SLAC – PUB – 3764
September 1985
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PHYSICAL ORIGIN OF TOPOLOGICAL MASS IN 2+1 DIMENSIONS*

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ABSTRACT

In this paper QED in 2+1 dimensions is studied. We offer a simple physical description of the mechanism that gives rise to anomalous fermion currents and the "topological" mass (Chern-Simmons term) in the effective gauge theory. The dynamical properties of the fermions are analyzed and explained in some detail. The physical properties responsible for these anomalous effects are identified and the mechanisms clarified. A novel result that follows from our discussion is that the ground state carries spin (S) as well as charge (Q) and that $S = Q/2$.

The relation of these effects to the 1+1-dimensional chiral anomaly and to some other topological features is discussed. The possible connection between QED in 2+1 dimensions and theories of the Quantum Hall effect is briefly studied. Even though both theories contain crossed-field currents, a detailed analysis of the structure of QED shows that they are different even at a physical level. Hence QED in 2+1 is <u>not</u> a suitable model for the Quantum Hall effect.

Submitted to *Nuclear Physics [FS]*

⋆ Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.
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1. Introduction

It was Schwinger who first realized that gauge invariance and massive gauge fields can coexist peacefully. He proposed the much celebrated "Schwinger model" or QED in 1+1 dimensions. This example allows the photons to become massive without spoiling gauge invariance and thereby sparked the search for mechanisms of a similiar type that operate in higher dimensional theories, so far with little success. The only well understood mechanism remains the Higgs phenomenon.

Schonfeld[1] and Deser, Jackiw and Templeton[2] realized that in 2+1 dimensions there exists a possible mass term for the gauge field Lagrangian that maintains gauge invariance,[3] however it breaks parity and time reversal. These authors showed that this mass term in non-Abelian theories has a deep topological meaning.[4] It is the Chern-Simmons (C-S) secondary characteristic class. The gauge field Lagrangian is $\mathrm{L} = \mathrm{L}_g + \mathrm{L}_{CS}$, where L_g is the usual term and L_{CS} is the C-S term that gives the gauge field a mass. Thus the name "topological mass".

Although the equations of motion are gauge invariant, L is not.[4] In the non-Abelian theory, under a "large" gauge transformation, the exponential of the action changes by a phase leading to the quantization of the mass parameter.[5]

Recently it has been shown that in gauge theories with fermions, such a topological mass term is induced by the interactions of these fermions,[6] and the long-distance effective gauge theory contains a C-S term.

That induced fermionic currents can lead to terms that violate parity and time reversal can be seen as follows. In 2+1 dimensions, besides the usual transverse structure, the current can have a contribution proportional to $\epsilon^{\mu\nu\rho}\partial_\nu A_\rho$ (ϵ is the totally antisymmetric tensor).[3] This leads directly to the C-S term. In non-Abelian theories, the C-S terms do not receive quantum corrections for topological reason.[4,5,7] What is perhaps surprising is that in the Abelian theory there are no corrections either,[8,9] even though the Abelian theory does not enjoy an interesting topological structure in 2+1 dimensions. The mass term (the Abelian part of the C-S term) does not lead to parameter quantization; the mass is not quantized. Recently it was also suggested by some others that these parity violating effects are related to the 1+1 dimensional chiral anomaly and zero modes of the Dirac operator.[10]

In the presence of an external background electric field, the parity abnormal contribution to the fermionic current leads to a current *perpendicular* to the electric field.[11]

This fact has motivated some authors to try to explain another fascinating phenomenon in condensed matter, the Quantum Hall Effect (QHE), using QED in 2+1 dimensions as a fundamental theory.[12,13] The hope is that somehow near the Fermi surface the effective theory may contain a C-S term. (However to the knowledge of the present authors this has not been shown.)

One of our aims is to use our detailed physical study of QED in 2+1 dimensions and then to compare it to the physics of the QHE. Our understanding of theories with a C-S term suggests that they are probably completely unrelated to the QHE.

In this paper we will offer a simple physical picture for the appearance of the parity abnormal effects. Certain topological aspects and the relation to the 1+1 dimensional chiral anomaly[10,11] are also clarified. We study in detail the spin properties of the theory and ground state quantum numbers. A novel relation between the charge and spin of the ground state is derived. Although we analyze in detail only the Abelian theory, we believe our arguments may be generalized to the non-Abelian case.

2. Generalities

We begin our study of $2+1$ dimensional theories by recalling certain essential features of the Poincaré group in $2+1$ dimensions.[14] These features are somewhat unfamiliar and their understanding will allow us to clarify the physics of interacting theories.

The Lorentz group in $2+1$ dimensions is $0(2,1)$, the full Poincaré group has six generators, three generalized translations and three generalized rotations (2 boosts and 1 rotation). This group has two Casimirs: the invariant mass (squared)

$$M^2 = P_\mu P^\mu \tag{1}$$

with P_μ the three-momentum, and the Pauli-Lubanski invariant "spin"

$$W = \epsilon^{\mu\nu\rho} M_{\mu\nu} P_\rho \tag{2}$$

where $M_{\mu\nu}$ is the generalized angular momentum tensor. In two space dimensions there is only one generator of angular momentum (rotations in the plane) corresponding to J_z, the projection of angular momentum onto the missing z-axis. The parity operation is

$$(t,x,y) \rightarrow (t,-x,y) \tag{3}$$

since changing the sign of both x and y amounts to a rotation. Therefore we see that J_z is a pseudoscalar in 2 space dimensions, as is W in Eq. (2). For a particle of invariant mass M,

$$W = M\,S_z\ ; \tag{4}$$

where S_z is the spin part of the angular momentum. These features have a profound impact on the dynamics in the plane; there is no helicity, for example.

For a massless particle, helicity is the only meaningful concept of spin, and hence massless particles are spinless. Thus we find the constraint

$$P_\mu P^\mu = 0 \Rightarrow W = 0 \ . \tag{5a}$$

One the other hand if a particle has spin, then it does possess a rest frame which can be seen as follows. Since the spin must be in the "z-direction", and since there is no helicity, the particle is definitely massive and

$$W \neq 0 \Rightarrow P_\mu P^\mu \neq 0 \ . \tag{5b}$$

Thus we see that angular momentum and spin have very peculiar properties in 2-space dimensions. Furthermore since there is only one generator of angular momentum, there is no non-Abelian Lie algebra that can restrict its possible eigenvalues, giving rise to the possibility of exotic statistics.[15] In the next two sections we will review certain features of the free theories, and clarify the subtleties that will be responsible for the interesting phenomena in the interacting theory.

