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FACTORIZATION PROPERTY OF THE DEUTERON*

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ABSTRACT

Using a simple field-theoretic model we show that, in the zero binding limit, the relativistic deuteron wave function has a cluster decomposition; *i.e.*, factors into two separate nucleon wave functions convoluted with a body wave function. The framework of the calculation is a Fock state expansion at equal time on the light-cone. Assuming a quark interchange mechanism, we then derive the deuteron reduced form factor at large momentum transfer, while recovering the standard impulse approximation form at small momentum transfer.

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1. Introduction

From the standpoint of quantum chromodynamics the deuteron is a complex dynamical system. At large distances the deuteron is evidently well described as a $J = 1,\ I = 0,\ Q = 1$ composite of two nucleon clusters with binding energy ~ 2.2 MeV, together with small admixtures of $\Delta\Delta$ and virtual meson components. However, at short distances, in the region where all six quarks overlap within a distance $R = 1/Q \to 0$, one can show rigorously that the deuteron state in QCD necessarily has "fractional parentage" $(1/9)\, np$, $(4/45)\, \Delta\Delta$, and 4/5 "hidden color" (nonnuclear) components.[1,2] In fact, at any momentum scale the deuteron cannot be described solely in terms of standard nuclear physics degrees of freedom, and in principle, any physical or dynamical property of the deuteron is modified by the presence of such non-Abelian components. For example, the standard "impulse approximation" form for the deuteron form factor (ignoring spin).

$$F_d(Q^2) = F_d^{\rm body}(Q^2)\ F_N(Q^2) \qquad , \tag{1.1}$$

where F_N is the on-shell nucleon form factor, cannot be precisely valid at any momentum transfer scale $Q^2 = -q^2 \neq 0$ because of hidden color components. More important, even if only the nucleon–nucleon component were important, Eq. (1.1) cannot be reliable for composite nucleons since the struck nucleon is necessarily off–shell in the nuclear wave function: $|k'^2 - k^2| \sim \frac{1}{2}Q^2$ (see Fig. 1). Thus in general one requires knowledge of the nucleon form factors $F_N(q^2, k^2, k'^2)$ for the case in which one or both nucleon legs are off-shell.[3] In QCD such amplitudes have completely different dynamical dependence compared to the on-shell form factors.

Although Eq. (1.1) has been used extensively in nuclear physics as a starting point for the analysis of nuclear form factors, its range of validity has never been seriously questioned. Certainly in the non-relativistic domain where target recoil and off-shell effects can be neglected, the charge form factor of a composite system can be computed from the convolution of charge distributions. However, in the general situation, the struck nucleon must transfer a large fraction of its momentum to the spectator system, rendering the nucleon state off-shell. As we shall show here, the region of validity of Eq. (1.1) for the deuteron is very small:

$$Q^2 < 2\,M_d\,\epsilon_d$$

i.e., $Q \lesssim 100$ MeV. However, in this region the nucleon form factor does not deviate significantly from unity,[4] so eq. (1.1) is of doubtful utility.

The deuteron form factor $F_d(Q^2)$, by definition, is the probability amplitude for the deuteron to stay intact after absorbing momentum transfer Q. If the deuteron is taken as a lightly-bound cluster of two nucleons, then the form factor contains the probability amplitudes for each nucleon to remain intact after absorbing momentum transfer $\sim q^\mu/2$. Thus, it is natural to factorize F_d in the form[5]

$$F_d(Q^2) = f_d(Q^2)\ F_N^2(Q^2/4) \qquad , \tag{1.2}$$

which defines the "reduced" form factor $f_d(Q^2)$. As shown in Ref. 1, QCD predicts $Q^2\, f_d(q^2) \cong$ const [modulo logarithmic modifications due to the running coupling constant anomalous dimensions of the nuclear wave function], which is in excellent agreement with experiment for $1 \leq Q^2 \lesssim 4$ GeV^2 (see Fig. 2). Thus it is interesting to understand the origin of the reduced form factor factorization, Eq. (1.2), from a fundamental point of view and to verify for which regime, if any, the standard impulse approximation form, Eq. (1.1), is valid or useful.

In order to study these questions, we construct a simple covariant and gauge-invariant dynamical model of the deuteron which allows an analysis of the effects of nucleon compositeness in the nuclear wave function. Within the framework of this simple model, which neglects hidden color components, we derive a cluster decomposition[6] property of the deuteron wave function and identify a transition region between forms (1.1) and (1.2). The important conclusion is that the impulse approximation (1.1) can only be valid in the nonrelativistic regime $Q^2 \lesssim 2M_d\,\epsilon_d$.

In order to focus on the essential points we will analyze a simple covariant model[7] which incorporates elements of the quark structure of the nucleon:

$$\mathcal{L}_I = g\ \phi_d\ \phi_N\ \phi_N + h\ \epsilon_{ijk}\ \phi_N\ q^i\ q^j\ q^k \quad . \tag{1.3}$$

Here g and h are the coupling constants of a deuteron to two nucleons and a nucleon to three quarks, respectively, and ϵ_{ijk} represents the SU(3) color singlet coupling. The quarks carry the electromagnetic current. This model gives an effective deuteron wavefunction with a factorized two-nucleon structure (see Sec. 2.1),

$$\Psi_d = \psi_d^{\text{body}} \ \times\ \psi_N \ \times\ \psi_N \ . \tag{1.4}$$

Since the relativistic deuteron form factor can be expressed as a convolution of initial and final light-cone Fock state wavefunctions,[8] the factorization of the wavefunction is the origin of form factor factorization in terms of nucleon form factors. Although the explicit model used here is simple, it will be clear from the structure of the proofs that the results can be generalized to the full QCD case.

