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TORSION IN SUPERSTRINGS*

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In this talk we discuss string theories on a background manifold with torsion.[1] The talk contains two parts. In the first part we discuss candidate vacuum configurations for ten-dimensional superstrings. We compactify these on $M_4 \times K$, where M_4 is four-dimensional and K some compact six-dimensional manifold. In particular we are interested in investigating the existence of solutions with non-zero torsion on K. The compactification problem is approached both from the effective field theory point of view and directly using string considerations.

The second part of the talk is devoted to the construction of string theories in curved space with torsion. We discuss both the Neveu-Schwarz-Ramond[2] type string and the Green-Schwarz[3] type string. Particular emphasis is put on the resulting constraints on space-time supersymmetry in the Green-Schwarz approach.

We use two-dimensional non-linear sigma models to describe the propagation of strings in background geometries with torsion. The background field can be understood as arising from condensation of infinite number of strings. Torsion can be viewed as the field strength associated with the vacuum expectation value of the anti-symmetric tensor field B_{mn} which appears in the supergravity multiplet. We show that if the background fields only include the metric and torsion, a consistent string theory requires torsion to vanish. The possibility remains that torsion is non-trivial when other background fields are included, e.g. gauge fields and dilaton.

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The effective ten-dimensional field theory which appears in the zero-slope limit $\alpha' = 0$ of the superstring is $N = 1$ supergravity coupled to super Yang-Mills matter. The low-energy theory has the supergravity transformations of the Chapline-Manton action[4] modified by the appropriate Chern-Simons terms introduced by Green and Schwarz.[5] The effective Lagrangian contains, even at the classical level, operators of arbitrarily high dimensions. These arise from integrating out the massive modes. Recently problems associated with this approximation were raised by Dine and Seiberg[6] and Kaplunovsky[7] on the basis of phenomenological considerations. If for the purpose of discussing the vacuum this truncation is questionable, then our analysis would need modification.

To prove that the only viable compactifications of the ten-dimensional manifold are on manifolds without torsion we take advantage of the analysis of Candelas, Horowitz, Strominger and Witten.[8] They have analyzed in detail the conditions for $N = 1$ supersymmetry and found that space-time M_4 must be flat Minkowski space. Form the supergravity transformation of the fermionic fields it follows that the compact manifold K must admit a covariantly constant spinor ϵ with respect to the connection $\Omega_m = \omega_m - 4\beta H_m$

$$\nabla_m(\Omega)\epsilon \equiv (\nabla_m(\omega) - \beta H_m)\epsilon = 0 \qquad \beta = \frac{3\sqrt{2}}{8}\, e^{2\phi} \ , \tag{1}$$

where $\nabla_m(\omega)$ is the covariant derivative with spin connection ω. In Eq. (1) H_m is defined through $H_m = H_{mnp}\gamma^n\gamma^p$, where H_{mnp} is the field strength associated with the antisymmetric field B_{mn}. The indices m, n, p refer to the compact manifold K and the γ's are the $O(6)$ Dirac matrices. From eq. (1) one can read off the torsion of the new connection

$$T_{mnp} = 4\beta\, H_{mnp} \ . \tag{2}$$

Candelas, Horowitz, Strominger and Witten showed that

$$H\epsilon = 0 \tag{3}$$

where $H = H_m\gamma^m$. However, they only studied in detail the case $H_{mnp} = 0$. Equation (3) can be satisfied without H_{mnp} being equal to zero. This corresponds to manifolds with torsion. Following the analysis of Ref. [8] it follows that the compact manifold admits a complex structure $f^m{}_n$ which is covariantly constant

$$\nabla_p(\Omega)\, f^m{}_n = 0 \ . \tag{4}$$

The complex structure can be built from the covariantly constant spinor ϵ. Therefore in order to preserve supersymmetry the compact manifold must be a hermitean manifold.

In Ref. [8] the following relation between the scalar curvature $R(\omega)$ and torsion T_{mnp} was derived

$$R(\omega) = \frac{16}{3}\,\beta^2 H_{mnp}H^{mnp} = \frac{1}{3}\,T_{mnp}T^{mnp} \ . \tag{5}$$

On manifolds with torsion it is straightforward to calculate the generalized Riemann tensor build from the connection Ω

$$R(\Omega)_{mnpq} = R(\omega)_{mnpq} + \nabla_p(\omega)T_{mnq} - \nabla_q(\omega)T_{mnp} + T_{rmp}T^r{}_{qn} - T_{rmq}T^r{}_{pn} \ . \tag{6}$$

For a totally antisymmetric torsion the generalized Ricci tensor is given by [1]

$$R(\Omega)_{mn} \equiv R(\Omega)^p{}_{mpn} = R_{mn}(\omega) - T^{pq}{}_m T_{pqn} + \nabla^p(\omega)T_{pmn} \ . \tag{7}$$