3. Free Fermions

We will choose to work with two component Dirac spinors, and leave the discussion of four components spinors to the last section. The Dirac algebra is then satisfied by the Pauli matrices and we choose the representation in which

$$\gamma^0 = \sigma_3 \quad \gamma^1 = i\sigma_1 \quad \gamma^2 = i\sigma_2 \tag{6a}$$

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} - i\epsilon^{\mu\nu\rho}\gamma_\rho \tag{6b}$$

$$\gamma_\sigma = g_{\sigma\alpha}\gamma^\alpha \ . \tag{6c}$$

We will also use the Euclidean version

$$\gamma^0 = i\sigma_3 \quad \gamma^1 = i\sigma^1 \quad \gamma^2 = i\sigma_2 \tag{7a}$$

$$\gamma^\mu\gamma^\nu = -\delta^{\mu\nu} - \epsilon^{\mu\nu\rho}\gamma_\rho \ .$$

From the Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{8}$$

it is found that the parity operation, Eq. (3), applied to the spinor ψ acts as

$$\mathcal{P}\psi(x,y,t)\mathcal{P}^{-1} = \sigma^1\psi(-x,y,t) \ . \tag{9}$$

Then from eq. (8) one finds that the mass term breaks parity, as well as time reversal. The angular momentum is, as usual,

$$M^{ij} = -i(x^i\partial_j - x^j\partial_i) + \tfrac{1}{2}\Sigma^{ij} \tag{10}$$

with

$$\Sigma^{ij} \equiv \tfrac{i}{2}\,[\gamma^i, \gamma^j] \ .$$

Using Eqs. (6), we find

$$M^{ij} = \epsilon^{ij} M \qquad M = -i(x\partial_y - y\partial_x) + \tfrac{1}{2}\,\sigma_3 \ . \tag{11}$$

Note that the spin density is given by

$$S = \tfrac{1}{2}\,\psi^+(x)\,\sigma_3\psi(x) = \tfrac{1}{2}\,\overline{\psi}(x)\,\psi(x) \ . \tag{12}$$

From Eq. (12) we see the reason why the mass term in Eq. (8) breaks parity; in two space dimensions the mass *couples to the spin density* which is a pseudoscalar (this is only true for two component spinors). The Hamiltonian equations reads ($\alpha_i = \gamma_0\gamma_i$, $i = 1,2$)

$$\left[-i\vec{\alpha}\cdot\vec{\nabla} + \sigma_3 m\right]\psi = E\psi \ . \tag{13}$$

We see that if $m > 0$ positive energy spinors have spin $s = +1/2$ and negative energy spinors $s = -1/2$ (the opposite for $m < 0$), therefore there is only one spin degree of freedom $s = +\frac{1}{2}\frac{mE}{|mE|}$.[2] Massless spinors are also spinless as has been discussed in the preceding section.

Classical Gordon Decomposition:

In order to understand and interpret the spin dependence of the fermion current, we use the Gordon decomposition. By using Eq. (8) in $J_\mu = e\overline{\psi}\gamma^\mu\psi$, the classical current becomes

$$J_\mu = -\frac{e}{2m}\,\overline{\psi}(i\overleftrightarrow{\partial}_\mu + 2A_\mu)\psi - \frac{ie}{4m}\,\frac{\partial}{\partial x_\mu}\,\overline{\psi}[\gamma^\nu,\gamma^\mu]\psi \ . \tag{14}$$

The first term is the convection or orbital part of the current and the second the spin or dipole part.

If a static magnetic field ($A_\mu(x,t) = (0,\vec{A}(x))$ is coupled to the current, then the spin part of Eq. (14) will couple to the gauge fields as

$$J_s^i = \frac{e}{2m}\,\epsilon^{ij}\,\frac{\partial}{\partial x_j}\,(\overline{\psi}\gamma^0\psi) \ , \tag{15a}$$

or

$$\vec{J}_s = \frac{e}{2m}\,\vec{\nabla}\times\rho \ , \tag{15b}$$

where we used Eq. (6a,c). (The notation is not perverse; it is correct in two space dimensions.)

Expressions (15a,b) are very surprising because the *magnetic moment* density is the *charge density*. This fact again is a consequence of the spin properties in 2-space dimensions for 2-component spinors.

Indeed, in the non-relativistic limit only the upper (large) component survives with only *one* spin degree of freedom; therefore the charge density and spin density are the same in this limit and the sign of the mass in (15a,b) determines the sign of the spin. This will be shown in more generality in a later section.

This fact has far reaching consequences. The situation in which a magnetic field polarizes spin is very familiar. In this case under discussion, however, the magnetic field will polarize charge. (See the later discussion of the interacting theory.)

4. Free electromagnetic field

We choose the Coulomb gauge

$$A_0 = 0 \qquad \vec{\nabla} \cdot \vec{A} = 0 \; . \tag{16}$$

Equation (16) clearly shows that the gauge field only has one degree of freedom in two dimensions. The magnetic field $B = \epsilon^{ij}\partial_i A_j$ is a pseudoscalar, since under parity (Eq. (3)) $A_x(x,y,t) \to -A_x(-x,y,t)$. The electromagnetic angular momentum

$$M = \int d^2x\, \vec{x} \times (\vec{E} \times B)$$

can be written as

$$M = -\int d^2x\; B(\vec{x} \cdot \vec{E}) \; . \tag{17}$$

This expression can be cast in terms of the "spin" and "orbital" parts[16]

$$M_s = \int d^2x\, \vec{E} \times \vec{A} \tag{18a}$$

$$M_0 = \int d^2x\, (\epsilon^{ij} A_j x_k \partial_i E_k) \; , \tag{18b}$$

where $\vec{E} = -d\vec{A}/dt$. Since by condition (16) there is only one polarization vector perpendicular to the wave vector, then $\vec{E} \times \vec{A} = 0$. This is of course in agreement with the discussion in Section 1 and the condition (5a) for massless particles.

Now let us suppose that we couple fermions to a background static gauge field $A_\mu = (0, \vec{A})$ producing a magnetic field B. We have argued in the preceding section that a magnetic field B will polarize charge density. This charge will in turn generate an electric field which is perpendicular to $\vec{A}$. Therefore we expect that $\vec{E} \times \vec{A} \neq 0$ and by Eq. (5b) the *gauge field will acquire a mass.* Now we will justify these physical arguments in more detail.

5. Interacting theory

We will consider the gauge fields as prescribed background fields and examine the fermionic degree of freedom that interacting with the gauge field through

$$(i\partial\!\!\!/ - A\!\!\!/ - m)\psi(\vec{x}) = 0 \ . \tag{20}$$

We will assume static gauge fields. Since the anomalous effects in the theory arise as parity violating contributions to the fermionic currents, we will study in detail the currents induced by the gauge fields. The normal ordered current is defined as

$$J_\mu(x) = \tfrac{1}{2}\gamma^\mu_{\alpha\beta}[\overline{\psi}_\alpha(x), \psi_\beta(x)] \ . \tag{21}$$

This definition removes the (infinite) contribution of the vacuum in the absence of the background fields.

Expanding the fermion fields in terms of positive and negative energy modes,

$$\psi(x) = \sum_{E_n>0} U_n(x) b_n + \sum_{E_n<0} V_n(x) d_n^+ \tag{22}$$

(here we have assumed that there are no $E = 0$ states) we find

$$\langle J^\mu(x)\rangle = \tfrac{1}{2}\left\{ \sum_{E_n<0} \overline{V}_n(x)\gamma^\mu V_n(x) - \sum_{E_n>0} \overline{U}_n(x)\gamma_\mu U_n(x) \right\} \ . \tag{23}$$

As we have argued, the spin structure of the fermions is responsible for the anomalous effects. Thus let us now study the spin dependent couplings to the gauge field in some detail. As usual we expect two types of spin terms, a spin-magnetic field term, and a "spin orbit" term.[17] The latter couples the spin directly to electric fields.