As we shall show in Sec. 2.2, if Q^2 is small the standard impulse approximation result (1.1) is recovered. However, at large Q^2 the factorization property (1.4) does not hold simultaneously for the initial and final wavefunctions and (1.1) fails. However, we can utilize the standard factorization of QCD for exclusive processes,

$$F_d(Q^2) = \int [dx][dy]\ \phi_d(x_i, Q)\ T_H^d(x_i, y_i, Q)\ \phi_d(y_i, Q)\ , \tag{1.5}$$

where T_H^d is the $6q + \gamma^* \rightarrow 6q$ hard scattering amplitude and[8]

$$\phi_d(x_i, Q) = \int^{(\vec{k}_\perp^2 < Q^2)} [d^2k_\perp]\ \Psi_d^{(Q)}\ (x_i, \vec{k}_{\perp i}) \tag{1.6}$$

is the deuteron distribution amplitude, the probability amplitude to find six quarks within a distance $1/Q$ in the deuteron wavefunction. If the hard scattering amplitude factorizes:

$$T_H^d = T_H^N\ \times\ T_H^N\ \times\ t_H\ , \tag{1.7}$$

then the *reduced* form factor factorization Eq. (1.2) immediately follows. The reduced amplitude t_H controls the fall-off of $f_d(Q^2)$. The hard scattering amplitude T_H is the perturbative amplitude for the six quarks to scatter from collinear to the initial two-nucleon configuration to collinear with the final two-nucleon configuration, where each nucleon has roughly equal momentum. We argue the dominant configuration for this recombination is the quark interchange plus one gluon exchange diagram. Note that in the case of color SU(3), where the gluon is a color octet, single–gluon exchange between the color–singlet nucleons is forbidden. Thus at lowest order in $\alpha_s(Q^2)$ there must also be an interchange of quarks between the nucleons in order to satisfy the color selection rules. This quark interchange model automatically satisfies factorization for the hard scattering amplitude (1.7), with

$$t_H \sim \frac{\alpha_s\,(Q^2)}{Q^2} \quad . \tag{1.8}$$

Using this quark interchange model we derive the reduced form factor defined in Eq. (1.2). This verifies the transition of the deuteron form factor from the standard impulse approximation to the reduced form.

One implication of this derivation of the reduced form factor using the quark interchange model is that the normalization of the reduced form factor can be approximately calculated in perturbative QCD theory without direct evaluation of the hard scattering amplitude T_H. Note that over 300,000 diagrams containing six fermion lines connected by five gluons are required to calculate T_H directly.[9] In the present calculation, the normalization of the reduced form factor is related to the deuteron wave function at small N–N separation[5] $\psi^d_{NR}(\vec{0})$. The relation between the normalization of the distribution amplitudes of the deuteron (A_d) and the nucleon (A_N) in principle could be used to determine the value of A_d and predict the normalization of $f_d(Q^2)$.

2. Factorization of Relativistic Nuclear Wavefunctions and Form Factors

2.1 The Deuteron Wave Function

In QCD the deuteron is a color-singlet composite of six-quark fields. Using light-cone quantization,[10] one can define a consistent Fock state basis at equal $\tau = t + z/c$ which defines the deuteron in terms of $|6q\rangle$, $|6q+g\rangle$, $|6q+q\bar{q}\rangle, \ldots$ components. Only one of the five color singlet configurations of six-quark corresponds to the usual $|NN\rangle$ nucleon-nucleon clustering. However, since the binding energy of the deuteron is very small, we shall assume that this $|6q\rangle = |NN\rangle$ configuration is by far dominant in the natural kinematic domain of the wavefunction. This structure is represented in its simplest form by the Lagrangian of Eq. (1.3). The resulting deuteron wavefunction is illustrated in Fig. 3. In terms of the light-cone variables

$$x_i = \frac{(k^0+k^z)i}{p^0+p^z}\ , \ \sum_{i=1}^{6} x_i = 1,\ \sum_{i=1}^{6} \vec{k}_{\perp i} = 0 \quad ,$$

the wave function has the form of a convolution:

$$\left(M^2 - \sum_{i=1}^{6} \frac{\vec{k}_{\perp i}^2 + m_i^2}{x_i}\right) \Psi_d(x_i, \vec{k}_{\perp i}) = \frac{g}{M^2 - \dfrac{\vec{\ell}_\perp + M_N^2}{y(1-y)}}\ \frac{1}{y}\ \frac{1}{1-y}$$

$$\times h^2 \left(\frac{1}{M^2 - \dfrac{\vec{\ell}_\perp + M_N^2}{(1-y)} - \sum_{i=1}^{3} \dfrac{\vec{k}_{\perp i}^2 + m_i^2}{x_i}} + \frac{1}{M^2 - \dfrac{\vec{\ell}_\perp^2 + M_N^2}{y} - \sum_{j=4}^{6} \dfrac{\vec{k}_{\perp j}^2 + m_j^2}{x_j}} \right), \tag{2.1}$$

where M, M_N and m_i are the masses of the deuteron, the nucleon, and the quarks, respectively, and the momentum-conserving delta function fixes $y = \sum_{i=1}^{3} x_i$ and $\vec{\ell}_\perp = \sum_{i=1}^{3} \vec{k}_{\perp i}$. If we define the function $\epsilon(y, \vec{\ell}_\perp)$,

$$\epsilon(y, \vec{\ell}_\perp) = M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{y(1-y)} \quad , \tag{2.2}$$

then $\epsilon(y, \vec{\ell}_\perp)$ measures the deuteron off-shell light-cone energy $\epsilon = p^+ \cdot \sum_{i=1}^{6} k_i^-$. The zero binding energy limit implies $\epsilon(y, \vec{\ell}_\perp) \to 0$. In the $\epsilon(y, \vec{\ell}_\perp) \to 0$ limit, $y \to 1/2$ and $\vec{\ell}_\perp \to 0$ since $M^2 \to 4M_N^2$. Thus we obtain approximate delta function behavior of $\epsilon^{-1}(y, \vec{\ell}_\perp)$ near the zero binding energy limit:

$$\epsilon^{-1}(y, \vec{\ell}_\perp) \sim \delta \left(y - \frac{1}{2} \right) \delta^2 \left(\vec{\ell}_\perp \right) \quad . \tag{2.3}$$

In this limit, the factor inside the parenthesis of the right hand side of Eq. (2.1) is given by

$$\frac{M^2 - \sum_{i=1}^{6} \frac{\vec{k}_{\perp i}^2 + m_i^2}{x_i}}{\left(M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{(1-y)} - \sum_{i=1}^{3} \frac{\vec{k}_{\perp i}^2 + m_i^2}{x_i} \right) \left(M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{y} - \sum_{j=4}^{6} \frac{\vec{k}_{\perp j}^2 + m_j^2}{x_j} \right)} \quad . \tag{2.4}$$

