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SUPERSYMMETRY AMONG FREE FERMIONS AND SUPERSTRINGS

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ABSTRACT

We give a complete classification of all supersymmetric theories of free massless two-dimensional fermions. This, in particular, implies a classification of all free-fermion representations of super Kac-Moody algebras. We show that these cannot be used to construct new string theories with unbroken supersymmetry in Minkowski space-time, other than the torus-compactifications of the known ten-dimensional superstrings. Assuming anti-de-Sitter spacetime could restore conformal invariance, we show how one could construct a string theory whose low-lying excitations form a multiplet of gauged $N = 8$ supergravity.

Supersymmetric string theories[1] have recently attracted a lot of attention[2] as candidates for a consistent unification of all interactions, including gravity.[3,1] A central and still open problem is how to build realistic four-dimensional string theories. In so doing, one must respect world-sheet reparametrization invariance and ensure the presence of massless space-time fermions. Together, these requirements impose severe restrictions on the allowed vaccua of the theory of strings.

In this paper we will examine the simplest possible proposal: Can a string theory in four flat dimensions be made consistent by adding only free fermions on the world-sheet, which carry all extra quantum numbers of the string? We must insist that these two-dimensional fermions provide a (non-linear) realization of world-sheet supersymmetry, which seems to be necessary for eliminating the tachyon.[1] Our results can be summarized as follows:

First we will give a complete classification of supersymmetric theories of free massless two-dimensional fermions. We will show that one such theory can be constructed for each pair of semi-simple Lie groups G, H with $H \subset G$ and G/H a symmetric space (this includes the case $G = H$). The symmetry group of these theories is H. This result is of mathematical interest in its own right since it implies a classification of all representations of super-Kac-Moody algebras[4,5] by free two-dimensional fermions. The theories we find include, but are more general than, the supersymmetric extension of the non-linear σ-models on group manifolds with Wess-Zumino term with coefficient one at their conformal fixed point.[6] Nevertheless, they are necessarily characterized by some symmetry group H and the corresponding super-Kac-Moody algebra. This then suffices to show that they cannot be used to reduce the critical dimension of Minkowski space-time in string theories, to less than ten, since they always lead to massive fermions, broken supersymmetry at the Planck scale, and furthermore they might be inconsistent. The argument is due to Friedan, Qiu and Shenker,[7] and is briefly reviewed in the second part of our paper. Thus, other than the known 10-dimensional superstrings, and their trivial torus-compactifications, no new string theories in flat space-time can be constructed by using only free world-sheet fermions. These considerations do not in general apply, if one allows a negatively curved (anti-de-Sitter) spacetime. Whether reparametrization invariance can then be ensured is an open, highly non-trivial question, but assuming it can, we will show in the last (conjectural) part of this paper, how to construct a string theory whose low-lying spectrum is that of gauged $N = 8$, $d = 4$ supergravity.

SUPERSYMMETRY AMONG FREE FERMIONS

Let us begin by considering N free Weyl-Majorana fermions in two dimensions, whose Euclidean action is:

$$S = \frac{1}{2} \int dz d\bar{z} \; \psi^A \partial_{\bar{z}} \psi^A \tag{1}$$

where $z = x + it$, $A = 1, \cdots, N$ and summation over repeated indices is implied. This action is invariant under:

$$\delta \psi^A = \eta^{ABC} \psi^B \psi^C \epsilon \tag{2}$$

(with ϵ an infinitesimal Grassman parameter) if and only if η^{ABC} is totally antisymmetric in its indices. We now prove a:

Theorem:

Transformation (2) is a supersymmetry if and only if the η^{ABC} are appropriately normalized structure constants of a semi-simple Lie group G.

Proof:

Indeed, let us first note that the current, which generates local transformations of type (2) can be written:

$$T_F(z) = \frac{1}{3} \, \eta^{ABC} \psi^A_{(z)} \psi^B_{(z)} \psi^C_{(z)} \tag{3}$$

where normal ordering is implied for operators multiplied at the same point. Now (2) is a real supersymmetry, if and only if the anticommutator of two such currents gives the energy-momentum density up to a possible c-number anomaly:

$$[T_F(z), T_F(w)]_+ = 2T_B(z)\delta(z-w) + c\text{-number} \tag{4}$$

where:

$$T_B(z) = \frac{1}{2}\,\psi^A \partial_z \psi^A$$

This condition is equivalent to requiring that the commutator of two supersymmetry transformations be a translation, as it should. Its validity can most easily be examined by means of the operator-product expansion:

$$\begin{aligned} T_F(z) \underset{z\to w}{T_F(w)} \sim\ & -\frac{2}{3}\,\frac{\eta^{ABC}\,\eta^{ABC}}{(z-w)^3} \\ & - 2\eta^{ACD}\,\eta^{BCD}\left[\frac{\psi^A(w)\,\psi^B(w)}{(z-w)^2} - \frac{\psi^A(w)\,\partial_w\,\psi^B(w)}{(z-w)}\right] \\ & + \eta^{ABE}\,\eta^{CDE}\,\frac{\psi^A(w)\,\psi^B(w)\,\psi^C(w)\,\psi^D(w)}{(z-w)} \\ & + \text{ regular terms} \end{aligned} \tag{5}$$

where we have here used Wick's theorem, and the free-fermion contraction:

$$\overbrace{\psi^A(z)\ \psi^B}(w) = \frac{\delta^{AB}}{z-w}$$

Since normal-ordered fermions anticommute, it follows immediately that the right-hand side of (5) agrees with (4) if and only if:

$$\eta^{ABE}\eta^{CDE} + \eta^{ACE}\eta^{DBE} + \eta^{ADE}\eta^{BCE} = 0 \tag{6a}$$

$$\text{and} \qquad \eta^{ACD}\eta^{BCD} = \frac{1}{2}\delta^{AB} \tag{6b}$$

The first condition is the Jacobi identity, implying that η^{ABC} are the structure constants of a Lie group G, while the second guarantees that G has no normal abelian subgroup, which completes the proof. Note that the η^{ABC} are related to the conventionally normalized structure constants f^{ABC} according to:

$$\eta^{ABC} = \frac{1}{\sqrt{2c(G)}} f^{ABC} \quad ; \qquad c(G) = f^{ABC} f^{ABC} / \dim\,(G)$$

with dim (G) the dimension of the Lie algebra of G.

The fact that supersymmetry can be non-linearly realized among free fermions was noticed for the first time in Ref. 4. The new result obtained above is that the structure constants of semi-simple groups G, exhaust all possible supersymmetries among free fermions. In particular, the non-linear transformation among fermions in the adjoint and any other representation r of a group G', proposed by di Vecchia et al.[4] is not in general a real supersymmetry. It can be modified into a real one if and only if r and adj G' can be combined to form the adjoint of a larger semi-simple group $G \supset G'$.

We will now consider the most general truncation of these free fermionic theories consistent with supersymmetry. Indeed, let us first recall that the algebra of the super-energy momentum tensor

$$T(z,\theta) = T_F(z) + \theta T_B(z)$$

is the well-known superconformal (or Virasoro-Neveu-Schwarz-Ramond) algebra,[8,9] uniquely characterized by the anomaly, that is the coefficient of the leading pole in the operator product-expansion of, for instance, the bosonic parts:

$$T_B(z) \underset{z \to w}{T_B(w)} \sim \frac{c}{2(z-w)^4} - \frac{2T_B(w)}{(z-w)^2} - \frac{\partial_w T_B(w)}{z-w} + \text{regular terms}$$

$$\text{with} \qquad c = \frac{1}{2} N = \frac{1}{2} \dim\,(G) \quad .$$

In addition, one has the infinite-dimensional super-Kac-Moody algebra of the supercurrents of G:

$$J^A(z,\theta) = \sqrt{\frac{1}{2}c(G)}\;\psi^A_{(z)} + \theta J^A(z) \quad \text{with} \quad J^A(z) = \frac{1}{2}\, f^{ABC}\psi^B_{(z)}\psi^C_{(z)}$$

likewise characterized by an anomaly that can be straight forwardly computed:

$$J^A(z)J^B(w) \underset{z\to w}{\sim} -\frac{\kappa\delta^{AB}}{(z-w)^2} - f^{ABC}\,\frac{J^C(w)}{z-w} + \text{ regular terms}$$

$$\text{with} \qquad \kappa = \frac{1}{2}\, c(G) \quad .$$

The free-fermion Hilbert space forms an irreducible representation of the (semidirect) product of these algebras, but a reducible in general representation of the superconformal algebra alone. Indeed, let us suppose that the fermions ψ^A can be separated in two sets, the "real fermions" ψ^a $(a = 1, \ldots, n)$ and the "pseudofermions" ψ^i $(i = n+1, \ldots, N = \text{dim } G)$, such that the latter always appear in pairs in the supersymmetry generator (3), i.e.