To begin with let us assume $A_\mu = (0, \vec{A}(x))$, so that only a static magnetic field $B(x) = \epsilon^{ij}\partial_i A_j(x)$ is present. We have argued before that such a B will induce charge in the system. From Eq. (23), this induced charge is

$$Q = -\tfrac{1}{2} \int d^2x \left\{ \sum_{E_n>0} U_n^+(x)\, U_n(x) - \sum_{E_n<0} V_n^+(x)\, V_n(x) \right\} \tag{24a}$$

or

$$Q = -\tfrac{1}{2}\,\eta \tag{24b}$$

where η is the (spectral) asymmetry between the positive and negative parts of the spectrum.[18] Using Eq. (20) we can guarantee solutions to the Dirac equation that have a given value of spin in the rest frame by writing for $m > 0$

$$U_{E>0}(x) = \left(1 - \frac{i\vec{\sigma} \times (\vec{\partial} + i\vec{A})}{E+m}\right) \begin{pmatrix} \Phi(\vec{x}) \\ 0 \end{pmatrix} , \tag{25a}$$

and

$$V_{E<0}(x) = \left(1 - \frac{i\vec{\sigma} \times (\vec{\partial} + i\vec{A})}{E-m}\right) \begin{pmatrix} 0 \\ \chi(x) \end{pmatrix} . \tag{25b}$$

Here $\Phi(x)$ and $\chi(x)$ are the large component for the positive and negative energy cases respectively. To treat the case $m < 0$ merely change $m \to -|m|$ and $U \leftrightarrow V$ in Eq. (25a,b).

Squaring the Dirac equation, (2), we find

$$\left[(i\vec{\partial} - \vec{A})^2 + \sigma_3 B(x)\right] \psi(\vec{x}) = (E^2 - m^2)\psi , \tag{26}$$

where ψ is $U(E>0)$ or $V(E<0)$.

Charge Generation:

For a magnetic field that is localized in space, we can consider Eq. (26) as defining a scattering problem. We see that Φ and χ, and hence U and V, of (25a,b) interact with B with opposite sign (of σ_3). For $m > 0$, positive energy states (Φ) will be repelled from the region where $B > 0$ and negative energy states (χ) will be attracted to this region. The opposite occurs for $B < 0$. Finally, these behaviors are reversed for $m < 0$ (see (25b)). This is of course in agreement with the fact that the sign of the spin is the same as the sign of the product mE.

Thus the sign of the magnetic field interaction depends upon the sign of the energy. This behavior then creates a charge polarization near the spatial regions where the magnetic fields are localized. There is an asymmetry in the spectrum if the total flux $F = \int d^2x\, B(x) \neq 0$ and by (24a) a net charge arises in the vacuum. In fact we will see in the next section that a localized magnetic field (vortex) will bind electrons; their energy will depend on the sign of the mass. The bound states in the spectrum will give rise to a charge density localized near B.

This charge density in turn will generate an electric field perpendicular to $\vec{A}$. The argument given in the preceding section then leads us to conclude that the photon will thereby acquire a mass.

Spin-Orbit Effects:

There is yet another interesting interaction between spin and the electromagnetic field; this is the "spin-orbit" interaction with an external electric field. To study this interaction we assume a static electrostatic potential $A_\mu = (A_0(x), \vec{0})$ that only depends on $r = |\vec{x}|$ such that $\vec{E}(r) = E(r)\,\vec{r}$ is the radial electric field. For our arguments we will assume that the fields are weak and localized in space. The assumptions of radial $\vec{E}(r)$ and weak fields serve the purpose of illustrating the physics clearly. The effects are in fact quite general. Writing $\psi = \binom{\psi_u}{\psi_d}$, the Dirac equation , defined by Eq.(2), becomes

$$(-\partial_x + i\partial_y)\psi_d = -(E - m - A_0)\psi_u \tag{27a}$$

and

$$(-\partial_x - i\partial_y)\psi_u = (E + m - A_0)\psi_d \ . \qquad (27b)$$

For positive energy spinors Eqs. (27ab) give rise to the second order equation for the spinor U_u (let us first assume $m > 0$).

$$\left\{ -\vec{\partial}_2 - \frac{1}{(E + m - A_0)} E(r) L_z + \cdots \right\} U_u = (E^2 - m^2) U_u \ , \qquad (28)$$

where $L_z = \vec{r} \times \vec{p}$ and the dots stand for the terms that do not depend on spin and angular momentum and $A_0 \ll E + m$ has been used. We recognize the second term on the left hand side of (28) as the usual spin-orbit coupling for a particle of spin $S_z = +\frac{1}{2}$. This is of course in agreement with the fact that positive energy spinors have $S_z = \frac{1}{2} \frac{m}{|m|}$. The effect of the interaction can be best visualized by writing in Eq. (28) $E \simeq m + \epsilon$ (this corresponds to a non-relativistic approximation), and expanding in ϵ.

Therefore we see that in the spatial regions where the electric field $E(r) > 0$, the energy decreases for $L_z > 0$ and it increases for $L_z < 0$. Therefore for $L_z > 0$ the particle is attracted by the electric field while for $L_z < 0$ it is repelled by it.

For negative energy electrons we find the analog of Eq. (28)

$$\left\{ -\partial^2 + \frac{1}{(E - m - A_0)} E(r) L_z + \cdots \right\} V_d = (E^2 - m^2) V_d \ . \qquad (29)$$

From (29) we recognize the spin orbit coupling for a particle of $S_z = -\frac{1}{2}$ as expected for negative energy electrons. Writing $E = -(m + \epsilon)$, we find that for $L_z > 0$, the ***absolute*** value of the energy increases and for $L_z < 0$ it decreases; it is the opposite behavior of the positive energy states. For $L_z < 0$ negative energy electrons are attracted to the electric field and for $L_z < 0$ they are deflected away. This situation is summarized in Fig. 1. Therefore we see, using the current given by Eq. (23), that if the positive and negative energy electrons scatter off a localized radial electric field there is a net azimuthal current due to this "spin-orbit" coupling. As usual the origin of this interaction is the magnetic field $B \sim m\vec{E} \times \vec{p}$ in the rest frame of the particle interacting with its spin (recall $m > 0$). This effect is of higher order in $\vec{p}/|E|$ as it must by Lorentz covariance. In the limit of weak fields, Eqs. (28), (29) can be seen to contain the usual Thomas factor.[17] This discussion is quite general and the physics it describes does not depend on the restriction to radial electric fields (or weak fields), save for the explicit coupling to angular momentum.

For the opposite sign of m the behavior is opposite to that described above. Thus these induced currents have a definite handedness determined by the sign of the fermion mass. Hence they generate a net magnetic moment in the system.

At this point we observe that if *two* spin values for the fermions were allowed for a given sign of E , the "spin orbit" coupling would induce two currents of opposite sign that would cancel each other, yielding zero total magnetic moment. Also, the same can be seen to happen in the case of the magnetic fields; a charge polarization of opposite signs would be induced by each spin component, thereby leading to zero net charge.