The numerator of the right hand side of Eq. (2.4) is cancelled by the factor on the left hand side of Eq. (2.1), so that in $\epsilon(y, \vec{\ell}_\perp) \to 0$ limit $\Psi_d(x_i, \vec{k}_{\perp i})$ is given by

$$\Psi_d(x_i,\vec{k}_{\perp i}) = \frac{g}{M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{y(1-y)}} \; \frac{1}{y} \; \frac{h}{M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{(1-y)} - \sum_{i=1}^{3} \frac{\vec{k}_{\perp i}^2 + m_i^2}{x_i}}$$
$$\times \frac{1}{1-y} \; \frac{h}{M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{y} - \sum_{j=4}^{6} \frac{\vec{k}_{\perp j}^2 + m_j^2}{x_j}} \quad . \tag{2.5}$$

Furthermore, if we change the variables:

$$\begin{aligned} z_i &= \frac{x_i}{y} \quad , \qquad \vec{k}'_{\perp i} = \vec{k}_{\perp i} - z_i \, \vec{\ell}_\perp \qquad (i = 1,2,3) \\ z_j &= \frac{x_j}{1-y} \quad , \qquad \vec{k}'_{\perp j} = \vec{k}_{\perp j} - z_j \, \vec{\ell}_\perp \qquad (j = 4,5,6) \quad , \end{aligned} \tag{2.6}$$

then

$$\begin{aligned} M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{(1-y)} - \sum_{i=1}^{3} \frac{\vec{k}_{\perp i}^2 + m_i^2}{x_i} &\rightarrow \frac{1}{y}\left(M_N^2 - \sum_{i=1}^{3} \frac{\vec{k}'^2_{\perp i} + m_i^2}{z_i} \right) \\ M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{y} - \sum_{j=4}^{6} \frac{\vec{k}_{\perp j}^2 + m_j^2}{x_j} &\rightarrow \frac{1}{1-y}\left(M_N^2 - \sum_{j=4}^{6} \frac{\vec{k}'^2_{\perp j} + m_j^2}{z_j} \right) \quad . \end{aligned} \tag{2.7}$$

Thus for $\epsilon \rightarrow 0$, Eq. (2.5) is reduced to

$$\Psi_d(x_i,\vec{k}_{\perp i}) = \frac{g}{M^2 - \frac{\vec{\ell}_\perp^2 + M_N^2}{y(1-y)}} \; \frac{h}{M_N^2 - \sum_{i=1}^{3} \frac{\vec{k}'^2_{\perp i} + m_i^2}{z_i}} \; \frac{h}{M_N^2 - \sum_{j=4}^{6} \frac{\vec{k}'^2_{\perp j} + m_j^2}{z_j}} \quad . \tag{2.8}$$

This is the expected factorized form of the deuteron wave function [Eq. (1.4)], since the last two terms of the right hand side of Eq. (2.8) are the nucleon wave functions $\psi_N(z_i, \vec{k}'_\perp)$ and $\psi_N(z_j, \vec{k}'_\perp)$. The new light–cone variables z_i and $\vec{k}'_{\perp i}$ are the light–cone momentum fractions and the transverse momenta in the nucleon frames. The first term of the right hand side of Eq. (2.8) is the "body" wave function $\psi_d^{\text{body}}(y, \vec{\ell}_\perp)$. This proves the factorization of the deuteron wave function in the zero binding energy limit:

$$\Psi_d(x_i, \vec{k}_{\perp i}) \;=\; \psi_d^{\text{body}}(y, \vec{\ell}_\perp)\; \psi_N(z_i, \vec{k}'_{\perp i})\; \psi_N(z_j, \vec{k}'_{\perp j}) \quad . \qquad (2.9)$$

2.2 The Impulse Approximation

The form factor of the deuteron is given exactly in terms of the light-cone Fock state expansion by[8] (a sum over Fock components is understood)

$$F_d(\vec{q}_\perp^{\,2}) = \sum_{a=1}^{6} e_a \int [dx] \int [d^2\vec{k}_{\perp i}]\; \Psi_d^*\left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\,\vec{q}_\perp\right)\; \Psi_d(x_i, \vec{k}_{\perp i}) \;, \qquad (2.10)$$

where $\vec{q}_\perp$ is absorbed by a^{th} quark, $\vec{q}_\perp^{\,2} = Q^2$, and

$$[dx] = \delta\left(1 - \sum_{i=1}^{6} x_i\right) \prod_{i=1}^{6} \frac{dx_i}{x_i} \quad ,$$

$$[d^2\vec{k}_{\perp i}] = 16\pi^3\, \delta^2\left(\sum_{i=1}^{6} \vec{k}_{\perp i}\right) \prod_{i=1}^{6} \frac{d^2\vec{k}_{\perp i}}{16\pi^3} \quad . \qquad (2.11)$$

In the last section we demonstrated the factorization of $\Psi_d(x_i, \vec{k}_{\perp i})$ for small $\epsilon(y, \vec{\ell}_\perp)$. If $|\vec{q}_\perp|$ is the order of $|\vec{\ell}_\perp|$ or $|\vec{k}_{\perp i}|$, then $\psi^* \left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\, \vec{q}_\perp\right)$ is factorized in the same way as $\psi(x_i, \vec{k}_{\perp i})$ since $\epsilon(y, \vec{\ell}_\perp + (1-y)\vec{q}_\perp)$ is almost the same as $\epsilon(y, \vec{\ell}_\perp)$. Thus for small q^2 the factorization of $\sum\limits_{a=1}^{6} \Psi_d^* \left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\, \vec{q}_\perp\right)$ is given by

$$
\begin{aligned}
\sum_{a=1}^{6} \Psi_d^* \left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\, \vec{q}_\perp\right) &= \psi_d^{*\,\mathrm{body}} \left(y, \vec{\ell}_\perp + (1-y)\vec{q}_\perp\right) \\
&\times \Bigg[\sum_{a=1}^{3} \psi_N^* \left(z_i, \vec{k}'_{\perp i} + (\delta_{ia} - z_i)\, \vec{q}_\perp\right) \psi_N^*(z_j, \vec{k}'_{\perp j}) \\
&+ \sum_{a=4}^{6} \psi_N^*(z_i, \vec{k}'_{\perp i})\, \psi_N^* \left(z_j, \vec{k}'_{\perp j} + (\delta_{ja} - z_j)\, \vec{q}_\perp\right) \Bigg] \; .
\end{aligned}
\tag{2.12}
$$