The reparametrization invariance of the compactified string theory demands that the two-dimensional non-linear sigma model must be conformally invariant. This means that the β-function must vanish. In this talk we will only consider the case of identically vanishing β-function. However, one can imagine not having $\beta \equiv 0$ but just a theory at a non-trivial fixed point of the β-function. All the results presented here are based on $\beta \equiv 0$. In particular the one loop β-function should vanish. The β-function has been studied by several authors.[9],[10] They have shown at one loop that the β-functions vanishes, with metric and torsion as background fields, when the generalized Ricci tensor vanishes

$$R(\Omega)_{mn} = 0 \ . \tag{8}$$

Using Eqs. (7) and (8) one finds that the background manifold must satisfy

$$R(\omega) = T^{mnp} T_{mnp} \ . \tag{9}$$

This is in contradiction with Eq. (5) unless

$$T_{mnp} = H_{mnp} = 0 \ . \tag{10}$$

Hence if the Ricci tensor $R(\Omega)_{mn}$ is required to vanish the background manifold cannot have any torsion. If the background fields include the dilaton and/or gauge fields Eq. (10) may no longer hold, accordingly our conclusions based on Eq. (10) may be modified.

Next we would like to give another proof of the above result. Equations (1) and (3) are the important constraints that guarantee that space-time M_4 is a Minkowski space and that the four-dimensional theory has $N = 1$ supersymmetry at the compactification scale. These constraints together with other constraints obtained in Ref. [8] have been analyzed and solved[1],[11] in the presence of torsion with a holonomy group $SU(3)$. The hermitean metric g of the compact six-dimensional manifold K must then satisfy

$$\begin{aligned} &\partial_i \, g^{i\bar{j}} = \partial_{\bar{j}} \, g^{i\bar{j}} = 0 \qquad i, j = 1, 2, 3 \\ &\det g = 1 \ . \end{aligned} \tag{11}$$

In Eq. (11) we have introduced complex coordinates. The condition (11) can be rewritten in terms of the curl of the metric

$$\partial_i \, g^{i\bar{j}} = 2\epsilon^{ik\ell} \, \overline{\epsilon^{jnm}} g_{\bar{n}k} \partial_i g_{\overline{m}\ell} \ . \tag{12}$$

On a Kähler manifold the metric satisfies the condition of Eq. (11). However, Eq. (11) admits more general solutions that include torsion. In the complex basis the only non-zero components of the torsion are

$$T_{ij\bar{k}} = \tfrac{1}{2}\,\partial_{[i}g_{j]\bar{k}} \qquad T_{\bar{i}\bar{j}k} = (T_{ij\bar{k}})^* \ . \tag{13}$$

From Eq. (13) it is clear that on a Kähler manifold the torsion vanishes. The Ricci tensor has the following form

$$R(\Omega)_{i\bar{j}} = -g^{m\bar{s}}(\partial_m\partial_{[\bar{s}}g_{\bar{j}]i} - \partial_i\partial_{[\bar{s}}g_{\bar{j}]m}) \ . \tag{14}$$

The antisymmetrization makes the generalized Ricci tensor to depend on the curl of g. Hence the Ricci tensor vanishes only on Kähler manifolds. Therefore, on manifolds with torsion the generalized Ricci tensor does not vanish. This is the same result as obtained above.

Let us emphasize that the conclusion of a vanishing torsion is based on the requirement that the Ricci tensor vanishes. This condition followed form the sigma model analysis for models that include just the metric and torsion. For more general sigma models with more background fields this condition may be relaxed and torsion need not vanish.

So far we have given two proofs that torsion on the compact manifold must vanish when the Ricci tensor vanishes. Since not very much is known about string theories it is instructive to derive the same equations from different points of view. Next we show how some of the equations of the effective field theory approach can be derived directly from the string theory. In the previous analysis of the effective field theory with torsion Eqs. (5) and (9) played a crucial role. Equation (9) followed from the demand that the generalized Ricci tensor vanishes. On the other hand Eq. (5) had to be satisfied for the theory to be supersymmetric at the compactification scale. In the string theory this equation arises from the demand that the central charge in the Virasoro algebra must have the same value as in a ten-dimensional supersymmetric string theory in flat space. If the central charge is changed, then the critical dimension of the theory is changed. Friedan and Shenker[12] have shown that if the critical dimension is changed there are no zero mass fermions in the theory and therefore supersymmetry is broken.