Since the spin orbit interaction responsible for these anomalous currents is related by Lorentz covariance to the spin-magnetic field coupling analyzed before, then the charge induced by B and the currents induced by $\vec{E}(r)$ transform as a three vector. Notice that in 2+1 dimensions the dual electromagnetic tensor ${}^*F^{\mu} = \frac{1}{2}\,\epsilon^{\mu\nu\rho}F_{\nu\rho}$ is a (pseudo) three-vector. The analysis carried out allows us to extract the following physical picture of the process in which anomalous charges and currents are induced. Suppose that a localized magnetic field (for simplicity it only depends on $|\vec{x}|$) is adiabatically switched on ($dB/dt = \dot{B}(r) \neq 0$). A charge density will accumulate near the spatial region where the magnetic field is localized. Also, because the Bianchi identity, one has

$$\partial_{\mu}\epsilon^{\mu\nu\rho}F_{\nu\rho} = 0 \tag{30a}$$

or

$$\dot{B}(r) = -\vec{\nabla} \times \vec{E} \ . \tag{30b}$$

An azimuthal electric field will, in turn, induce a radial current through the "spin orbit" coupling. Thus by gauge invariance and (30a,b),

$$\dot{\rho}(r) \sim \dot{B}(r) \tag{31a}$$

or

$$-\vec{\nabla} \cdot \vec{J} = -\vec{\nabla} \times \vec{E} \ . \tag{31b}$$

Hence these imply the relation

$$J_\mu \sim \epsilon^{\mu\nu\rho} F_{\nu\rho} \ . \tag{32}$$

Now dimensional arguments suggest that the coefficient in front of the right hand side in (32) is proportional to the coupling constant e (dimension $[\text{length}]^{1/2}$). The ϵ-symbol reflects the spin nature of the anomalous current. The physical situation described by Eqs. (30,31) is depicted in Fig. 2, and proceeds as follows.

As the (localized) B field is switched on, currents flow in from the boundaries. A charge density is built up near the B-field region. This implies a depletion of charge near the boundaries of the system. This situation can also be thought of as charge (of opposite sign) flowing *out* of the spatial boundaries. The total system including the charge density near the origin plus the boundary charges, *remains* neutral as charges are redistributed. No charge is created. In a physical situation a two dimensional magnetic vortex can be visualized as a section of a long cylindrical solenoid, however for a long but finite solenoid there is always return flux that plays the role of the "antivortex". As the length of this solenoid is taken to infinity the two dimensional physics is recovered, but the return flux has been pushed to the boundaries (at infinity) in the two dimensional plane. The boundary charges are distributed near this "anti-vortex".

Perturbation Theory:

A direct computation of the induced currents yields

$$\langle J_\mu \rangle = [g_{\nu\mu} p^2 - p_\mu p_\nu] A_\mu(p) \pi^{(1)}(p^2, m^2) + im\epsilon^{\mu\nu\rho} p_\rho A_\nu \pi^{(2)}(p^2, m^2) \tag{33}$$

where

$$\pi^{(1)}(p^2, m^2) = \frac{1}{2\pi} \int_0^1 d\alpha \, \frac{\alpha(1-\alpha)}{[-p^2(1-\alpha) + m^2]^{1/2}} \tag{34a}$$

and

$$\pi^{(2)}(p^2, m^2) = \frac{1}{4\pi} \int_0^1 d\alpha \, \frac{1}{[-p^2\alpha(1-\alpha) + m^2]^{1/2}} \; . \tag{34b}$$

There is an ambiguity in the regularization of the vacuum polarization tensor arising from the fermion-antifermion loop. To obtain expressions (33,34) we have used gauge invariant projection.[2] The ambiguity arises from a linear ultraviolet divergence proportional to $g_{\mu\nu}$ i.e. a gauge dependent term. This ambiguity is also reflected in the second term as a finite local counterterm, whose value depends on the regularization scheme.

The parity abnormal term in expression (33) can also be seen to have its origin in the spin (or dipole) contribution to the currents by looking at the discontinuity (spectral function) of the corresponding Feynman diagram. This discontinuity can be computed as usual[19] by setting the internal electron and positron lines on mass shell. The resulting spectral function is given by (up to kinematical factors)

$$\begin{aligned} \rho(p^2, m^2) \sim \int \frac{d^3k}{(2\pi)^3} \, \frac{d^3q}{(2\pi)^3} \delta^3(k+q-p) \\ \times \left\langle 0 | J_\mu | e(q) e^+(k) \right\rangle \, \left\langle e^+(k) e(q) | J_\nu | 0 \right\rangle \; . \end{aligned} \tag{35}$$

Since the internal lines are on mass shell we can use the Gordon decomposition of the currents given by Eq. (14) (for both vertices). Using (6abc) it is straightforward to see that the term proportional to the ϵ-symbol (parity violating) arises from the spin-convection and spin-spin terms of the cross products in (35). The convection-convection term contributes to the symmetric $(g_{\mu\nu}p^2 - p_\mu p_\nu)$ part only.

This behavior can be emphasized by temporarily adding an effective point anomalous magnetic moment interaction to the Lagrangian of the form

$$\mathrm{L}_m = \frac{\kappa}{4m}\, \overline{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu} \ . \tag{36}$$

Since the spin part of the current is proportional to $(1+\kappa)$, the term proportional to the ϵ-symbol in (35) has the same overall factor. Thus the anomalous behavior is due to the spin, and vanishes if one chooses $\kappa = -1$.

The induced current (33) is obtained as

$$\langle J_\mu \rangle_A = \frac{\delta \Gamma_{\mathrm{ind}}[A_\mu]}{\delta A_\mu} \ , \tag{37}$$

where $\Gamma_{\mathrm{ind}}[A_\mu]$ is the induced Euler-Heisenberg effective action resulting from the integration of the fermionic degrees of freedom. Functionally integrating Eq. (37) and restoring the coupling constant, the long-distance effective Lagrangian is obtained as[6]

$$\mathrm{L}_{\mathrm{eff}} = -\tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{16\pi}\, \frac{m}{|m|}\, \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \ . \tag{38}$$

The second piece in (38) is the (parity violating) Abelian part of the Chern-Simmons term.[2] It corresponds to a gauge invariant mass $M = \frac{e^2}{8\pi}\frac{m}{|m|}$ for the photon. In the Landau gauge $(\partial_\mu A_\mu = 0)$ the equations of motion are

$$-\partial^2 A^\nu + M\epsilon^{\nu\rho\sigma}\partial_\rho A_\sigma = 0 \ . \tag{39}$$

Writing

$$A^\mu = \epsilon^\mu(k)e^{ik\cdot x} \qquad k_\mu \epsilon^\mu(k) = 0 \ , \tag{40}$$

it is found[7] for on shell photons $(k^2 = M^2)$ the solution to Eqs. (39), (40) in the rest frame $k_\mu = (M, \vec{0})$ is

$$\epsilon^\mu(k) = \frac{1}{\sqrt{2}} \left(0, 1, \frac{iM}{|M|}\right) . \tag{41}$$

