac{1}{8\pi}\int d\tau \int d\sigma\ \varepsilon^{\alpha\beta}\partial_\alpha X^i (A^M_i F^M_{ab} S^{ab}_\beta - i\omega^{ab}_i R^{ab}_\beta) \tag{19}$$

Adding (19) to (17) and eliminating the auxiliary fields by their equations of motion we get the following extra terms in the action besides the four fermion coupling:

$$-\frac{i}{32\pi} F^M_{ab}[A^M_k \partial_\beta X^k \bar{\lambda}^a \rho_\alpha \lambda^b \varepsilon^{\alpha\beta} - \omega^{ab}_i \partial_\beta X^i \bar{\psi}\rho_\alpha T^M \psi \varepsilon^{\alpha\beta} + \frac{\alpha'}{4}\partial_\alpha X^i \partial^\alpha X^k A^M_k \omega^{ab}_i]. \tag{20}$$

The addition of these new terms, as well as the replacement of S by H in the σ-model action destroys the naive $N = \frac{1}{2}$ supersymmetry of the action. This symmetry, however, is anomalous,* since it involves field dependent phase transformation of the chiral fermions:

$$\begin{aligned} \delta\lambda^a &= e^a_{i,j}(i\epsilon\lambda^j)\lambda^i - e^a_i(\partial_\tau - \partial_\sigma)X^i\epsilon \\ \delta\psi^s &= (-\epsilon\lambda^i A^M_i)(T^M)_{st}\psi^t \end{aligned} \tag{21}$$

It is conceivable that the extra terms added to the lagrangian in order to restore the ten dimensional local gauge and Lorentz invariance will also restore the two dimensional $N = \frac{1}{2}$ supersymmetry.

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⋆ This has been noted by Attick, Dhar and Ratra in a different context[12].

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