[This result becomes invalid if $|\vec{q}_\perp|$ is much larger than $|\vec{\ell}_\perp|$ since $\epsilon(y, \vec{\ell}_\perp + (1-y)\vec{q}_\perp)$ is then non-negligible.] The integrating weight is also simply decomposed:

$$
\int [dx] \int [d^2\vec{k}_{\perp i}] = \int_0^1 \frac{dy}{y(1-y)} \int \frac{d^2\vec{\ell}_\perp}{16\pi^3} \int [dz]_i \int [d^2\vec{k}'_\perp]_i \int [dz]_j \int [d^2\vec{k}'_\perp]_j \; , \tag{2.13}
$$

where

$$[dz]_i = \delta\left(1 - \sum_{i=1}^{3} z_i\right) \prod_{i=1}^{3} \frac{dz_i}{z_i} \quad ,$$

$$[dz]_j = \delta\left(1 - \sum_{j=4}^{6} z_j\right) \prod_{j=4}^{6} \frac{dz_j}{z_j} \quad ,$$

$$[d^2\vec{k}'_\perp]_i = 16\pi^3\,\delta^2\left(\sum_{i=1}^{3} \vec{k}'_{\perp i}\right) \prod_{i=1}^{3} \frac{d^2\vec{k}'_{\perp i}}{16\pi^3} \quad ,$$

$$[d^2\vec{k}'_\perp]_j = 16\pi^3\,\delta^2\left(\sum_{j=4}^{6} \vec{k}'_{\perp i}\right) \prod_{j=4}^{6} \frac{d^2\vec{k}'_{\perp j}}{16\pi^3} \quad . \tag{2.14}$$

Thus Eq. (2.10) becomes

$$\begin{aligned}
F_d(\vec{q}^{\,2}_\perp) \;=\; & \int_0^1 \frac{dy}{y(1-y)} \int \frac{d^2\vec{\ell}_\perp}{16\pi^3}\, \psi_d^{*\text{body}}\left(y, \vec{\ell}_\perp + (1-y)\vec{q}_\perp\right)\, \psi_d^{\text{body}}(y, \vec{\ell}_\perp) \\
& \times \left[\sum_{a=1}^{3} e_a \int [dz]_i \int [d^2\vec{k}'_\perp]_i\, \psi_N^*\left(z_i, \vec{k}'_{\perp i} + (\delta_{ia} - z_i)\,\vec{q}_\perp\right)\, \psi_N(z_i, \vec{k}'_{\perp i})\right. \\
& \times \int [dz]_j \int [d^2\vec{k}'_\perp]_j\, \psi_N^*(z_j, \vec{k}'_{\perp j})\, \psi_N(z_j, \vec{k}'_{\perp j}) \\
& + \sum_{a=4}^{6} \int [dz]_i \int [d^2\vec{k}'_\perp]_i\, \psi_N^*(z_i, \vec{k}'_{\perp i})\, \psi_N(z_i, \vec{k}'_{\perp i}) \\
& \times \left. e_a \int [dz]_j \int [d^2\vec{k}'_\perp]_j\, \psi_N^*\left(z_j, \vec{k}'_{\perp j} + (\delta_{ja} - z_j)\,\vec{q}_\perp\right)\, \psi_N(z_j, \vec{k}'_{\perp j})\right] \\
& = \sum_N F_N(\vec{q}^{\,2}_\perp)\, F_d^{\text{body}}(\vec{q}^{\,2}_\perp) \quad ,
\end{aligned} \tag{2.15}$$

where the body form factor $F_d^{\rm body}(\vec{q}_\perp^{\,2})$ is defined by

$$F_d^{\rm body}(\vec{q}_\perp^{\,2}) = \int_0^1 \frac{dy}{y(1-y)} \int \frac{d^2\vec{\ell}_\perp}{16\pi^3}\, \psi_d^{*\,\rm body}\left(y, \vec{\ell}_\perp + (1-y)\vec{q}_\perp\right)\, \psi_d^{\rm body}(y, \vec{\ell}_\perp)\ . \tag{2.16}$$

Equation (2.16) is the same form as Eq. (1.1). This proves the impulse approximation at small $|\vec{q}_\perp|$ for $q_\perp^2$ of order of $|\vec{\ell}_\perp^{\,2}|$ or $|\vec{k}_{\perp i}^{\,2}|$.

2.3 Reduced Form Factor

When $|\vec{q}_\perp|$ becomes large, $|\vec{q}_\perp| \gg |\vec{\ell}_\perp|$ or $|\vec{k}_{\perp i}|$, then the impulse approximation (2.15) breaks down since $|\, \epsilon\left(y, \vec{\ell}_\perp + (1-y)\vec{q}_\perp\right) \,|$ becomes large and $\Psi_d^*\left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\,\vec{q}_\perp\right)$ cannot be factorized in the same way as $\psi(x_i, \vec{k}_{\perp i})$. However, even in the case $|\vec{q}_\perp| \gg |\vec{\ell}_\perp|$ or $|\vec{k}_{\perp i}|$, the deuteron after absorbing $\vec{q}_\perp$ must be a bound state of two nucleons since the target remains intact by the definition of the form factor. Thus the quarks of the deuteron must exchange momentum so that a large fraction of $\vec{q}_\perp$ can be transferred from the quark which absorbs $\vec{q}_\perp$ to the quarks of the other nucleon. In QCD theory, the momentum transfer is due to gluon exchange. The dominant lowest order contribution to the evolution kernel is represented by the one gluon exchange diagrams shown in Fig. 4. Since the gluon is a color octet in SU(3) color group, quarks must be interchanged between the nucleons in order to satisfy the color selection rules.