This state has spin (in the rest frame)

$$S_z = \frac{M}{|M|} = \frac{m}{|m|} . \tag{42}$$

Notice the correlation between the spin of the photon in Eq. (42) and the spin of the fermions $S_z = \frac{1}{2} \frac{m}{|m|}$. Indeed this is no accident. The intermediate state in the vacuum polarization tensor is an electron and a *positron both* of spin $S_z = \frac{1}{2} \frac{m}{|m|}$ since angular momentum is additive in two space dimensions (there being just one generator), in the photon rest frame the intermediate state has total spin $S_z = \frac{m}{|m|}$ and this is the spin acquired by the photon through its coupling to the current. The existence of the photon spin and its mass are of course in agreement with the arguments presented in Section 1. The equation of motion of the effective theory described by Eq. (38) when coupled to an external sources is

$$\partial_\mu F^{\mu\nu} + \frac{M}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = J^\nu . \tag{43}$$

The solutions to these equations fall off exponentially at distances $|\bar{x}| \gg M^{-1}$. It is easy to see that if J^ν corresponds to an external static charge density, both electric and magnetic fields are screened but[2,7]

$$\int B \, d^2x = -\frac{Q_{\text{ext}}}{M} \tag{44}$$

where Q_{ext} is the external charge. This effect can be understood at the level of the fermionic theory that gives rise to the Chern-Simmons term.

The external charge density (assumed to be localized in space) produces an electric field. The fermions interact with this electric field ($\vec{E}$) through the "spin orbit" coupling as described before, giving rise to currents perpendicular to $\vec{E}$. These currents in turn generate a magnetic field. Since these currents have definite handedness they give rise to definite flux.

If the external sources are currents, these currents generate a magnetic field; this in turn polarizes charges that generate an electric field. Therefore the effective theory with the Chern-Simmons term describes a sort of superconductor; electric and magnetic fields are screened. However the screening mechanism is different from the usual Meissner effect since the source of (parity abnormal) currents are electric fields. Indeed it is more than a superconductor, since external (heavy) charges induce the formation of magnetic vortices of width of order $|M|^{-1}$ with finite magnetic flux.

6. The 1+1 dimensional chiral anomaly or the anomalous Gordon decomposition, and zero modes

In this section we will clarify the relation between the chiral anomaly in 1+1 dimensions and the Chern-Simmons term in 2+1 dimensions.[10] This dimensional reduction comes about by considering static background fields. We now construct the Gordon decomposition for the charge density in the interacting theory, since the classical Gordon decomposition, Eq. (14), does not take proper account of quantum effects. To this end we define the spinors to be eigenstates of the Dirac Hamiltonian in the $A_0 = 0$ gauge with only magnetic fields present:

$$[-i\vec{\alpha}\cdot\vec{\nabla} - \vec{\alpha}\cdot\vec{A}(\vec{x}) - \gamma^0 m]\psi_n = E_n\psi_n \ . \tag{45}$$

We will study the induced charge density $J^0(\vec{x})$, and also define the "chiral" current as a function of the energy E by

$$J_i^5(x,E) = T_r\left\langle x\right|\frac{\alpha_i\gamma_0}{H-E}\left|x\right\rangle = \sum_n \frac{\psi_n^+(x)\alpha_i\gamma_0\psi_n(x)}{E_n - E} \ , \tag{46}$$

where the trace is over Dirac indices.

The variable E will allow us to fully explore the spectral representation of the currents.[20] For example

$$\begin{aligned}\frac{1}{\pi}\,\mathrm{Im}\,J_i^5(x,E+i\eta) &= T_r\,\langle x|\alpha_i\gamma_0\delta(H-E)|x\rangle \\ &= \sum_n \psi_n^+(x)\alpha_i\gamma_0\psi_n(x)\delta(E_n-E)\ .\end{aligned} \tag{47}$$

Using Eq. (45) and its adjoint with the representation (46) for the current J_i^5, we find

$$\frac{1}{2m}\,i\partial_i J_i^5(x,E) = \frac{1}{m}\sum_n \frac{\psi^+(x)E_n\psi_n(x)}{E_n-E} - \sum_n \frac{\psi_n^+(x)\psi_n(x)}{E_n-E} \tag{48}$$

or

$$\frac{1}{2m}\,\left\langle i\partial_i J_i^5(x)\right\rangle_E = \frac{E}{m}\,\left\langle\overline{\psi}\psi\right\rangle_E - \left\langle\overline{\psi}\gamma^0\psi\right\rangle_E + \frac{1}{m}\sum_n \psi_n^+\gamma^0\psi_n\ . \tag{49}$$

The last term is ambiguous, being of the form $(0\times\infty)$. We subtract Eq. (49) in the limit $E\to\infty$ and write the *anomalous Gordon decomposition* as

$$\left\langle i\partial_i J_i^5\right\rangle_E = 2E\left\langle\overline{\psi}\psi\right\rangle_E - 2m\left\langle\overline{\psi}\gamma^0\psi\right\rangle_E - 2\ \text{Anomaly} \tag{50a}$$

$$\text{Anomaly} = \lim_{E\to\infty}\ E\left\langle\overline{\psi}\psi\right\rangle_E \tag{50b}$$

where for a general Γ :

$$\left\langle\overline{\psi}\Gamma\psi\right\rangle_E = \sum_n \frac{\psi_n^+\gamma^0\Gamma\psi_n}{E_n-E}\ . \tag{50c}$$

The last term of Eq. (50a) takes into account the short distance behavior of the theory. This result can also be obtained directly from the usual gauge invariant point-split definition of the current. Notice that the "anomaly" is in the *spin-density*. This should be expected since for $m=0$ the Hamiltonian in Eq. (45)

is equivalent to the Dirac operator in Euclidean space (chosen to be hermitian) in which the α matrices ($\alpha_i = \gamma_0\gamma_i \ i = 1,2$) play the role of the (hermitian) γ matrices in Euclidean 1+1 dimensions. The adjoint spinor in this representation is ψ^+ (instead of $\overline{\psi}$ in Minkowski space) and γ^0 plays the role of γ^5. Therefore the 1+1 Euclidean chiral density $\langle\psi^+\gamma^5\psi\rangle$ is the spin density $\langle\overline{\psi}\psi\rangle$ in Minkowski space in 2+1 dimensions. The mass term in (45) plays the role of a pseudoscalar coupling.

Also the variable E in (46) plays the role of a mass term for the (hermitian) 1+1 dimensional Euclidean "Dirac operator" H. Therefore we identify the "anomaly" in Eqs. (50) with the axial anomaly in Euclidean 1+1 dimensions.

$$\text{Anomaly} = \frac{1}{4\pi}\,\epsilon^{ij}\partial_i A_j = \frac{B}{2\pi}\ . \tag{51}$$

But we want to stress the fact that in 2+1 dimensions the anomaly is present in the *spin-density* (this will be illustrated shortly in perturbation theory).

