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GAUGE INVARIANCE IN QUANTUM GRAVITY*

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ABSTRACT

We show that in quantum gravity the mass counterterm for any matter field is gauge dependent. Nevertheless, full use of the invariances of the theory still guarantees the possibility to construct, order by order in perturbation, a gauge invariant S-matrix.

1. INTRODUCTION

Although a satisfactory theory of gravity is still unknown, people have often tried to use the experience gained by the study of Yang-Mills theories in order to quantize Einstein's theory of General Relativity. The standard procedure is to add a gauge-fixing term, compute the corresponding Faddeev-Popov ghost Lagrangian and expand the action around the flat-space Minkowski metric. In close analogy to the Yang-Mills case, one finds that the resulting effective Lagrangian is invariant under B.R.S. transformations.[1] Leaving aside the problem of renormalizability and introducing an appropriate regularization scheme, it is generally believed that this method will yield a gauge invariant S-matrix.[♯1] The purpose of this paper is to point out an important peculiarity of gravity as compared to Yang-Mills theories. Namely, we shall show that the mass counterterm which one should introduce for any particle in a gravitational field, depends on the gauge one uses for the coordinate invariance. This result will be established by an explicit one loop calculation in section 2 below. In section 3 we shall prove that this is consistent with the Ward identities of the theory. Finally, in section 4, we shall show that, in spite of this result, full use of the invariances of the theory, still guarantees the existence of a gauge-invariant S-matrix. Our results are valid both in ordinary theory of gravity as well as the higher derivative, conformally invariant Weyl's theory. The problems of renormalizability and/or unitarity will not be discussed.

♯1 Strictly speaking, in gravity, like in any other theory containing massless particles, there is no S-matrix. However, one expects to be able to construct gauge invariant quantities, like transition probabilities, which are free from infrared divergencies.

2. A SAMPLE COMPUTATION

We shall compute in this section the two point function of a scalar field coupled to a gauge field. We shall start with the well-known case of an electromagnetic or a Yang-Mills field and then proceed to that of a gravitational field.

At the one-loop level the diagrams contributing to the self-energy of a scalar field are shown in fig. 1. At this level there is no difference between Abelian or non-Abelian theories. The sum of the two diagrams gives a function $g^2 \sum(p^2, m^2, \xi, \epsilon)$ where m is the mass of the scalar particle, g is the coupling constant, ξ is the gauge-fixing parameter and ϵ is the cut-off. The statement that the mass counterterm is gauge independent means that

$$\frac{\partial}{\partial \xi} \sum (m^2,\ m^2,\ \xi,\ \epsilon) = 0\ . \tag{2.1}$$

We shall verify (2.1) by explicit computation. The effective Lagrangian density is given by:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}\, F_{\mu\nu}^2 + (D_\mu \phi)\, (D^\mu \phi)^* - m^2 \phi \phi^* - \frac{1}{2\xi}\, G(A)^2 \tag{2.2}$$

where $G(A)$ is the gauge-fixing term which will be taken to depend on the gauge field A_μ through $\partial_\mu A^\mu$. The photon propagator is then of the form:

$$iD_{\mu\nu}(k) = \frac{1}{k^2} \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \xi\, \frac{k_\mu k_\nu}{k^4}\, \widetilde{D}(k^2) \tag{2.3}$$

where $\widetilde{D}(k^2)$ depends on the detailed form of $G(A)$. Since we are interested in checking (2.1), it is sufficient to use only the ξ-dependent part of $D_{\mu\nu}$ in the diagrams and verify that the resulting $\widetilde{\Sigma}(p^2, m^2, \xi, \epsilon)$ vanishes for $p^2 = m^2$. A convenient choice for a family of gauges $G(A)$ is given by:

$$G(A) = \frac{\Box + \mu^2}{\mu^2} \, \partial_\mu \, A^\mu \tag{2.4}$$

with μ^2 an arbitrary parameter with dimensions of mass^2. This yields a $\widetilde{D}(k^2)$ in (2.3):

$$\widetilde{D}(k^2) = \frac{\mu^4}{(k^2 - \mu^2)^2} \; . \tag{2.5}$$

The advantage of this choice is that the ξ-dependent part of any diagram is both ultraviolet and infrared finite. The two diagrams of fig. 1 give:

$$\begin{aligned} \widetilde{\Sigma}\left(m^2, m^2, g, \xi\right) \sim \int d^4k \; & \frac{(2p-k)^\mu \, (2p-k)^\nu \, k_\mu \, k_\nu}{k^4 (k^2 - \mu^2)^2 \, [(p-k)^2 - m^2]} \Bigg|_{p^2 = m^2} \\ & - \int d^4k \; \frac{k^2}{k^4 (k^2 - \mu^2)^2} = 0 \; . \end{aligned} \tag{2.6}$$