$$f^{ijk} = f^{iab} = 0 \tag{7}$$

The pseudofermion number $(-)^{F_{pseudo}}$ is then multiplicatively conserved by supersymmetry transformations, so that the Hilbert space can be consistently truncated to even pseudofermion parity:

$$(-)^{F_{\text{pseudo}}} = +1 \quad .$$

Conditions (7) imply that the real fermions transform in the adjoint of a subgroup $H \subset G$, such that G/H is a symmetric space. The truncated Hilbert space is then an irreducible representation of the semi-direct product of the superconformal algebra, and the super-Kac Moody algebra of H with anomalies

$c = \frac{1}{2} \dim G$, and $\kappa' = \frac{1}{2}c(G)$, as can be easily computed. Note that since the supercurrents $J^i(z,\theta)$, corresponding to G/H, have odd pseudofermion parity, they do not exist as operators in the truncated theory: the basic superfields are then the supercurrents of H. Thus the truncation has broken the G-invariance down to H. We have then proven a:

Lemma:

Given the free fermion supersymmetric theory (1), a new supersymmetric model can be constructed for any subgroup H, such that G/H is a symmetric space, by imposing even parity on all fermions other than those transforming in the adjoint of H. This truncation breaks the invariance from G to H.

Note that for $G/H = O(M+1)/O(M)$ the pseudofermions transform in the fundamental representation of $O(M)$, and we recover the Wess-Zumino model on the group manifold of $O(M)$ with unit Wess-Zumino coefficient.[6] More generally one should expect that in the truncated theory the pseudofermions can be effectively bosonized, but we won't examine this here.

As is well known, given a two-dimensional superconformal model, one can construct another local field theory, the spin-model, by introducing an extra (Ramond) sector in which the fermionic parts of all superfields are double-valued,[7] and then projecting onto even ("real") fermion number. In a cylindric geometry, the Ramond sector corresponds to periodic boundary conditions[9] for the fermionic parts of the superfields, so that in the models we are considering here, one has $n = \dim H$ fermionic zero modes in the Ramond sector. This is of crucial importance when one tries to construct consistent string theories, as we will now proceed to explain.

APPLICATION TO SUPERSTRINGS

Following the approach of Polyakov,[10] to write down a consistent string theory, one starts with a two-dimensional scale-invariant supersymmetric field theory and, to ensure world-sheet (super)-reparametrization invariance, couples it to the two-dimensional graviton and gravitino. These latter are auxiliary fields and do not propagate, provided the two-dimensional supergravity has vanishing trace (conformal) anomaly. In the simplest case of D free real superfields (X^μ, ψ^μ) with $\mu = 1, 2, \ldots, D$ for instance, the anomaly cancellation condition is:

$$-26 + 11 + \frac{3}{2}\, D = 0 \tag{8}$$

where the graviton and gravitino ghosts in the superconformal gauge contribute -26 and 11 respectively, and each free boson and Majorana fermion 1 and 1/2. The solution is the well-known $D = 10$ supersymmetric string. The critical dimension does not change in the heterotic model,[11] whose beautiful construction is based on the observation that supersymmetry among only the left-moving modes is sufficient in order that the resulting string-theory be tachyon-free. Thus both the gravitino and the superpartners of the string coordinates are left Weyl-Majorana fermions. There now exists a potential two-dimensional gravitational anomaly,[12,11] however, whose cancellation is equivalent to requiring that the trace (conformal) anomaly separately cancel among left and right-moving modes. This brings us back to condition (8) and the critical dimension $D = 10$. In what follows we will concentrate on left-right symmetric models, although our discussion can be easily taken over to the heterotic case.

In order to reduce the critical dimension of the string theory let us now consider, in addition to the D string super-coordinates (X^μ, ψ^μ), extra free world-sheet fermions,[13] among which supersymmetry is (non-linearly) realized. Following our previous discussion, these fermions are then in the adjoint of a semi-simple Lie group G, whose dimension is fixed by the anomaly-cancellation condition:

$$\frac{1}{2}\, \dim G = \left(15 - \frac{3}{2}\, D\right) \tag{9}$$

so that $\dim G = 18$ in $D = 4$ dimensions, 12 in $D = 6$ and so on.