From Eq. (45) we can write

$$\langle\overline{\psi}\psi\rangle_E = m\,\mathrm{Tr}\,\frac{1}{H^2-E^2} + E\,\mathrm{Tr}\,\frac{\gamma_0}{H^2-E^2} \tag{52a}$$

$$\langle\overline{\psi}\gamma^0\psi\rangle_E = E\,\mathrm{Tr}\,\frac{1}{H^2-E^2} + m\,\mathrm{Tr}\,\frac{\gamma_0}{H^2-E^2}\ . \tag{52b}$$

It is easy to see from gauge invariance that only the contribution from the lowest order term in an expansion in A , i. e., the free vacuum, is non-zero for the first terms on the right hand side of (52a,b). These terms cancel each other in Eq. (50a) and therefore the second terms in (52a,b) are the interesting ones. The *induced* charge and spin, that is, the difference between the above quantities in the interacting and free case, satisfy

$$\langle\overline{\psi}\psi\rangle_E = \frac{E}{m}\ \langle\overline{\psi}\gamma^0\psi\rangle_E\ . \tag{52c}$$

From now on expectation values are understood as the difference between the interacting and free field values. Therefore Eq. (50a) becomes

$$\tfrac{1}{2}\ \langle i\partial_i J_i^5\rangle_E = \left(\frac{E^2}{m} - m\right)\ \langle\overline{\psi}\gamma^0\psi\rangle_E - \frac{B(x)}{2\pi}\ . \tag{53}$$

By integrating over all space, the left hand side is a surface term that integrates to zero (there are no massless particles) and hence

$$Q_E = \frac{m}{E^2 - m^2}\ \Phi \tag{54}$$

with

$$Q_E = \int d^2x\ \langle\overline{\psi}\gamma^0\psi\rangle_E \qquad \Phi = \int d^2x\, \frac{B}{2\pi}\ .$$

The total charge is

$$Q = \int \frac{dE}{2\pi i}\ Q_E = \frac{m}{|m|}\ \frac{\Phi}{2}\ . \tag{55}$$

In addition, from (54) and (47) we find

$$\frac{1}{\pi}\ \mathrm{Im}\ Q_E(E + i\eta) = \frac{\Phi}{2}\ [\delta(m+E) - \delta(m-E)]\ . \tag{56}$$

Therefore we conclude that *only* the threshold states $E^2 = m^2$ contribute to the charge and charge density. By the relation (52) and (12) these states contribute to the spin density an amount $S_z = Q/2$, therefore we find that there is a net spin *induced* in the ground state, (the *induced* spin is defined by subtracting its value in the vacuum without external fields).

From Eqs. (54) and (56) we see that there are Φ states at $E^2 = m^2$. This can be understood by means of the Atiyah-Singer index theorem as follows. The Hamiltonian in Eq. (45) can be cast as

$$H = -i\not{D} + \sigma^3 m \tag{57}$$

where $i\not{D}$ is the Euclidean Dirac operator in 1+1 dimensions. In an external background field with non-vanishing flux $i\not{D}$ has zero modes, the number of these modes is given by the total flux (Φ).[20] Since $\{\sigma_3, i\not{D}\} = 0$ these modes are eigenstates of σ_3, i.e. they are chiral, and therefore they are eigenstates of H in (57) with eigenvalues $E = \pm m$. These are the threshold states that contribute to the charge, but since they are eigenstates of σ_3 they also contribute to the spin in the ground state.

These "zero modes" are bound and/or threshold continuum states. Their wave function has been given in Refs. (11,20,21) for certain vortex configurations. They are localized near the vortex. Since these are threshold states they have zero binding energy and the wave function falls off algebraically at long distances. These states are topological nonlocal configurations.

We now proceed to compute in perturbation theory the charge and spin (as a function of the variable E) for static background fields. This will illuminate the *spin-anomaly* discussed in Eq. (50b) and some subtleties associated with symmetric integration of divergent quantities. Since the external fields are static there is no energy transferred to the fermion loop. The propagator in Euclidean space is

$$S(E, \vec{k}) = \frac{\gamma_0 E + \not{k} + m}{k^2 + E^2 + m^2} \ . \tag{58}$$

A straightforward computation of the fermion loop in momentum space ,after subtracting the (infinite) contribution of the trivial vacuum, yields

$$\langle \overline{\psi}\psi(p) \rangle_E = \frac{E}{4\pi} \, B(p) \, \pi(p^2, M^2) \tag{59a}$$

$$\langle \overline{\psi}\gamma^0\psi(p) \rangle_E = \frac{m}{4\pi} \, B(p) \, \pi(p^2, M^2) \tag{59b}$$

$$p_i \, \langle J_i^5(p) \rangle_E = \frac{B(p)}{2\pi} \, [1 - M^2 \pi(p^2, M^2)] \ , \tag{59c}$$

where $p^2 = (\vec{p})^2$, and

$$M^2 = E^2 + m^2 \qquad \pi(p^2, M^2) = \int_0^1 d\alpha \, \frac{1}{[p^2\alpha(1-\alpha) + M^2]} \ . \tag{60}$$

The reader may recognize that Eq. (60) is the invariant amplitude of the vacuum polarization tensor in the massive Schwinger model in Euclidean space and with mass $(M^2)^{1/2}$. The relations (59) verify the anomaly equations (50) and (53) in Euclidean space. It should be mentioned that the (infinite) contribution to $\langle\overline{\psi}\psi\rangle$ and $\langle\overline{\psi}\gamma^0\psi\rangle$ from the vacuum in absence of background fields cancels in (50a) (as it must for a free theory). The integral over E of the total spin is

$$\int \frac{dE}{2\pi} \, \langle\overline{\psi}\psi(\vec{p}=0)\rangle_E = \frac{\Phi}{2}\int \frac{dE}{2\pi} \, \frac{E}{E^2+m^2} = 0 \, . \tag{61}$$

The integral is logarithmically divergent but vanishes by symmetric integration; this is the origin of $(0 \times \infty)$ in Eq. (49).

Performing this integral in the complex E plane closing the contour along a semicircle at infinity, we see that the contribution from this semicircle exactly cancels the contribution from the pole at $E = \pm im$. The contribution from the circle at infinite is precisely the anomaly in (50b). Subtracting this contribution from eq. (61) in order to define $\langle\overline{\psi}\psi\rangle$ unambiguously, we see that the total spin induced by these states at $E = \pm m$ (zero modes) is given by the 1+1 dimensional chiral anomaly. This is the same subtraction as performed in the chiral anomaly case.

Since these are threshold states, from Eq. (45) we see that they are eigenstates of σ_3 (γ_0), i.e. they have definite spin (for a radial magnetic vortex they are also eigenstates of orbital angular momentum). There are Φ states at threshold (see Eq. (56)) each one with spin $S_z = \frac{1}{2}\,\frac{m}{|m|}$, these are the states that induce the asymmetry in the spectrum and give rise to the charge. Then by Eqs. (24) we find that Q and S_z satisfy

$$S_z = \frac{Q}{2} = \frac{\Phi}{4}\,\frac{m}{|m|} \, . \tag{62}$$

The factor 1/2 has its origin in fermion fractionalization. Therefore the asymmetry in the spectrum that gives rise to the vacuum charge (see Eqs. (24)) is also responsible for the spin in the vacuum.[22]

The fact that charge and spin of the vacuum state may be related is perhaps not surprising. Under a 2π rotation, we expect the ground state wave function to change as $|0\rangle \to (-1)^F |0\rangle = e^{2\pi J} |0\rangle$ where F is the fermion number and J the total angular momentum ($J = \ell + S$). Since in the Abelian Theory the charge and fermion number are the same we find $Q/2 = S$ as in Eq. (62).