In recent papers[13],[14] the critical dimension was computed for group manifolds with torsion. The torsion corresponds to the Wess-Zumino term which must be included in order to preserve conformal invariance of the theory.[15] For a purely bosonic string the critical dimension on $SU(N)$ is given by[13]

$$D = 26 - d_c = \frac{(N^2-1)k}{N+k} = d_G - \frac{N(N^2-1)}{k} + O\left(\frac{1}{k^2}\right) \tag{15}$$

where $d_G = N^2 - 1$ is the dimension of $SU(N)$ and k is the integer coefficient of the Wess-Zumino term. To make contact with the previous analysis one needs the relation between the string tension and the integer k[13]

$$\alpha' = \frac{2}{k} \ . \tag{16}$$

One can rewrite Eq. (15) in terms of curvature and torsion on the manifold

$$26 - (d_c + d_G) = \alpha'(-3R + T^2) + O(\alpha'^2) \ . \tag{17}$$

From Eq. (17) it follows that the critical dimension remains unchanged if $R = \frac{1}{3}T^2$. A similar analysis can be performed for the supersymmetric case using the computation of the critical dimensionality for this case.[13] Again one finds that $R = \frac{1}{3}T^2$ is needed to ensure that the critical dimension does not shift. To all orders in the string tension the critical dimension for group manifolds is given by

$$26 - (d_C + d_G) = -\frac{(3R - T^2)\alpha' d_G}{d_G + \alpha'(3R - T^2)} \ . \tag{18}$$

Therefore, if the dilaton is set to zero, the only way to preserve supersymmetry at the compactification scale is to have zero torsion. Recently Callan, Martinec, Perry and Friedan[16] have studied the non-linear sigma model with metric, torsion and dilaton background fields.

Next we would like to construct string theories on background manifolds with torsion. As we discussed earlier the requirement that the resulting four-dimensional effective field theory has $N = 1$ supersymmetry determines a lot of properties of the compact manifold. In particular the manifold must admit a covariantly constant spinor with respect to the connection with torsion Ω and consequently a covariantly constant complex structure. Furthermore, the metric must be hermitean. When the torsion vanishes the relevant manifolds are Ricci flat Kähler manifolds.[8] When torsion is included one should consider manifolds satisfying Eq. (11).

Recently two-dimensional supersymmetric non-linear sigma models with torsion have been analyzed.[17] It has been shown that the sigma model has an $N = 2$ supersymmetry if the manifold is hermitean and if the complex structure is covariantly constant relative to the connection that includes torsion.

The general structure of the action after elimination of the auxiliary fields is[17]

$$\begin{aligned} I(X,\lambda) = \frac{1}{2}\int d^2\sigma \Big[& g_{mn}\partial_\mu X^m \partial^\mu X^n + \frac{3}{2}\, B_{mn}\epsilon^{\mu\nu}\partial_\mu X^m \partial_\nu X^n \\ & + i g_{mn}\overline{\lambda}^m_+ /D^+ \lambda^n_+ + i g_{mn}\overline{\lambda}^m_- /D^- \lambda^n_- \\ & + \frac{1}{4}\, R^+_{mnpq}(\overline{\lambda}^m_+ \rho_\mu \lambda^n_+)(\overline{\lambda}^p_- \rho^\mu \lambda^q_-) \Big] \end{aligned} \tag{19}$$

where the ρ_μ's are the two-dimensional Dirac matrices and $\lambda_\pm$ are Majorana-Weyl spinors. In Eq. (19) $\pm$ refer to right-handed and left-handed fermions respectively. The antisymmetric tensor B_{mn} is the potential associated with the torsion $T_{mnp} = -B_{[mn,p]}$. Note that this definition of torsion does not include the Chern-Simons terms. It differs form the torsion that appears in the effective field theory $T = dB - \frac{1}{30}\omega_Y + \omega_L$. These extra terms involve a compensating dimensionful parameter. Such a parameter is the slope parameter $\alpha' \sim (\ell\, Planck)^{-2}$. In fact the Chern-Simons terms appear in the next order of the loop expansion in the sigma model with the coefficient α'.[16] This is a necessary consequence of Lorentz and gauge invariance as discussed by Green and Schwarz.[3]

The action (19) is invariant under the supersymmetry transformation

$$\delta X^n = \delta_+ X^n + \delta_- X^n = \overline{\epsilon}_+ \lambda_-^n + \overline{\epsilon}_- \lambda_+^n$$
$$\delta \lambda_\pm^n = -i/\partial X^n \epsilon_\mp - \Gamma^n_{\pm mp} \lambda_\pm^m \delta_\pm X^p \ . \tag{20}$$

The sigma model has another supersymmetry

$$\delta X^n = \delta_+ X^n + \delta_- X^n = f^n_{-m} \overline{\epsilon}_+ \lambda_-^m + f^n_{+m} \overline{\epsilon}_- \lambda_+^m$$
$$\delta(f^n_{\pm m} \lambda_\pm^m) = -i/\partial X^n \epsilon_\pm - \Gamma^n_{\pm mp} f^m_{\pm \ell} \lambda_\pm^\ell \delta_\pm X^p \tag{21}$$

provided the complex structure $f^n{}_m$ is covariantly constant.