Next, we consider the spectrum of string excitations; in the covariant formulation, the mass-shell condition follows from the requirement of conformal invariance of physical states:[7]

$$\left(L_0^{\text{matter}} + L_0^{\text{ghost}}\right) |\text{phys}\rangle = 0 \tag{10}$$

Here $L_0^{\text{ghost}} = \oint \frac{dz \cdot z}{2\pi i} T_B^{\text{ghost}}(z)$ with T_B^{ghost} the energy-momentum tensor of all ghost-fields in the superconformal gauge, while L_0^{matter} is similarly defined for all remaining (non-ghost) fields. The contribution of L_0^{ghost} in Eq. (10) is universal: it only depends on the ghost content, but not on the particular physical state, nor the specific details of the string theory. It can be computed by requiring all ghost-destruction operators to annihilate physical states, with the result $L_0^{\text{ghost}} = -1/2$, $-5/8$ and -1 for the Neveu-Schwarz, Ramond and non-supersymmetric (in the case of heterotic strings) sectors respectively.[7,14] The contribution of L_0^{matter} on the other hand, equals the conformal weight of the field that creates the physical state out of the vacuum. This must include the field $: e^{iP^\mu X_\mu} :$ of conformal weight $+\frac{1}{2}P^2$, that injects momentum into the state in order to represent the Poincaré algebra of bosonic zero modes, and, in the Ramond sector only, a spin-field[7] Θ of conformal weight h_θ, which represents the $(D + \dim H)$-dimensional Clifford algebra of fermionic zero-modes:*

$$[\psi_0^\mu, \psi_0^\nu]_+ = g^{\mu\nu}$$

$$[\psi_0^a, \psi_0^b]_+ = \delta^{ab}$$

$$[\psi_0^\mu, \psi_0^a]_+ = 0$$

The mass-shell condition for the lowest-lying states in the Ramond sector therefore reads:

$$\left(+\frac{1}{2}P^2 + h_\theta - \frac{5}{8}\right) \left| \begin{matrix} \text{lowest} \\ \text{Ramond state} \end{matrix} \right\rangle = 0 \tag{11}$$

Assuming that the spin-field is constructed out of the fermionic parts of the

* Our convention for the signature of $g^{\mu\nu}$ is $(- + + +)$, so that the invariant mass is $-P^2$.

superfields alone, we can easily calculate h_θ by noting that the energy-momentum tensor of the $\ell \equiv D +$ dim H free fermions can be written in the form:

$$T(z) = \frac{1}{\ell - 1} : J^\alpha(z) J^\alpha(z) :$$

where $J^\alpha(z)$ are the $O(1,\, \ell - 1)$ currents and normal-ordering is with respect to the current-quanta. It is then straightforward to compute the commutator $[L_0^{\mathrm{matter}}, \Theta]$ using the fact that Θ transforms in the lowest spinor representation of $O(1, \ell - 1)$, with the result:[15]

$$h_\theta = \frac{c(\mathrm{spinor})}{\ell - 1} = \frac{\ell}{16} \tag{12}$$

where c(spinor) is the Casimir of the spinor representation. Combining (11) and (12) we finally deduce that the condition for the existence of massless space-time fermions, and unbroken supersymmetry at the Plank scale reads:

$$\dim H = 10 - D \tag{13}$$

This, together with the anomaly-cancellation condition, Eq. (9), implies:

$$\frac{\dim G}{\dim H} = 3$$

As can be shown by inspection, the only symmetric space satisfying this condition is:

$$\frac{G}{H} = \prod^{10-D} \otimes O(3) \Big/ \prod^{10-D} \otimes O(2)$$

which can be easily seen, by bosonization, to correspond to a trivial compactification of the ten-dimensional superstring on a $(10 - D)$-dimensional torus.[1] Note, however that in the fermionic construction given here, the radii of the torus are automatically equal to one (in units where $2\alpha' = 1$ with α' the Regge slope). This then completes our demonstration that no new string theories with massless fermions in Minkowski space-time can be constructed by using only free world-sheet fermions in addition to the string supercoordinates.