In two space dimensions, a net angular momentum in the ground state *does not break* rotational invariance.

7. The massless limit

So far we have only studied the case of massive fermions. The long distance properties of the effective theory are not perturbative in the sense that the perturbative expansion is in terms of $e^2/\sqrt{-p^2}$ where p^2 is a typical transferred momentum. Hence we will have to resort to the non-perturbative arguments elaborated in the previous sections. As was argued before *free* massless fermions are spinless in 2 space dimensions,this can be understood as follows.

For $m = 0$ the (free) Dirac Hamiltonian in Eq. (13) anticommutes with σ_3. Therefore eigenstates of H are combinations of "spin up" and "down" with equal probability. Therefore the expectation value of σ_3 in energy eigenstates is zero.

The electromagnetic interactions, however, treats the spin components differently.

For the case of the electric field analyzed before, Eqs. (27) still hold (with $m = 0$) and the steps leading to Eqs. (28) and (29) for U_u and V_d hold without modification.

For a space dependent A_0 the eigenspinors are no longer equal mixture of spin up and down. The "spin-orbit" interaction is still at work and currents are induced. In the case of the magnetic fields, the massless limit is also smooth. The threshold states at $E = \pm m$ responsible for the spectral asymmetry and induced charge will become "zero modes" in the $m = 0$ case (i.e. zero energy states). The total charge is only sensitive to $m/|m|$, i.e. whether these states have positive or negative energy.

Hence the charge and spin quantum numbers are insensitive to the mass, however since these states are at zero energy, there is the usual ambiguity in defining the charge in the ground state. Perturbation theory cannot know about these zero modes and averages over the spin configurations thereby yielding zero as a result for the charge.[23] A small mass lifts the degeneracy.

It has been found that if the massless theory is regulated using a heavy Pauli-Villars regulator, a parity breaking Chern-Simmons term is obtained in the limit of the regulator masses taken to infinity.[6] In fact these heavy regulators detect the presence of the zero modes by lifting the degeneracy; the answer will depend on the sign of the regulator mass since it is this mass that is responsible for removing the degeneracy.

Hence we claim that the physics of the massless theory can be understood as a limiting case of a massive theory once proper account of possible zero energy states has been made.

In the presence of non-trivial gauge configurations (vortices, etc.) the vacuum carries spin and charge and since these states are eigenstates of spin Eq. (62) is satisfied. By the same arguments of Lorentz covariance, charges induced by B and currents induced by $\vec{E}$ form a three-vector. However we must mention a subtlety involved with the massless case. In the case of a vortex configuration or any localized background gauge field, there is *no gap* in the spectrum. Not only are the charge and spin ill-defined quantities but there is no natural "zero" in the spectrum. For long range gauge fields the massless Dirac Hamiltonian has a conformal symmetry. This behavior may be responsible for large infrared fluctuations and may indicate that Q and S are only expectation values and *not* sharp quantum numbers.

It is evident that the massless case needs a more careful and thorough treatment than that given above.

8. The four component theory

An alternative formulation of the theory consists of working with four component spinors as in 3+1 dimensions. The Dirac algebra is represented by the four dimensional γ matrices $\gamma_0, \gamma_1, \gamma_2$. However now the spin is $S_z = \sigma_3 \otimes \mathbf{1}$ ($\mathbf{1}$ = unit 4×4 matrix). As a consequence a brief analysis indicates that *positive* and *negative* energy spinors have both values of spin, therefore positive and negative energy states are no longer asymmetric with respect to the spin-electromagnetic field interaction. A fermionic theory with two species of Dirac fermions with masses of opposite signs (not necessarily the same value) will give rise to opposite induced charges and currents cancelling the anomalous contribution to the effective theory. Again this is because the two species have opposite values of spin.

Hence there are no parity anomalies in these theories since the parity anomalies are associated with the spin asymmetry in 2+1 dimensions. This is in agreement with Refs. 2 but we want to stress again the fact that the spin is playing the fundamental role. Although we have studied the Abelian theory, we believe that the general physical arguments will still hold in the non-Abelian theories, insofar as the anomalous effects are related to the spin properties rather than internal (color, flavor) symmetries.

9. Relation to the Quantum Hall effect (or lack thereof) and some other systems

As was mentioned in the introduction, one of our motivations for studying theories with a Chern-Simmons term or topological mass is to understand their possible relation to the Quantum Hall effect (QHE). Let us briefly recall some of the basic concepts involved in the analysis of QHE. The reader is urged to consult some literature on the subject especially Refs. 24, 25, 26 which provide a very physical picture of the situation.

The QHE is observed in very thin films of semiconductor material in inversion layers (Mosfet junctions). The experimental situation corresponds to very strong (constant) magnetic fields perpendicular to the two dimensional film, very low temperatures and high mobilities. For the usual QHE (not the fractional QHE), only an effective non-interacting two-dimensional electron gas is studied. The *Schrödinger* equation obeyed by the electrons is

$$H\psi = E\psi \ , \ H = \frac{1}{2m^*} \left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\mathcal{E}y \tag{63}$$

where m^* is an effective mass and $\mathcal{E}y$ gives a constant electric field in the y direction. The solutions of (63) in a constant magnetic field are the usual Landau levels.

For very low temperatures and strong magnetic fields only the lowest Landau levels are considered. This condition is satisfied for

$$kT \ll \hbar w_c \tag{64}$$

where w_c is the cyclotron frequency. Notice that in H in (63) there is no Pauli interaction $(\sigma_3 B)$. Indeed by condition (64) the spin excitations are suppressed, therefore spin *does not play a fundamental* role in QHE, unlike the fermionic theories studied in the preceding sections.

The induced currents are given by

$$J_i = -\frac{e}{2m^*}\,\psi^* i \overleftrightarrow{\partial}_i \psi - \frac{e}{m^*}\,\psi^*\psi A_i \ . \tag{65}$$

Hence it corresponds to the *convection* part of the Gordon decomposition in Eq. (14). Therefore the Hall current arises only from the convection contribution. This is a *crucial* difference between the physics of QHE and that of theories with a Chern-Simmons term.

In the QHE it is a *Lorentz force* that is responsible for the interesting effects, and notice that in the absence of a perpendicular magnetic field (in the laboratory frame) the currents induced by the electric field are parallel to it. We stress once more that spin does not play any important role in QHE, unlike theories with parity violating currents which are produced by spin effects.

A further discrepancy can be seen at the level of the definition of the currents in both theories, given by Eqs. (21,23) and (65) respectively.

As an example of the different physics described by the two theories, consider the case of a static electric field and no magnetic field. In the theory with parity violating currents (C-S) the 'spin-orbit" coupling generates a current perpendicular to the electric field, it is a spin current. In the non relativistic (Schrödinger theory) there is only a convection current parallel to $\vec{E}$.