This is the expected result. It is known (see also next section) that it is valid to all orders in perturbation for both Abelian and non-Abelian theories and, furthermore, it is essential in the proof of the gauge invariance of the S-matrix.

We turn now to quantum gravity. The diagrams are still the same with the graviton replacing the gauge boson. The relevant part of the effective Lagrangian is given by:

$$\mathcal{L} = \sqrt{-g} \left\{ -\frac{2}{\kappa^2} \, R + \frac{1}{2} \, g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) - \frac{m^2}{2} \, \phi^2 \right\} - \frac{1}{2\xi} \, (G^\mu)^2 \tag{2.7}$$

where g is the determinant of $g_{\mu\nu}$, R the scalar curvature, κ^2 the gravitational coupling constant and G_μ the gauge-fixing function. We expand around Minkowski flat space and we write:

$$\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} \tag{2.8}$$

which yields:

$$\sqrt{-g} = 1 + \frac{1}{2}\, h^{\alpha}_{\alpha} + \frac{1}{8}\left(h^{\alpha}_{\alpha}\, h^{\beta}_{\beta} - 2 h^{\alpha}_{\beta}\, h^{\beta}_{\alpha}\right) + \ldots \quad . \tag{2.9}$$

The indices are contracted with the flat space metric $\eta^{\mu\nu}$. We choose again a family of gauges of the form:

$$G^{\nu}(h) = \frac{\Box + \mu^2}{\mu}\, \partial_{\mu}\, h^{\mu\nu} \tag{2.10}$$

and the ξ-dependent part of the graviton propagator becomes:

$$\begin{aligned}\widetilde{D}(k)_{\mu\nu,\lambda\rho} &= i\xi\ \frac{1}{k^2}\ \frac{\mu^2}{(k^2-\mu^2)^2} \\ &\times \left[2P^{(1)} + 3P^{(0-s)} - \sqrt{3}\left(P^{(0-sw)} + P^{(0-ws)}\right) + P^{(0-w)}\right]_{\mu\nu,\lambda\rho}\end{aligned} \tag{2.11}$$

where the P-tensors are defined by:

$$P^{(1)}_{\mu\nu,\lambda\rho} = \frac{1}{2}\ \left(\theta_{\mu\lambda}\,\omega_{\nu\rho} + \theta_{\mu\rho}\,\omega_{\nu\lambda} + \theta_{\nu\rho}\,\omega_{\mu\lambda} + \theta_{\nu\lambda}\,\omega_{\mu\rho}\right) \tag{2.12a}$$

$$P^{(0-s)}_{\mu\nu,\lambda\rho} = \frac{1}{3}\ \theta_{\mu\nu}\,\theta_{\lambda\rho} \tag{2.12b}$$

$$P^{(0-sw)}_{\mu\nu,\lambda\rho} = \frac{1}{\sqrt{3}}\ \theta_{\mu\nu}\,\omega_{\lambda\rho} \tag{2.12c}$$

$$P^{(0-ws)}_{\mu\nu,\lambda\rho} = \frac{1}{\sqrt{3}}\ \omega_{\mu\nu}\,\theta_{\lambda\rho} \tag{2.12d}$$

$$P^{(0-w)}_{\mu\nu,\lambda\rho} = \omega_{\mu\nu}\,\omega_{\lambda\rho} \tag{2.12e}$$

and

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \tag{2.13a}$$

$$\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \tag{2.13b}$$

The calculation of the two diagrams now is straightforward. We find:

$$\begin{aligned}\widetilde{\Sigma}\left(m^2, m^2, \xi\right) \sim & \int d^4k \\ & \times \left. \frac{\left[p^\mu (p-k)^\nu - \frac{m^2}{2}\eta^{\mu\nu}\right]\left[p^\lambda (p-k)^\rho - \frac{m^2}{2}\eta^{\lambda\rho}\right] \widetilde{D}_{\mu\nu,\lambda\rho}(k)}{(k^2 - 2pk)} \right|_{p^2=m^2} \\ & + \frac{m^2}{8}\int d^4k \left[\eta^{\mu\nu}\eta^{\lambda\rho} - 2\eta^{\mu\lambda}\eta^{\nu\rho}\right] \widetilde{D}_{\mu\nu,\lambda\rho}(k) \\ = & -\xi m^2 \mu^2 \int \frac{d^4k}{k^2(k^2-\mu^2)^2} \neq 0 \; .\end{aligned} \tag{2.14}$$

This result means that if we compute the scalar field two-point function using, for example, dimensional regularization followed by minimal subtraction, the position of the pole of the resulting propagator will be gauge dependent. This contradicts all our experience from Q.E.D. or Yang-Mills field theories[♯2] and, at first sight, it seems to jeopardize the construction of a gauge invariant S-matrix. It is therefore remarkable that, as we shall show in the following sections, a physically meaningful and gauge invariant S-matrix can, nevertheless, be obtained.

♯2 The gauge dependence of the mass counterterm occurs also in Q.E.D. or Yang-Mills theories, but only for gauge-fixing functions nonlinear in the fields.[2] The novel feature of gravity is its appearance even for linear gauge-fixing.

3. THE WARD IDENTITIES

The result of the previous section, strange as it may seem, is in fact consistent with the invariances of the theory, as can be shown by a study of the Ward identities.

Let $\mathcal{L}_{\rm inv}$ be a coordinate invariant Lagrangian density of a scalar field coupled to the gravitational field. We do not have to specify its detailed form for the argument of this section, except to say that, in the purely gravitational sector, it includes the Einstein term $R\sqrt{-g}$, a bare cosmological constant $\Lambda\sqrt{-g}$ and possibly other terms containing higher derivatives of the metric tensor. The Ward identities will be obtained by using the B.R.S. invariance of the effective action, so we introduce the gauge-fixing Lagrangian:

$$\mathcal{L}_G = -\frac{1}{2\xi}\, G_\mu^2 + \overline{C}_\mu \; s \; G^\mu \tag{3.1}$$

where s denotes the generator of the B.R.S. transformation[1] and $\overline{C}_\mu$ the Faddeev-Popov antighost. The simplest choice for the gauge-fixing term G^μ is:

$$G^\mu = \partial_\nu \; h^{\mu\nu} \quad . \tag{3.2}$$

For the general argument of this section there is no need to introduce special convergence factors like the one of eq. (2.10). Under a B.R.S. transformation the various fields of (3.1) transform like:

$$sh^{\mu\nu} = [\partial^\mu C^\nu + \partial^\nu C^\mu - \eta^{\mu\nu}\,\partial_\alpha C^\alpha + h^{\alpha\nu}\,\partial_\alpha C^\mu + h^{\alpha\mu}\,\partial_\alpha C^\nu - \partial_\alpha\,(C^\alpha\, h^{\mu\nu})] \tag{3.3a}$$

$$s\phi = -C^\alpha\;\partial_\alpha\;\phi \tag{3.3b}$$

$$sC^{\mu} = -\left(\partial_{\alpha} C^{\mu}\right) \; C^{\alpha} \tag{3.3c}$$

$$s\overline{C}^{\mu} = -\frac{1}{\xi} \, G^{\mu} \quad . \tag{3.3d}$$

A peculiarity of gravity is that, although gauge invariance forces $h^{\mu\nu}$ to be massless, it does not prevent it from taking a nonzero vacuum expectation value. In our case, diagrams of the form of fig. 2 are divergent and require a subtraction. The 1-PI part of them can be put equal to zero with a suitable renormalization condition, but the corresponding contributions in connected Green functions are ambiguous,[3] containing expressions of the form 0/0 due to the graviton propagator at zero momentum. In other words, gravity coupled to massive matter fields, when quantized around flat space, requires an infrared regulator, even for Green functions. We choose to introduce a "soft" breaking of coordinate invariance of the form th^{μ}_{μ} where t is a parameter with dimensions mass^4. The final form of the effective Lagrangian is therefore:

$$\mathcal{L} = \mathcal{L}_{\text{inv}} + \mathcal{L}_G + th^{\mu}_{\mu} \tag{3.4}$$

supplemented with the renormalization condition:

$$\langle \, h^{\mu\nu} \, \rangle = 0 \tag{3.5}$$

which ensures that flat space is a solution of the equations of motion. At the tree level (3.5) gives a relation

$$\frac{1}{2} \, \Lambda + t = 0 \quad . \tag{3.6}$$

At higher orders it requires the introduction of a counterterm of the form $\delta\Lambda\sqrt{-g}$. After the condition (3.5) is enforced we can take the limit $t \to 0$ thus recovering a massless graviton.

We now introduce sources and write an effective action:

$$S = \int d^n x \left\{ \mathcal{L} + \sum_F J_F \, F + \sum_{F \neq \overline{C}_\mu} J_F^s \, sF + Q \, G^\mu \, \overline{C}_\mu \right\} \tag{3.7}$$

where the sum over F extends to all fields, J_F^s are the sources for the B.R.S. transformed fields and the last term with an anticommuting constant Q has been introduced for convenience.[4] The generating functional of the Green functions is now given by:

$$Z\,[J,\, J^s,\, Q] = \frac{\int \mathcal{D}[F] \; \exp\{-S[J,\, J^s,\, Q]\}}{\int \mathcal{D}[F] \; \exp\{-S[J = J^s = Q = 0]\}} \tag{3.8}$$

and that of connected Green function by:

$$W = -i \; \ln Z \quad . \tag{3.9}$$

The classical fields $\hat{F}$ are defined by:

$$\hat{F} = \frac{\delta W}{\delta J_F} \quad . \tag{3.10}$$

A functional Legendre transformation gives the generating functional of 1-PI diagrams. In our case it is convenient to define

$$\Gamma = W + \int d^n x \; \left\{ -\sum_F J_F \, \hat{F} + \frac{1}{2\xi} \, \hat{G}^{\mu^2} - Q \hat{G}^\mu \hat{\overline{C}}_\mu - t \hat{h}^\mu_\mu \right\} \quad . \tag{3.11}$$

In order to give a precise meaning to all these formal expressions we shall proceed in two steps: First we define an intermediate renormalized theory in which all Green functions are computed, order by order in perturbation theory, as sums of Feynman diagrams using dimensional regularization and minimal subtraction. At a second stage renormalization conditions will be imposed. In this section only (3.5) will be considered. Our results will be valid also for the case of the nonrenormalizable Einstein theory.

We are now in a position to write the Ward identities. B.R.S. invariance is broken in (3.7) by the sources as well as the linear term. In the standard way we obtain for the intermediately renormalized Green functions without external ghost lines, at the limit $t \to 0$:[♮3]

$$2\xi \frac{\partial \Gamma}{\partial \xi} = \int \frac{\delta \Gamma}{\delta \hat{h}^{\mu\nu}(x)} \frac{d}{dQ} \frac{\delta \Gamma}{\delta J^{s}_{\mu\nu}(x)} dx + \int \frac{\delta \Gamma}{\delta \hat{\phi}(x)} \frac{d}{dQ} \frac{\delta \Gamma}{\delta J^{s}_{\phi}(x)} dx \quad . \tag{3.12}$$

This is our basic equation.[5] Our first result will be to show that $\delta\Lambda$, the cosmological constant counterterm which is determined by the condition (3.5) at the limit $t \to 0$, is ξ-independent. We take the derivative of (3.12) with respect to $\hat{h}^{\rho\sigma}$ and set all classical fields and sources equal to zero. We obtain:

$$\begin{aligned} 2\xi \frac{\partial}{\partial \xi} \frac{\delta \Gamma}{\delta \hat{h}^{\rho\sigma}(y)} = &\int \frac{\delta^2 \Gamma}{\delta \hat{h}^{\mu\nu}(x)\, \delta \hat{h}^{\rho\sigma}(y)} \frac{d}{dQ} \frac{\delta \Gamma}{\delta J^{s}_{\mu\nu}(x)} dx \\ &+ \int \frac{\delta \Gamma}{\delta \hat{h}^{\mu\nu}(x)} \frac{d}{dQ} \frac{\delta^2 \Gamma}{\delta J^{s}_{\mu\nu}(x)\, \delta \hat{h}^{\rho\sigma}(y)} dx \quad . \end{aligned} \tag{3.13}$$