The equation of motion for $\psi\left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\,\vec{q}_\perp\right)$ is given by

$$\left[M^2 - \sum_{i=1}^{6} \frac{\left\{\vec{k}_{\perp i} + (\delta_{ia} - x_i)\,\vec{q}_\perp\right\}^2 + m_i^2}{x_i}\right] \Psi_d\left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\,\vec{q}_\perp\right)$$

$$= \int [dw]\,[d^2\vec{j}_\perp]\; V\left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\vec{q}_\perp;\, w_j, \vec{j}_{\perp j}\right)\; \Psi_d(w_j, \vec{j}_{\perp j}) \;. \tag{2.17}$$

The factorization of $\Psi_d(w_j, \vec{j}_{\perp j})$ for low relative momenta is already proved in Sec. 2.1 [see Eq. (2.9)];

$$\begin{aligned}\Psi_d(w_i, \vec{j}_{\perp i}) = \psi_d^{\text{body}}(y, \vec{\ell}_\perp)\; \psi_N &\left(\frac{w_i}{\sum\limits_{i=1}^{3} w_i}\;,\; \vec{j}_{\perp i} - \frac{w_i}{\sum\limits_{i=1}^{3} w_i}\sum_{i=1}^{3} \vec{j}_{\perp i}\right) \\ \times\, \psi_N &\left(\frac{w_j}{\sum\limits_{j=1}^{3} w_j}\;,\; \vec{j}_{\perp j} - \frac{w_j}{\sum\limits_{j=4}^{6} w_j}\sum_{j=4}^{6} \vec{j}_{\perp j}\right) \;.\end{aligned} \tag{2.18}$$

The body wave function $\psi_d^{\text{body}}(y, \vec{\ell}_\perp)$ behaves like a delta function near the zero binding energy limit [see Eq. (2.3)]

$$\psi_d^{\text{body}}(y, \vec{\ell}_\perp) = 16\pi^3\, \delta\left(y - \frac{1}{2}\right)\; \delta^2\left(\vec{\ell}_\perp\right)\; \psi_{NR}^{d}(\vec{0}) \qquad , \tag{2.19}$$

where

$$\psi_{NR}^{d}(\vec{0}) = \int dy\, \frac{d^2\vec{\ell}_\perp}{16\pi^3}\; \psi_d^{\text{body}}(y, \vec{\ell}_\perp) \qquad . \tag{2.20}$$

Thus the integration in Eq. (2.17) is trivial and the variables $w_i, \vec{j}_{\perp i}$ are fixed for the quark interchange model:

$$\begin{aligned} w_i &= x_i \qquad , \\ \vec{j}_{\perp i} &= \vec{k}_{\perp i} + \{\, y\delta_{ia} + (1-y)\delta_{ib} - x_i \,\}\ \vec{q}_\perp \qquad , \end{aligned} \tag{2.21}$$

where a and b are indices of two interchanged quarks. Using Eq. (2.21), we can prove that Eq. (2.18) reduces to

$$\begin{aligned} \Psi_d(w_i, \vec{j}_{\perp i}) = &\ \psi_d^{\text{body}}(y, \vec{\ell}_\perp) \\ &\times\ \psi_N\left(z_i, \vec{k}'_{\perp i} + (\delta_{ai} - z_i)\, y\, \vec{q}_\perp\right) \\ &\times\ \psi_N\left(z_j, \vec{k}'_{\perp j} + (\delta_{bj} - z_j)\,(1-y)\, \vec{q}_\perp\right) \quad . \end{aligned} \tag{2.22}$$

By substituting Eq. (2.22) into Eq. (2.17), we obtain the factorization of:

$$\begin{aligned} &\sum_{a=1}^{6} \Psi_d\left(x_i, \vec{k}_{\perp i} + (\delta_{ia} - x_i)\,\vec{q}_\perp\right) = \left(\sum_{a=1}^{3}\sum_{b=4}^{6} + \sum_{a=4}^{6}\sum_{b=1}^{3}\right) \frac{x_a}{x_a - 1} \\ &\times\ \frac{1}{\vec{q}_\perp^{\,2}} V\left(x_i,\ (\delta_{ia} - x_i)\vec{q}_\perp;\ x_j, \{y\delta_{ja} + (1-y)\delta_{jb} - x_j\}\,\vec{q}_\perp\right) \\ &\times\ \psi_N\left(z_i, \vec{k}'_{\perp i} + (\delta_{ia} - z_i) y \vec{q}_\perp\right) \\ &\times\ \psi_N\left(z_j, \vec{k}'_{\perp j} + (\delta_{bj} - z_j)(1-y)\vec{q}_\perp\right)\ \psi_{NR}^{d}(\vec{0}) \quad , \end{aligned} \tag{2.23}$$

where the kernel V can be obtained by calculating the diagrams shown in Fig. 4. The weak binding of the deuteron forces $y \sim \frac{1}{2}$. On the average we expect the struck and interchanged quark to have roughly the same x. Using this approximation we obtain the factorization of the form factor from Eq. (2.10):

$$\begin{aligned} F(\vec{q}_\perp^{\,2}) &= \frac{C}{\vec{q}_\perp^{\,2}} \mid \psi^d_{NR}(\vec{0}) \mid^2 \\ &\times \left[\sum_{a=1}^{3} \int [dz]_i \, [d^2\vec{k}'_\perp]_i \, \psi^*_N \left(z_i, \vec{k}'_{\perp i} + (\delta_{ia} - z_i) \frac{\vec{q}_\perp}{2} \right) \, \psi_N(z_i, \vec{k}'_\perp) \right. \\ &\times \left. \sum_{b=4}^{6} \int [dz]_j \, [d^2\vec{k}'_\perp]_j \, \psi^*_N \left(z_j, \vec{k}'_{\perp j} + (\delta_{jb} - z_j) \frac{\vec{q}_\perp}{2} \right) \psi_N(z_j, \vec{k}'_\perp) + (a \leftrightarrow b) \right] \\ &= f_d(\vec{q}_\perp^{\,2}) \; F_N^2 \left(\vec{q}^{\,2}/4\right) \qquad , \end{aligned} \tag{2.24}$$

where the reduced form factor $f_d(\vec{q}_\perp^{\,2})$ is defined by

$$f_d(\vec{q}_\perp^{\,2}) = \frac{C}{\vec{q}_\perp^{\,2}} \mid \psi^d_{NR}(\vec{0}) \mid^2 \qquad , \tag{2.25}$$

and C is determined by value of the kernel V. More generally, we may iterate the wavefunction wherever large momentum transfer is required and in this way build up the entire T_H contribution to the form factor, as in Eq. (1.6). Equation (2.2) is thus the same form as Eq. (1.2). This proves the transition of the form factor at large $|\vec{q}_\perp|$ $(|\vec{q}_\perp| \gg |\vec{\ell}_\perp|$ or $|\vec{k}_{\perp i}|)$ from the impulse approximation form to the reduced form.