These considerations do not in general apply, if we allow the D-dimensional space-time to be negatively curved. The anomaly cancellation condition, Eq. (9), would then change into a set of functional equations that are only known to leading order in the σ-model loop expansion,[16] an expansion that is not valid when the space-time curvature is itself of order the Regge slope α. Even though we are, thus, unable to decide the issue of conformal invariance, let us for the moment assume that a solution with four-dimensional anti-de-Sitter spacetime, and $H = O(4)$ (and for instance $G = O(5)$ or $SU(4)$) exists.† We will then show, as a concluding remark, that the lowest lying string excitations have the correct quantum numbers to form the multiplet of $N = 8$ gauged supergravity.[17]

Indeed, the lowest lying states in the left Neveu-Schwarz sector consist of a vector, and six scalars in the adjoint of $H = O(4)$. The lowest-lying left Ramond state, on the other hand, is an $O(1,9)$ spinor. To find its transformation properties under $H \simeq O(3) \times \overline{O(3)}$, note first that one can take $\psi_\circ^\mu$, $\eta^{ia}\psi_\circ^a$ and $\bar{\eta}^{ja}\psi_\circ^a$ as the generators of the ten-dimensional Clifford algebra, where η^{ia} and $\bar{\eta}^{ja}$ are the tensors that project a vector of O(3) or $\overline{O(3)}$ out of the adjoint of $H = O(4)$, and satisfy the orthonormality properties $\eta^{ia}\eta^{ja} = \bar{\eta}^{ia}\bar{\eta}^{ja} = \delta^{ij}$ and $\eta^{ia}\bar{\eta}^{ja} = 0$. Thus a vector of O(1, 9) breaks into a direct sum of vectors of Lorentz = O(1, 3), O(3) and $\overline{O(3)}$. Consequently a spinor of O(1, 9) transforms as a (spinor, spinor, spinor) of $O(1,\ 3) \times O(3) \times \overline{O(3)}$ or, equivalently a (spinor, fundamental) of $O(1,\ 3) \times H$.

Now the states of the closed string are obtained by taking the direct product of the identical left and right sectors, and thus carry indices of the symmetry group $H_{\text{left}} \times H_{\text{right}}$. Consider for instance the massless vector bosons: as shown in Fig. 1 these transform in the $(1, 6) \oplus (6, 1) \oplus (4, 4)$ representation of $O(4)_{\text{left}} \times O(4)_{\text{right}}$, which is precisely how the adjoint of O(8) (or O(4, 4)) decomposes under these groups. The reader can likewise convince himself that all states have the correct

† To lowest order in the σ-model loop expansion, the anomaly cancellation condition gives the minimal equations of $N = 2,\ D = 10$ nonchiral supergravity. These admit no solution of the above type, so that higher order terms are crucial for stabilizing, if at all, this solution.

quantum numbers to form the multiplet of $N = 8$, $D = 4$ supergravity with an explicity gauged $O(8)$ (or $O(4, 4)$) symmetry. It is amusing to observe that, without any reference to strings, anti-de-Sitter space-time is anyway required for a consistent formulation of gauged supergravities. Note also that, in the heterotic version, the above construction would lead to an $N = 4$ gauged supergravity coupled to a Yang-Mills theory with a rank-22 group.

A final comment: it would be interesting to further investigate the existence of such a string theory, in relation to the question of whether requiring an unbroken supersymmetry can guarantee the tree-level vanishing of the cosmological constant.[18]

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Note added in proof:

After completion of this work, we received a preprint by Goddard, Nahm and Olive[19] where they show that the free-fermion energy-momentum tensor can be expressed as the square of the currents of a group G, if the fermions transform as the symmetric space (G'/G) generators for some group G'. We thank P. Goddard for bringing this reference to our attention.

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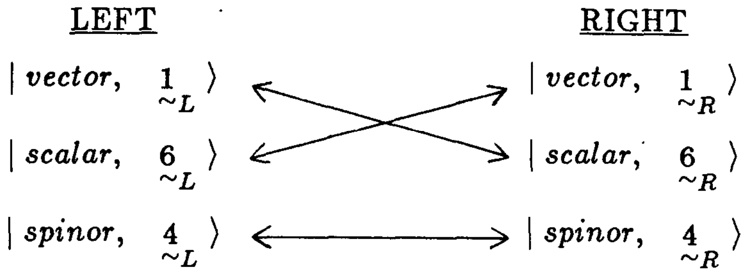

Fig. 1

Fig. 1: Schematic description of the construction of the massless vector bosons by taking the direct product of left and right moving states (as shown by the arrows). We explicitly indicate the Lorentz transformation properties as well as the representation of the states with respect to $H_{\text{left}} \times H_{\text{right}}$.