We now impose the renormalization condition (3.5). The first term on the r.h.s. vanishes because it is proportional to the inverse graviton-graviton two-point function at zero momentum. So does the second term because, precisely (3.5) puts the graviton tadpole to zero. It follows that[5]

$$\frac{\partial}{\partial \xi} \frac{\delta \Gamma}{\delta \hat{h}^{\rho\sigma}} \bigg|_{\Lambda=0} \propto \frac{\partial \delta \Lambda}{\partial \xi} = 0 \quad . \tag{3.14}$$

We conclude that (3.5) does not introduce any new gauge dependence in the effective action.

Armed with this result we proceed to the study of the gauge dependence of the mass counterterm. We take the second derivative of (3.12) with respect to $\hat{\phi}$

♮3 1-PI Green functions do not need the t-infrared regulator.

and let all sources go to zero. The last term becomes proportional to the inverse ϕ propagator which vanishes for some value $p^2 = \overline{m}^2$, $\overline{m}^2$ being the value of the ϕ-mass at the order we are working. (Remember, no renormalization condition has been imposed.) Therefore (3.12) gives:

$$2\xi \frac{\partial}{\partial \xi} \overline{m}^2 = r \; \frac{d}{dQ} \frac{\delta \Gamma}{\delta J^s_{\mu\nu}} \bigg|_{J,\ldots=0} \int dx \; \frac{\delta^3 \Gamma}{\delta \hat{h}^{\mu\nu}(x) \; \delta \hat{\phi}(x_1) \; \delta \hat{\phi}(x_2)} \bigg|_{p_1^2=p_2^2=\overline{m}^2} \tag{3.15}$$

where r is the residue of the pole $\overline{m}^2$. The first factor is given by a sum of vacuum diagrams and can be parametrized as:

$$\frac{d}{dQ} \frac{\delta \Gamma}{\delta J^s_{\mu\nu}} \bigg|_{J,\ldots=0} = \tau \, \eta^{\mu\nu} \tag{3.16}$$

with τ being a constant which has an expansion in powers of $\hbar$ and whose first nonzero term is of order $\hbar$. The second factor is the graviton-scalar three point function with a zero momentum graviton and on-shell scalars. Its expansion in number of loops starts at zero order and equals $-im^2 + O(\hbar)$. We conclude that the l.h.s. of (3.15) is different from zero already at the one-loop level, a result which is confirmed by the explicit calculation of the previous section.

It is instructive to trace the origin of the null result in ordinary gauge theories. For example, in scalar electrodynamics we obtain a Ward identity which looks almost the same as (3.12) with photon lines replacing the graviton ones and the corresponding changes in the ghosts. However, now the term in (3.16) vanishes because there is no way to have a vacuum diagram with just one free index. Physically our result can be understood with the following argument: The mass of a particle interacting with a gauge field is an external parameter which should not depend on the particular gauge choice for the latter, while the mass-term of the same particle in a gravitational field becomes part of the interaction Lagrangian.

4. THE GAUGE INVARIANCE OF THE S–MATRIX

The purpose of this section is to show that the gauge dependence of the mass counterterm, which we established in the previous sections, does not prevent the construction of a gauge invariance S-matrix. Naturally, physical masses are gauge invariant and this must be imposed by appropriate renormalization conditions. The gauge dependence of the counterterms introduces some additional complication and the standard proofs must be modified accordingly.

The most direct way to prove the gauge invariance of the S-matrix in a gauge theory is to impose on-shell renormalization conditions for all arbitrary parameters of the theory[♮4] and use the B.R.S. Ward identity in order to establish a parametric differential equation[1,6] of the form:

$$\left(\xi \frac{\partial}{\partial \xi} + \sum_i \Delta_i\right) W^R = 0 \tag{4.1}$$

where W^R is the on-shell renormalized generating functional and the Δ_i's are a set of unphysical insertions. Therefore, the S-matrix elements, identified by the appropriate amputation at the physical masses, are ξ-independent. It is clear that the gauge dependence of the mass-counterterms, which we found in gravity, spoils the counting on which eq. (4.1) was based.