In the full QCD analysis, the iteration of the gluon exchange kernel leads to a logarithmically evoluting distribution amplitude which replaces $\psi^d_{NR}(\vec{0})$.

At large Q^2 the gluon exchange kernel generates other color singlet configuration of six quarks, so that the approximation that the deuteron only consists of a nucleon pair breaks down. The complete calculation of the deuteron form factor thus requires the inclusion of these other components. The reduced form factor prediction is useful for incorporating non-leading power law corrections, but it does not include the hidden color contributions of the deuteron wavefunction (see Fig. 5).

The definition of $f_d(Q^2) = F_d(Q^2)/F_N^2(Q^2/4)$ provides a convenient tool for comparing QCD with experiment since it correctly removes the effects of nucleon compositeness for the part of the deuteron wavefunction which consists of two nucleons. More generally QCD predicts at large Q^2

$$f_d(Q^2) = \frac{\alpha_s(Q^2)}{Q^2} \sum_{n=0}^{\infty} a_n \left(\ell n\, Q^2/\Lambda^2\right)^{\Gamma_n} \times [1 + \mathcal{O}(\alpha_s(Q^2), m^2/Q^2)] \; ,$$

where the Γ_n are determined from the difference of deuteron and nucleon anomalous dimensions. Here $\Gamma_0 = -\frac{3}{4}\,\frac{C_F}{\beta}$. Since $(\ell_n Q^2/\Lambda^2)$ is slowly varying, the essential test of QCD in the deuteron is the prediction $f_d(Q^2) \sim 1/Q^2$ for the leading helicity zero to helicity zero form factor, and that the other non-zero helicity deuteron form factors are relatively power law suppressed at large momentum transfer.

3. Discussion and Conclusion

In the zero binding limit, the light-cone Fock state wavefunction naturally decomposes into a product form of cluster wavefunctions. This result (Eq. (2.9)) is closely related to the cluster decomposition theorem for scattering amplitudes proved in Ref. 5. Thus the nuclear wavefunction to a good approximation contains as factors a product of on-shell nucleon wavefunctions, but only in the near on-shell regime where the relative momentum of the nucleons is small. The factorization of light-cone wavefunctions leads, as we have shown, to various forms of factorization for the nuclear form factor. At low $Q^2 < 2M_d\epsilon_d$, the usual impulse approximation result is valid. The region of validity of this form though is limited to momentum transfers smaller than the inverse size of the nucleons where the struck nucleon can remain nearly on-shell by virtue of the nuclear Fermi motion. In this domain, the nucleon form factor is still nearly point-like $F_N(Q^2) \sim 1$. At larger Q^2, the kinematics of the boosted recoil nucleus forces the struck nucleon off-shell and the traditional form of factorization becomes useless. Fortunately, in this domain the reduced form factor result becomes approximately valid, replacing the impulse approximation as a valid starting point for QCD phenomenology. We have also discussed a simple quark interchange model. Using this model one can not only avoid the enormous labor[8] (300,000 diagrams) required to calculate the hard scattering amplitude directly, but it also allows one to connect the reduced form factor with the phenomenological value $\psi^d_{NR}(\vec{0})$, the deuteron body wave function at origin.

Acknowledgements

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Figure Captions

Fig. 1. Representation of the deuteron form factor according to the standard nuclear physics impulse approximation. Here $|\, k'^2 - k^2 \,| = |\, 2k \cdot q + q^2 \,| \sim Q^2/2$ since $k \sim p/2$ or $k' \sim \frac{1}{2}(p+q)$.

Fig. 2. (a) Comparison of the asymptotic QCD prediction $f_d(Q^2) \propto 1/Q^2 \, [\ell n(Q^2/\Lambda^2)]^{-1+\Gamma_0}$ with data for the reduced deuteron form factor, where $F_N(Q^2) = (1 + Q^2/0.71 \text{ GeV}^2)^{-2}$. The normalization is fixed at the $Q^2 = 4 \text{ GeV}^2$ data point. [(b) Comparison of the prediction $[1+(Q^2/m_0^2)]f_d(Q^2) \propto [\ell n(Q^2/\Lambda^2)]^{-1+\Gamma_0}$ with the above data. The value $m_0^2 = 0.28 \text{ GeV}^2$ is used.

Fig. 3. The diagrammatic kernel equation of the relativistic deuteron wave function in the light–cone frame. The effective ϕ^3–type interaction [see Eq. (1.3)] provides the clustering of two separate nucleons.

Fig. 4. The lowest order diagrams of the quark interchange model.

Fig. 5. QCD contribution included in analysis of the reduced form factor. The gluon contributions to the deuteron wave function indicated by dotted lines lead to hidden color components and are not included.

