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COMPACTIFIED SUPERSTRINGS AND TORSION*

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ABSTRACT

String theories on a background manifold with torsion are constructed and investigated in the light cone gauge with holonomy group $H \subseteq SU(3)$. The appropriate non-linear sigma model is constructed on a hermitian manifold. The relationship between the Neveu-Schwarz-Ramond and the Green-Schwarz versions of the theory is discussed. Under the assumption that the β-function vanishes identically we show that the only viable compactification is on a manifold with zero torsion. This result is obtained from effective field considerations as well as directly from string considerations.

1. Introduction

In this paper we investigate various aspects of string theories on a background manifold with torsion. The paper contains two major parts. In the first part we study candidate vacuum configuration for ten-dimensional superstrings. We compactify these theories on $M_4 \times K$ where M_4 is four-dimensional space-time and K some compact six dimensional manifold. In particular we are interested in investigating the existence of solutions with non-zero torsion on K. We approach the compactification problem both from the effective field theory point of view and directly using string considerations.

The second part of the paper is devoted to the construction of string theories in curved space with torsion. We discuss both the Neveu-Schwarz-Ramond [1] type of string where the fermions carry vector indices and the Green-Schwarz [2] type of string where the fermions carry spinorial indices. Particular emphasis is put on the resulting constraints on space-time supersymmetry in the Green-Schwarz approach. Our analysis is mainly in the light cone gauge although some aspects of the covariant approach are discussed in the Appendix.

We use two-dimensional non-linear sigma models to describe the propagation of strings in background geometries with torsion. The background field can be understood as arising from condensation of infinite number of strings. Torsion can be viewed as the field strength associated with the vacuum expectation value of the anti-symmetric gauge field B_{nm} which appears in the supergravity multiplet. If the background fields only include the metric and torsion, a consistent string theory requires torsion to vanish. The possibility remains that torsion is nontrivial when other background fields are included, e.g. gauge fields and dilaton.

The effective ten-dimensional field theory which appears in the zero-slope limit $\alpha' = 0$ of the superstring is $N = 1$ supergravity coupled to super Yang-Mills matter. The low-energy theory has the supergravity transformations of the Chapline-Manton action [3] modified by the appropriate Chern-Simons terms introduced by Green and Schwarz [4]. The effective Lagrangian contains, even at the classical level, operators of arbitrarily high dimensions. These arise from integrating out the massive string modes. The truncation to the modified Chapline-Manton action is therefore strictly justified only if the radius of compactification measured in string units is much larger than one.

The paper is organized as follows. Section 2 is devoted to the proofs that, within our set of assumptions (identically zero β-function and only metric and torsion background fields), the only viable compactifications of the ten-dimensional theory are on manifolds without torsion. To prove this result from the effective field theory we take advantage of the analysis of Candelas, Horowitz, Strominger and Witten [5], who have analyzed in detail the conditions for $N = 1$ supersymmetry. They found that space-time M_4 must be flat Minkowski space and that the compact manifold K must admit a covariantly constant spinor ϵ. Furthermore, they showed that ϵ is an eigenspinor of H with zero eigenvalue, where

$$H = H_{mnp}\gamma^m\gamma^n\gamma^p \qquad m,n,p = 1,\ldots,6 \; . \tag{1.1}$$

The indices m,n,p refer to the compact manifold K and the γ's are the $O(6)$ Dirac matrices. H_{mnp} is the field strength associated with the antisymmetric field B_{mn} and can be identified with torsion. We reanalyze the conditions of Candelas, Horowitz, Strominger and Witten [5] with non-zero torsion. We show that if the generalized Ricci tensor vanishes, then the compact manifold cannot have any torsion.

From string considerations using the sigma model description the vanishing of the torsion arises as a clash between the demand of having zero β-function and having the flat space value for the central charge in the Virasoro algebra. At one loop the β-function vanishes provided the generalized Ricci tensor is zero. The central charge in the Virasoro algebra must remain unchanged in order not to shift the critical dimensionality away from ten [6].

In Section 3 we start the second part of our paper. We give a general discussion of string theories on a background manifold with torsion. In view of the first part one may ask why consider such theories at all. First we should remember that the string considerations of the previous section were carried within this framework. Moreover, the conclusions of Section 2 rely on a perturbative approach to the non-linear sigma model. In particular one can imagine not having $\beta \equiv 0$ but just a theory at a non-trivial fixed point of the β-function. Furthermore, in the presence of more background fields, such as dilaton and/or gauge fields the condition $\beta = 0$ need not be satisfied by a Ricci flat manifold. In that case torsion may be nontrivial. Hence, the vanishing Ricci tensor may not be a feature of all conceivable string models. Alternatively we can imagine relaxing the condition $H\epsilon = 0$. This amounts to considering string theories with a cosmological constant [5].