Alternatively, we can use the same B.R.S. Ward identity to prove that a change in ξ is equivalent to a change in the source terms[7,8] and, in an ordinary gauge theory, this does not affect the physical S-matrix. However, this is no more true in quantum gravity because such a change induces, in particular, a change in the vacuum energy which is precisely given by the term represented in eq. (3.16). We see that in either case we need one more equation in the form of a new Ward identity in order to be able to control the total gauge dependence of the S-matrix.

♮4 We consider, for simplicity, the case of a spontaneously broken gauge theory where on-shell renormalization conditions do not introduce any infrared divergences.

The remarkable result is that quantum gravity has enough invariance to provide us with precisely one extra Ward identity as we shall show next.

The most rigorous way to obtain these equations is to follow the method of ref. [6] and write the complete set of parametric differential equations of the theory. In our case they include the renormalization group equation and two equations which control the gauge dependence of the Green functions.[♯5] We shall not present this computation explicitly because it is rather lengthy and not very illuminating. Furthermore it can only be performed to all orders in a higher derivative theory because Einstein's theory being nonrenormalizable contains an infinite number of arbitrary parameters. The final result (for the generating functional of connected Green functions without external ghost lines) reads:

$$\left[\mu\,\frac{\partial}{\partial\mu}+\beta_i^{(\mu)}\,\frac{\partial}{\partial g_i}+\beta_\kappa^{(\mu)}\kappa\,\frac{\partial}{\partial\kappa}+\beta_\xi^{(\mu)}\xi\,\frac{\partial}{\partial\xi}+\sum_F\gamma_F^{(\mu)}\int J_F\,\frac{\delta}{\delta J_F}\right]W=0 \tag{4.2a}$$

$$\left[\xi\,\frac{\partial}{\partial\xi}+\beta_i^{(\xi)}\,\frac{\partial}{\partial g_i}+\sum_F\gamma_F^{(\xi)}\int J_F\,\frac{\delta}{\delta J_F}+\delta^{(\xi)}\int J_{\mu\nu}\,\frac{d}{dQ}\,\frac{\delta}{\delta J^s_{\mu\nu}}+\tau^{(\xi)}\int\frac{\delta}{\delta\tilde{J}^\mu_\mu}\right]W$$
$$=\alpha^{(\xi)}\int J^\mu_\mu \tag{4.2b}$$

$$\left[\kappa\,\frac{\partial}{\partial\kappa}+\beta_i^{(\kappa)}\,\frac{\partial}{\partial g_i}+\sum_F\gamma_F^{(\kappa)}\int J_F\,\frac{\delta}{\delta J_F}+\delta^{(\kappa)}\int J_{\mu\nu}\,\frac{d}{dQ}\,\frac{\delta}{\delta J^s_{\mu\nu}}+\tau^{(\kappa)}\int\frac{\delta}{\delta\tilde{J}^\mu_\mu}\right]W$$
$$=\alpha^{(\kappa)}\int J^\mu_\mu \tag{4.2c}$$

where μ is the renormalization point, κ the gravitational coupling constant defined in (2.7), g_i are the various coupling constants and masses and F denotes all fields; $\beta_i^{(a)}$, $\gamma_F^{(a)}$, $\delta^{(a)}$, $\tau^{(a)}$ and $\alpha^{(a)}$ are calculable functions of the coupling

♯5 The technical reason why we obtain two equations instead of one can be traced to the fact that in gravity the quantum field $g^{\mu\nu}$ is expanded around the classical value $\eta^{\mu\nu}$ rather than zero. Were it not the case, gravity would had contained one parameter less since Newton's coupling constant could have been absorbed into a rescaling of the fields.

constants and $\widetilde{J}_{\mu\nu}$ stands for an amputated source. The insertions corresponding to $\gamma_F^{(a)}$ and $\delta^{(a)}$ are unphysical (they do not contribute to the S-matrix) while the insertion corresponding to $\tau^{(a)}$ is physical. Equation (4.2a) is the renormalization group equation and the linear combination of (4.2b) and (4.2c) which does not contain the physical insertion gives the gauge independence of the S-matrix.