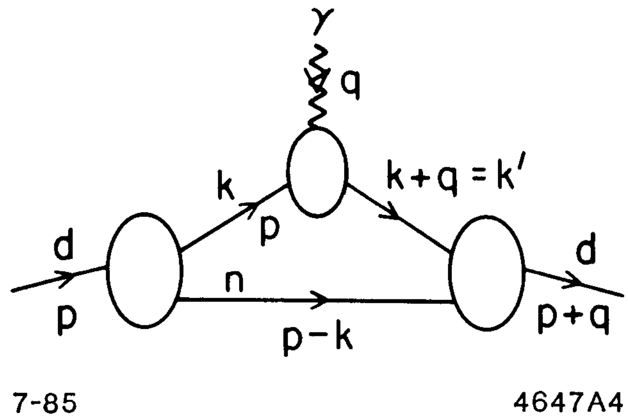


Fig. 1

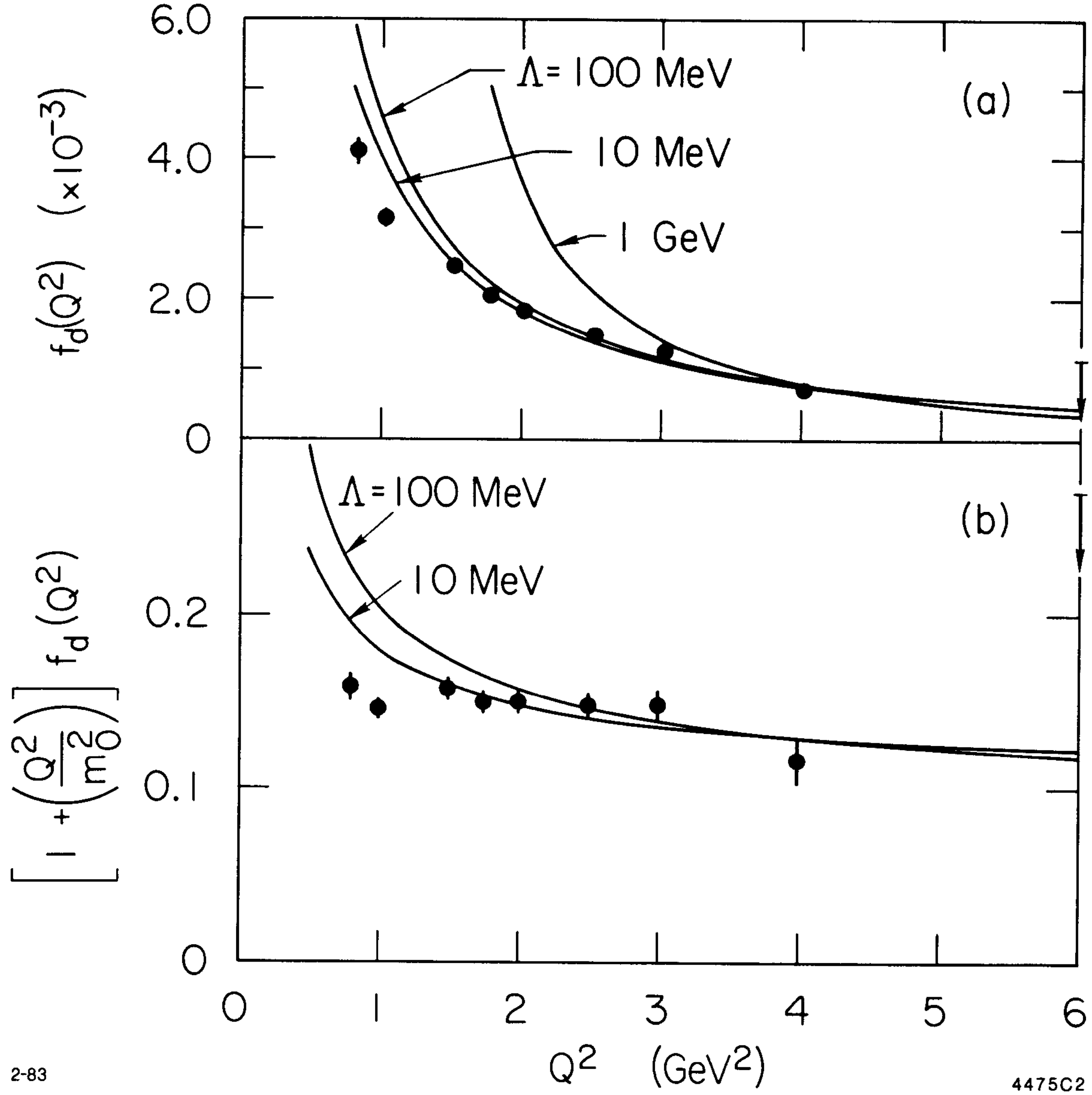
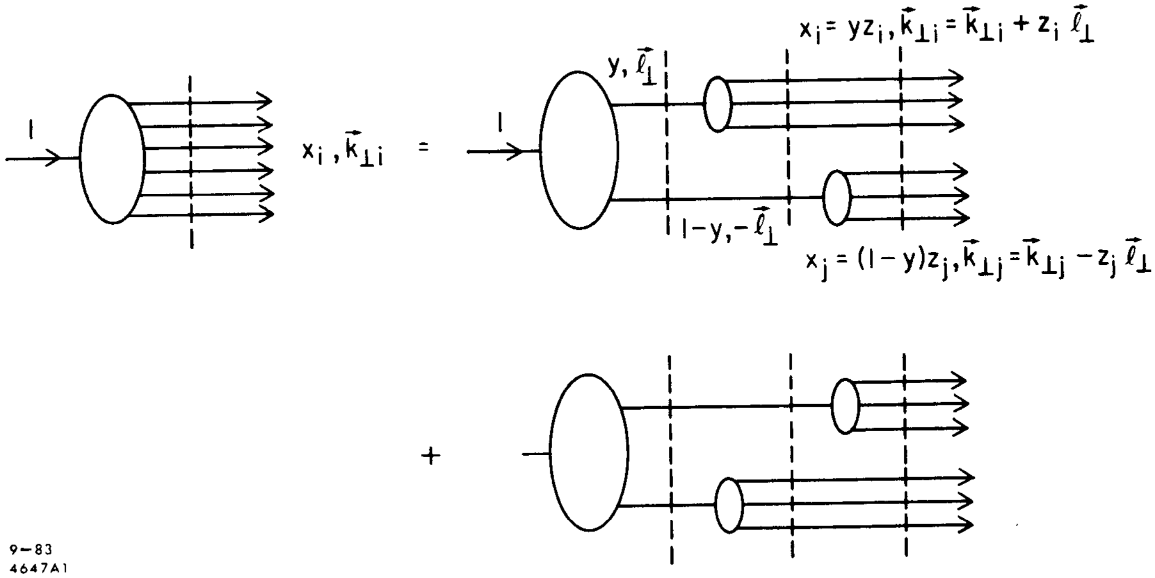
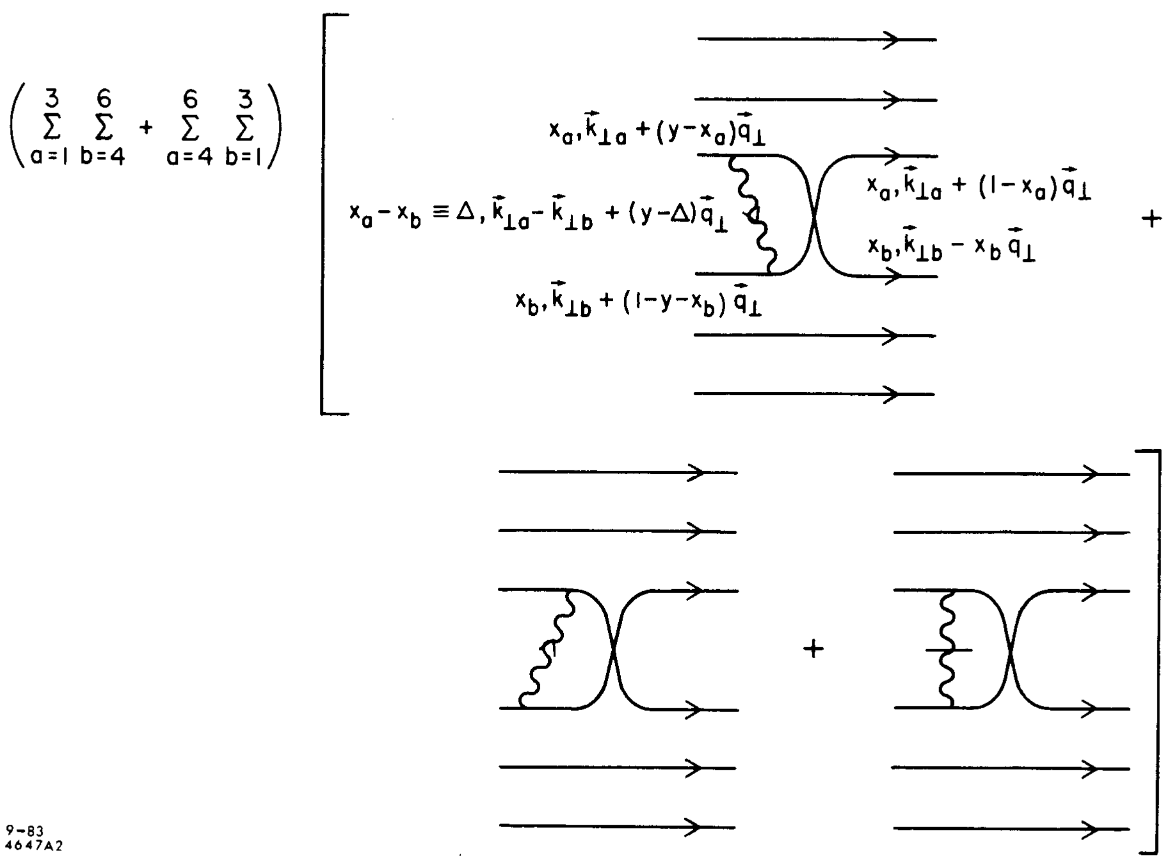