The Hall current is purely convection, whereas the Chern-Simmons is due to the spin contribution. The real problem in the QHE is to explain the plateaus in the Hall conductivity. The present understanding emphasizes the role of impurities and localized states.[24,25,26] Clearly C-S theories cannot account for these effects.

For the fractional QHE there is the accepted belief that electron-electron interactions are responsible for cusps in the free energy at rational filling factors. Clearly these features cannot be reproduced by theories with a C-S term.

Furthermore while the materials used in QHE are semiconductors and screening effects are small and material dependent, the C-S theory has superconducting properties, i.e. electric and magnetic fields are *completely* screened. Notice for example that Laughlin's argument[25] of gauge invariance will *not* hold in the C-S theory because the screening properties will preclude any adiabatic charge transport. We then agree with the result of Abouelsaood in that theories with a (C-S) term have very little (if anything) to do with QHE.[27]

Some possible condensed matter applications of the phenomena described by parity violating (C-S) theories are beginning to emerge. Recently a very interesting model has been proposed by Semenoff.[28] It is a "two-dimensional" honeycomb lattice of graphite (interaction between layers is neglected as they are very weak) with one atomic electron per site and two degeneracy points per Brillouin zone. Unfortunately, the fermionic spectrum on the lattice suffers the problem of species doubling and the anomalous effects cancel between the two species.

Perhaps fermion bound states on magnetic vortices can be observed in superfluid Helium III. Some interesting properties on these systems have been reported in Ref. 29 and 30. Perhaps they may offer an experimental setting for observation of strange quantum numbers. A clear signature of a theory with (C-S) is that (heavy) charged impurities will form magnetic vortices of finite width and flux given by Eq. (44).

Recently two dimensional fermionic theories have been proposed as effective models to describe the physics of disordered electronic systems with degenerate bands.[31] Fradkin has suggested that there are real materials that can be studied with these models.[32]

10. Conclusions and some open questions

We have clarified the physical mechanisms that give rise to a "topological mass" or Chern-Simmons terms induced by fermions in 2+1 dimensions.

It is argued that the spin properties of the fields are responsible for the parity anomaly in the induced fermion currents and the ensuing Chern-Simmons term in the effective action for the gauge fields. These spin properties are dynamical and are unique for two components fermions in 2+1 dimensions. The parity anomaly is completely determined by these dynamical features and not by topology.,

Topology comes about in the quantum numbers of the vacuum and is determined by nontrivial configurations like magnetic vortices. In the presence of this configuration the vacuum carries charge Q and spin S and we find the novel result

$$\tfrac{1}{2}Q = S$$

It is this spin in the ground state that is responsible for the anomalous breaking of parity. We also investigated some topological properties of these excitations.

We have argued that theories with a Chern-Simmons term do not seem to be qualitatively equivalent to the situation in the Quantum Hall effect. The physics of the former being completely determined by the spin structure, while in the latter spin is frozen out and is not important. We gave further sources of disagreements between the two phenomena.

While we only studied in detail the Abelian theory we argued that the physics of the non-Abelian theory may be similar. However this remains to be studied further as well as the possibility of extending the arguments to higher odd-dimensional spaces. We did not attempt to study the global anomaly in the non-Abelian case in detail. Perhaps there is also a simple physical picture to understand it. Another problem that we did not address is the sharpness of the ground state charge and spin. Magnetic vortices possess long range gauge fields and it is not clear whether charge and spin fluctuations are small. Study of some of these problems is in progress.

ACKNOWLEDGEMENTS

We would like to thank Professor Don Yennie for enjoyable and illuminating conversations on the QHE. We also thank S. Yankielowicz, E. Fradkin, W. Dittrich, P. Nason, C. Schmidt and H. Partovi for interesting discussions and suggestions. R.Y. thanks the Rothschild Fellowship for financial support.

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FIGURE CAPTIONS

1. "Spin-orbit" couplings. Positive and negative energy particles scattering off a radial electric field. (The black dot is the source of $\vec{E}(((r))$. There is an azimuthal electric current induced.

2. Adiabatic change in a magnetic field ($\dot{B} \neq 0$) localized near the origin. Charges are accumulated near the origin. The azimuthal electric field (Faraday's law) induces a radial current through the "spin-orbit" interaction.

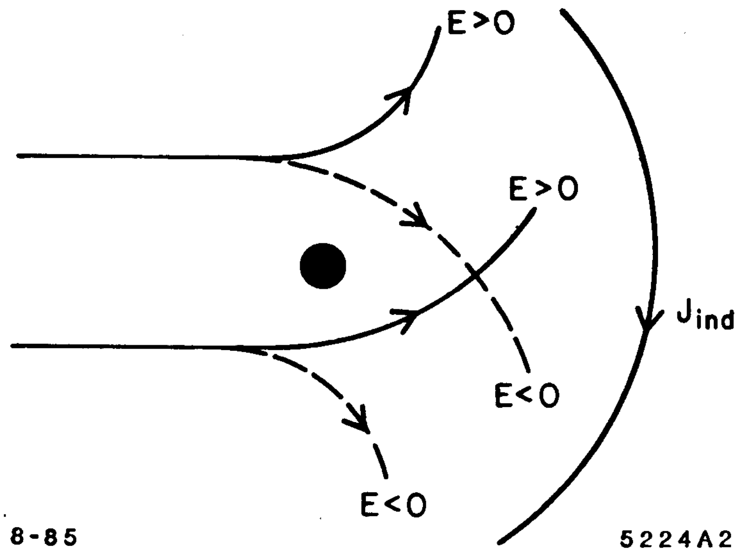

Fig. 1

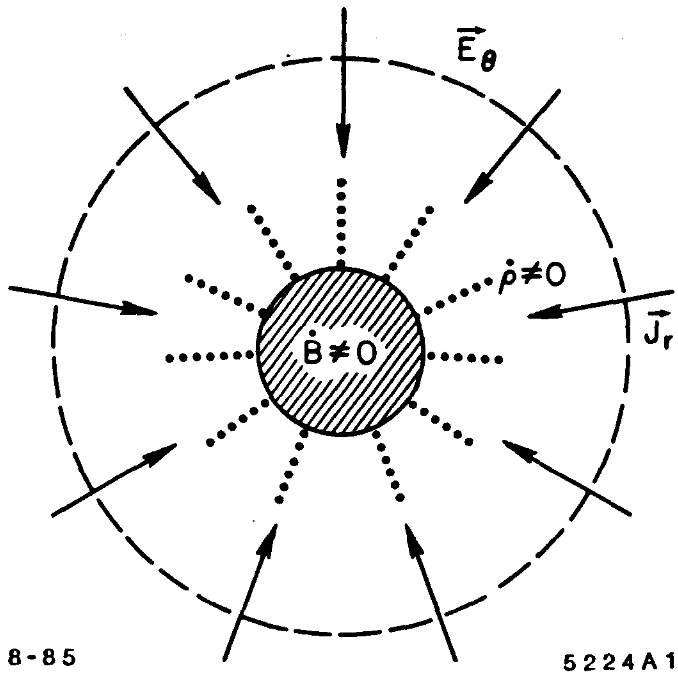

Fig. 2