The conformally invariant non-linear sigma model has another type of supersymmetry. This supersymmetry is the partner of the local Kac-Moody transformation and has the form

$$\delta X^i = 0 \qquad \delta \lambda_\pm^i = \delta_\pm^i \ . \tag{22}$$

Next we would like to elevate the two-dimensional supersymmetry to a space-time supersymmetry. We will work in ten dimensions. When the compact manifold is flat this amounts to going from the Neveu-Ramond-Schwarz version of the string theory to the Green-Schwarz superstring. For non-trivial curved background with torsion, we will use the light cone gauge to relate the Neveu-Ramond-Schwarz type of string theory to the Green-Schwarz superstring. The Green-Schwarz version of the action (19) is given by

$$\begin{aligned} I = \int d^2\sigma \Bigg[& \frac{1}{2} g_{ij} \partial_\alpha X^i \partial^\alpha X^j + \tfrac{1}{2} \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j \\ & + \frac{i}{4} \overline{S}_+ \gamma_+ \rho^\alpha D_\alpha^+ S_+ + \frac{i}{4} \overline{S}_- \gamma_+ \rho^\alpha D_\alpha^- S_- \\ & + \frac{1}{4} R^+_{ijk\ell} \overline{S}_+ \gamma_+ \gamma^i \gamma^j \rho^\alpha S_+ \overline{S}_- \gamma_+ \gamma^k \gamma^\ell \rho_\alpha S_- \Bigg] \end{aligned} \tag{23}$$

where $S_+(S_-)$ is a right (left)-moving fermion. In flat space the action (23) has two eight component supersymmetries

$$\delta X^i = (p^+)^{-1/2}\sqrt{2}\,\bar{\epsilon}\gamma^i S \tag{24a}$$

$$\delta S^\gamma = i\frac{(p^+)^{-1/2}}{\sqrt{2}}\,(\gamma_-\gamma_M(\rho\cdot\partial x^M)\epsilon)^\gamma \tag{24b}$$

and

$$\delta X^i = 0 \qquad \delta S^\alpha = \delta^\alpha \tag{25}$$

where ϵ and δ are eight component real spinors of $O(8)$. In curved space the transformation (24b) gets modified by terms of the form $\Omega_i\gamma_- S\,\delta X^i$. The action (23) is invariant under the δ supersymmetry of Eq. (25) provided that δ is covariantly constant. In this case the quadratic term is automatically invariant and the quartic term is invariant since $R^{\pm}_{mnpq}\gamma^p\gamma^q\delta = 0$. The δ-supersymmetry is analogous to the transformation of Eq. (22). In curved space one cannot implement the full eight component ϵ supersymmetry. To see this let us study the relation between the action in Eq. (19) and that in Eq. (23) assuming $SU(3)$ holonomy. In the $SU(3)$ basis the fermions have the form

$$S^A = \begin{pmatrix} \psi^\alpha \\ \chi_1 \\ \psi_\alpha \\ \chi_2 \end{pmatrix} \qquad \begin{array}{l} A = 1,\ldots,8 \\ \\ \alpha = 1,2,3\ . \end{array} \tag{26}$$

This corresponds to the following decomposition of the spinor representation under $SU(8) \supset SO(6) \supset SU(3)$

$$8(\text{spinor}) \rightarrow 4 + \overline{4} \rightarrow 3 + 1 + \overline{3} + 1\ . \tag{27}$$

Under this decomposite the action in Eq. (23) takes the form of that in Eq. (19). For details see Ref. [1]. Since the action of Eq. (19) is a sigma model in two dimensions, it is clear that it cannot have the full ϵ-supersymmetry. To see what part survives let us focus on the compact part of the background manifold. The spinor ϵ has a similar decomposition under $SU(3)$ as S given in Eq. (26). The action for the superstring is invariant when the triplet ϵ^{α} and antitriplet ϵ_{α} vanish.

The supersymmetries of the curved space are associated with the singlet parameters $\epsilon_1 = \epsilon_2^*$. This means that the spinor ϵ must be covariantly constant. This is of no surprise if one studies the relationship of the supersymmetry transformations of the covariant action to the supersymmetry transformations of the light cone action of the Green-Schwarz string in flat space.[1] For the argument to be applicable to the present case one needs the covariant form of the Green-Schwarz action in curved space.[18]

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