We start our analysis in Section 3 with two-dimensional non-linear sigma models on hermitian manifolds. These theories have an $N = 2$ supersymmetry in two dimensions provided certain conditions are satisfied [7]. We review the important features of these models, in particular the condition for supersymmetry. In the case in which the theory has a vanishing β-function, the sigma model can be viewed as a string theory of the Neveu-Schwarz-Ramond type. Next we use these theories to construct in ten dimensions a string theory of the Green-Schwarz type in the light cone gauge. In the flat space both of these theories have two eight-component supersymmetries, ϵ and δ [5]. It is essential to have both type of supersymmetries to ensure $N = 1$ supersymmetry [5] in four dimensions. The δ supersymmetry generalizes immediately to the curved space provided δ is covariantly constant. However, also the full eight component ϵ supersymmetry does not survive in general. The leftover supersymmetry is determined by the holonomy group of the compact manifold. For manifolds with $SU(3)$ holonomy with or without torsion two of the eight supersymmetries survive. The full eight-component supersymmetry can be recovered in curved space only if the torsion parallelizes the Riemannian connection. In this case the holonomy group is trivial. The $SU(3)$ holonomy group turns out to be also very important for demonstrating the equivalence of the Green-Schwarz action and the Neveu-Schwarz-Ramond action in a curved background.

Section 4 is devoted to an explicit example of a theory of the type discussed in Section 3. We study compactification on a four-dimensional hermitian manifold $S^3 \times S^1$ with torsion. The torsion corresponds to a Wess-Zumino term [8] which parallelizes K. The model will allow us to follow and observe explicitly some of the general considerations of Sections 2 and 3. In particular the model gives rise to a cosmological constant. We present arguments that compactifications on $M \times K$ with M maximally symmetric cannot be obtained from a consistent string theory unless M is flat. Compactifications which lead to a cosmological constant may turn out to be of interest as cosmological vacuum configurations. Section 5 contains our conclusions and some remarks.

In the Appendix we discuss some aspects of the covariant formulation and its connection to the light cone formulation. The covariant string action in flat space has both rigid and local supersymmetries. We show that the rigid supersymmetry becomes the δ supersymmetry in the light cone, while the ϵ supersymmetry is a combination of the rigid and the local supersymmetries.

2. String Compactification on Manifolds with Torsion

2.1 Effective Field Theories

As we mentioned in the Introduction, the effective field theory approach is valid whenever the compactification scale measured in string units is much larger than one. In this case one can neglect the higher derivative terms which appear in the effective ten-dimensional theory.* The supersymmetry transformation

* Recently some problems associated with this approximation were raised by Dine and Seiberg [9] and Kaplunovsky [10] on the basis of phenomenological considerations. If for the purpose of discussing the vacuum this truncation is questionable, then our analysis would need modification.

laws are identified with the Chapline-Manton supergravity transformation [3] modified by Green and Schwarz [3]. In the vacuum sector the fermions transform as follows

$$
\begin{aligned}
\delta\psi_\mu &= \nabla_\mu\epsilon + \frac{\sqrt{2}}{32}\, e^{2\phi}(\gamma_\mu\gamma_5 \otimes H)\epsilon \\
\delta\psi_m &= \nabla_m\epsilon + \frac{\sqrt{2}}{32}\, e^{2\phi}(\gamma_m H - 12 H_m)\epsilon \\
\delta\lambda &= \sqrt{2}(\gamma^m\nabla_m\phi)\,\epsilon + \frac{1}{8}\, e^{2\phi} H\epsilon \\
\delta\chi^a &= -\frac{1}{4}\, e^{\phi} F^a_{mn}\gamma^{mn}\epsilon\ .
\end{aligned}
\tag{2.1}
$$

The Greek indices refer to four-dimensional space M_4 and the Latin indices refer to the compact space K. The tensor H_{mnp} is the field strength associated with the antisymmetric tensor field B_{mn} which appears [5] in the supergravity multiplet. H_m is defined through $H_m = H_{mnp}\gamma^n\gamma^p$. The invariance of the vacuum under supersymmetry transformation demands that the variation of all the fermi fields in Eq. (2.1) vanishes. Candelas, Horowitz, Strominger and Witten [5] have shown that

$$
H\epsilon = 0 \tag{2.2}
$$

when M_4 is maximally symmetric. Furthermore they showed using Eq. (2.1) that the four-dimensional space-time M_4 must be flat Minkowski space. However, they only studied the case $H_{mnp} = 0$. Equation (2.1) can be satisfied without H_{mnp} being equal to zero. From the transformation laws of Eq. (2.1) we learn that the case $H_{mnp} \neq 0$ corresponds to a background compact manifold with torsion. In particular from the vanishing of $\delta\psi_m$ it follows that ϵ is a covariantly constant spinor with respect to the connection $\Omega_m = \omega_m - 4\beta H_m$

$$
\nabla_m(\Omega) \equiv (\nabla_m(\omega) - \beta H_m)\epsilon = 0 \qquad \beta = \frac{3\sqrt{2}}{8}\, e^{2\phi}\ , \tag{2.3}
$$