A more transparent method consists of exhibiting all the invariances of the theory. The B.R.S. invariant action we considered in the last section is:

$$S_0 \equiv \int d^n x \left\{ \mathcal{L}_{\text{inv}} - \frac{1}{2\xi}\, G_\mu^2 + \overline{C}_\mu\, s\, G^\mu \right\} \tag{4.3}$$

with G_μ given by (3.2). In the Landau gauge $\xi \to 0$ we can rewrite (4.3) introducing an auxiliary field b^μ:[7]

$$S_0 \to S_0' \equiv \int d^n x \;\left\{ \mathcal{L}_{\text{inv}} + b^\mu\, G_\mu + \overline{C}_\mu\, s\, G^\mu \right\} \quad . \tag{4.4}$$

The B.R.S. transformation properties (3.3) remain the same except that for the anti-ghost eq. (3.3d) which is replaced by:

$$s\overline{C}^{\,\mu} = b^\mu \tag{4.5a}$$

$$sb = 0 \quad . \tag{4.5b}$$

Notice that (4.5) is not the most general action consistent with B.R.S. invariance. One can add the infrared regulator h_μ^μ with an arbitrary coefficient t. Indeed, the action

$$S_1 = t \int d^n x\; h_\mu^\mu \tag{4.6}$$

is also B.R.S. invariant because the transform of $\int d^n x\, h_\mu^\mu$ vanishes for $\partial_\nu\, h^{\mu\nu} = 0$. On the other hand S_1 is not invariant under coordinate transformations. The corresponding phenomenon in Q.E.D. is the possibility of adding a photon mass-term $\sim A_\mu^2$ which, in the Landau gauge $\partial_\mu\, A^\mu = 0$, also yields a B.R.S. invariant although gauge noninvariant action.

In gravity, however, this freedom will give a new Ward identity. In a somewhat heuristic way, it can be viewed as corresponding to the residual invariance of S_0' once the Landau condition $\partial_\mu h^{\mu\nu} = 0$ has been imposed. Indeed it is easy to see that one can still make coordinate transformations with parameter

$$\omega^\mu(x) = \omega\, x^\mu \tag{4.7}$$

with constant ω. This is a dilatation which is an exact symmetry of S_0' in n dimensions provided we assign to the various fields the transformation properties:

$$\delta h^{\mu\nu} = -(n-2)\,\eta^{\mu\nu} - [(n-2) + x^\alpha\,\partial_\alpha]\,h^{\mu\nu} \tag{4.8a}$$

$$\delta F = -\,[d_F + x^\alpha\,\partial_\alpha]\,F \tag{4.8b}$$

where F stands for all other fields, i.e. the ghosts C^μ and $\overline{C}^\mu$, the auxiliary field b^μ as well as matter fields ϕ (scalar), ψ (spinor) and A_μ (gauge) with the following dimensions: $d_c = d_{\bar{c}} = 0$; $d_b = 1$; $d_\phi = d_\psi = 0$; $d_A = 1$. At an arbitrary gauge the invariance under (4.7) is broken by the gauge-fixing term $\frac{1}{2\xi}\,G_\mu^2$ as well as the infrared regulator (4.6). The difference with Q.E.D. lies in the fact that, for the latter, in the Landau gauge there is essentially no invariance left. Indeed, if $\partial_\mu A^\mu = 0$, one can still perform transformations $A_\mu \longrightarrow A_\mu + \partial_\mu \chi$ with $\chi(x)$ satisfying $\Box\chi = 0$. If we do not want to change the asymptotic behavior of A_μ, which is assumed to vanish at infinity, χ must be a constant. On the other hand (4.8a) is an acceptable transformation. The constant term $-(n-2)\eta^{\mu\nu}$ is just the reflection of the fact that we are perturbing around a nonzero value of the field $g^{\mu\nu}$. For $\sqrt{-g}\,g^{\mu\nu}$ the corresponding transformation is $\delta\sqrt{-g}\,g^{\mu\nu} = -(n-2)\sqrt{-g}\,g^{\mu\nu} - x^\alpha\partial_\alpha\sqrt{-g}\,g^{\mu\nu}$. Therefore, we can use the invariance under (4.8) in order to obtain a new Ward identity. We thus obtain a system of two Ward identities, B.R.S. and dilatation, which is equivalent to the one we obtain from the parametric differential equations. For the dimensionally regularized generating functional of connected Green functions without external ghost lines we get:

$$-2\xi\,\frac{\partial W}{\partial \xi}=\int (J_{\mu\nu}+t\eta_{\mu\nu})\,\frac{d}{dQ}\,\frac{\delta W}{\delta J^{s}_{\mu\nu}}d^{n}x+\int J_{\phi}\,\frac{d}{dQ}\,\frac{\delta W}{\delta J^{s}_{\phi}}\,d^{n}x \quad (4.9)$$

$$\begin{aligned}-2(n+2)\,\xi\,\frac{\partial W}{\partial \xi}&=\int (J_{\mu\nu}+t\eta_{\mu\nu})\,(n-2+x^{\alpha}\,\partial_{\alpha})\,\frac{\delta W}{\delta J_{\mu\nu}}\,d^{n}x \qquad (4.10)\\ &+\int J_{\phi}\,(d_{\phi}+x^{\alpha}\,\partial_{\alpha})\,\frac{\delta W}{\delta J_{\phi}}\,d^{n}x+(n-2)\int J^{\mu}_{\mu}\,d^{n}x\,.\end{aligned}$$

It is instructive to study the limit $t\to 0$ in W. Only diagrams with a zero momentum graviton propagator will survive in the terms proportional to t, so we obtain:

$$\begin{aligned}-2\xi\,\frac{\partial W}{\partial \xi}&=\int J_{\mu\nu}\,\frac{d}{dQ}\,\frac{\delta W}{\delta J^{s}_{\mu\nu}}\,d^{n}x+\int J_{\phi}\,\frac{d}{dQ}\,\frac{\delta W}{\delta J^{s}_{\phi}}\,d^{n}x\\ &-\frac{d}{dQ}\,\frac{\delta \Gamma}{\delta J^{s}_{\mu\nu}}\bigg|_{J_{F}=J^{s}_{F}=0}\int \frac{\delta W}{\delta \tilde{J}^{\mu\nu}}\,d^{n}x \qquad (4.11)\end{aligned}$$

$$\begin{aligned}-2(n+2)\,\xi\,\frac{\partial W}{\partial \xi}&=\int J_{\mu\nu}\,(n-2+x^{\alpha}\,\partial_{\alpha})\,\frac{\delta W}{\delta J_{\mu\nu}}\,d^{n}x \qquad (4.12)\\ &+\int J_{\phi}\,(d_{\phi}+x^{\alpha}\,\partial_{\alpha})\,\frac{\delta W}{\delta J_{\phi}}\,d^{n}x-(n-2)\int\left[\frac{\delta W}{\delta \tilde{J}^{\mu}_{\mu}}+J^{\mu}_{\mu}\right]\,d^{n}x\end{aligned}$$

where the functional derivatives with respect to $\tilde{J}_{\mu\nu}$ represent terms in which the corresponding graviton propagator is amputated. Using (3.16) we see that the last term on the r.h.s. of (4.11) is proportional to $\int \delta W/\delta \tilde{J}^{\mu}_{\mu}$. We can now combine (4.11) and (4.12) in order to eliminate this term. The resulting equation expresses the gauge dependence of W in terms of "unphysical insertions." It follows that we can introduce renormalized parameters, masses and coupling constants, which are functions of the bare ones, the gauge parameter ξ and the cut-off $\epsilon = 4 - n$, such that the physical S-matrix, expressed in terms of these renormalized quantities, is gauge independent. We want to emphasize that this result is valid order by order in the usual perturbation theory. The gauge dependence of the mass counterterms may have a different significance in another expansion scheme.[9]

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FIGURE CAPTIONS

Fig. 1. One loop self energy diagrams of a scalar field.

Fig. 2. Graviton tadpole diagrams.

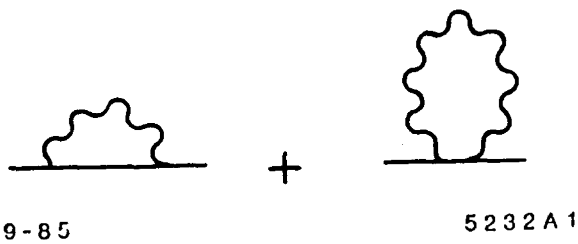

Fig. 1

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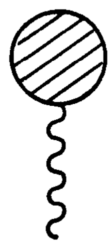

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Fig. 2