6.0
4.0
2.0
0
$f_d(Q^2)$ $(\times 10^{-3})$
Λ = 100 MeV
10 MeV
1 GeV
(a)
0.2
0.1
0
$\left[1+\left(\frac{Q^2}{m_0^2}\right)\right] f_d(Q^2)$
Λ = 100 MeV
10 MeV
(b)
0
1
2
3
4
5
6
Q^2 (GeV^2)
2-83
4475C2

Fig. 2



1
$x_i, \vec{k}_{\perp i}$
=
1
$y, \vec{\ell}_\perp$
$1-y, -\vec{\ell}_\perp$
$x_i = yz_i, \vec{k}_{\perp i} = \vec{k}_{\perp i} + z_i \vec{\ell}_\perp$
$x_j = (1-y)z_j, \vec{k}_{\perp j} = \vec{k}_{\perp j} - z_j \vec{\ell}_\perp$
+
9-83
4647A1

Fig. 3



$\left(\sum_{a=1}^{3} \sum_{b=4}^{6} + \sum_{a=4}^{6} \sum_{b=1}^{3} \right)$
$x_a, \vec{k}_{\perp a} + (y - x_a)\vec{q}_\perp$
$x_a - x_b \equiv \Delta, \vec{k}_{\perp a} - \vec{k}_{\perp b} + (y-\Delta)\vec{q}_\perp$
$x_b, \vec{k}_{\perp b} + (1-y-x_b)\vec{q}_\perp$
$x_a, \vec{k}_{\perp a} + (1-x_a)\vec{q}_\perp$
$x_b, \vec{k}_{\perp b} - x_b \vec{q}_\perp$
+
+
9-83
4647A2

Fig. 4

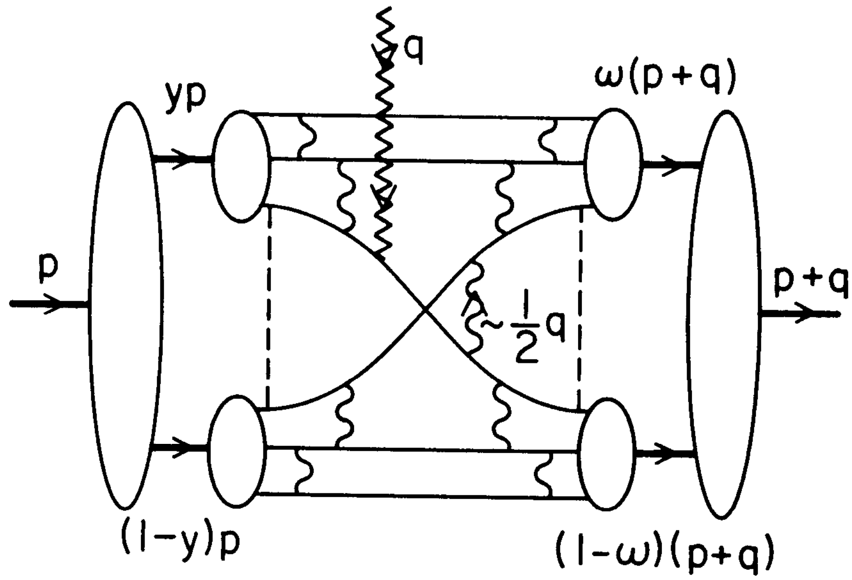


Fig. 5