where $\nabla_m(\omega)$ is the covariant derivative with spin connection ω. From Eq. (2.3) we can read off the torsion of the new connection

$$T_{mnp} = 4\beta\, H_{mnp} \ . \tag{2.4}$$

Following the analysis of Ref. [5] we learn that the manifold admits a complex structure $f^m{}_n$ which is covariantly constant

$$\nabla_p(\Omega)\, f^m{}_n = 0 \ . \tag{2.5}$$

As in the case of zero torsion the complex structure can be built from the covariantly constant spinor. Also in Ref. [5] the following relation between the scalar curvature $R(\omega)$ and torsion H_{mnp} was derived

$$R(\omega) = \frac{16}{3}\,\beta^2 H_{mnp}H^{mnp} = \frac{1}{3}\,T_{mnp}T^{mnp} \ . \tag{2.6}$$

Note that the existence of complex structure and hermitian metric indicate that in the general case the manifold is hermitian.

On manifolds with torsion it is straightforward to calculate the generalized Riemann tensor built from the connection Ω

$$R(\Omega)_{mnpq} = R(\omega)_{mnpq} + \nabla_p(\omega)T_{mnq} - \nabla_q(\omega)T_{mnp} + T_{rmp}T^r{}_{qn} - T_{rmq}T^r{}_{pn} \ . \tag{2.7}$$

For totally antisymmetric T_{mnp}, the Riemann tensor $R(\Omega)_{mnpq}$ has the following symmetry properties

$$R_{mnpq} = R_{[mn]\,[pq]} \tag{2.8}$$

where the bracket indicates antisymmetrization of the indices. This allows the definition of only one type of generalized Ricci tensor

$$R(\Omega)_{mn} \equiv R(\Omega)^p{}_{mpn} = R_{mn}(\omega) - T^{pq}{}_m T_{pqn} + \nabla^p(\omega)T_{pmn} \ . \tag{2.9}$$

If Eq. (2.8) is not satisfied there are three different Ricci tensors that one can define corresponding to the different ways of contracting the indices.

The reparametrization invariance of the compactified string theory demands that the two dimensional non-linear sigma model must be conformally invariant [11]. This means that the β-function must vanish. It has been shown [8],[12] at one-loop level that, in a theory where the only background fields are the metric and torsion this is satisfied when the generalized Ricci tensor vanishes

$$R(\Omega)_{mn} = 0 \ . \tag{2.10}$$

In general both the metric and the potential of the torsion get one-loop counter terms proportional to the symmetric and antisymmetric parts of the generalized Ricci tensor. For detailed discussion on this point see section 3.

Using Eqs. (2.9) and (2.10) we find that the background manifold must satisfy

$$R(\omega) = T^{mnp}T_{mnp} \ . \tag{2.11}$$

This is in contradiction with Eq. (2.6) unless

$$T_{mnp} = H_{mnp} = 0 \ . \tag{2.12}$$

Hence if the Ricci tensor $R(\Omega)_{mn}$ is required to vanish the background manifold cannot have any torsion. If the background fields include the dilaton and/or gauge fields (2.10) may no longer hold, accordingly our conclusions based on (2.10) may be modified.

Next we would like to give another proof of the above result. Equations (2.3) and (2.4) are the important constraints that guarantee that our space-time M_4 is Minkowski space and that the four-dimensional theory has $N = 1$ supersymmetry at the compactification scale. These constraints together with other constraints obtained in Ref. [5] have been analyzed and solved [13] in the presence of torsion. The hermitian metric g on the compact six-dimensional manifold K must then satisfy

$$\partial_i(g^{i\bar{j}} \det g) = \partial_{\bar{j}}(g^{i\bar{j}} \det g) = 0 \qquad i,j = 1,2,3 \tag{2.13}$$

and

$$\det g = |f(z)|^2 \ . \tag{2.14}$$

The analytic function $f(z)$ can be mapped locally to $f(z) = 1$ by a general coordinate transformation. In Eq. (2.13) and (2.14) we have introduced complex coordinates

$$\begin{aligned} z^i &= \frac{1}{\sqrt{2}} \left(x^i - ix^{i+3}\right) \\ \bar{z}^i &= \frac{1}{\sqrt{2}} \left(x^i + ix^{i+3}\right) \end{aligned} \qquad i = 1,2,3 \ . \tag{2.15}$$

The condition in Eq. (2.13) can be rewritten in terms of the curl of the metric

$$\partial_i(\det g\, g^{i\bar{j}}) = 2\epsilon^{ik\ell}\epsilon^{\bar{j}\bar{n}\bar{m}}\ g_{\bar{n}k}\partial_i\ g_{\bar{m}\ell} \ . \tag{2.16}$$

On Kahler manifold the metric satisfies the relation

$$\partial_i g_{\bar{m}\ell} - \partial_\ell g_{\bar{m}i} = 0 \tag{2.17}$$

and therefore the conditions (2.13) are satisfied for Kahler metrics. However, Eq. (2.13) admits more general solutions that include torsion. Below we show that the Ricci tensor vanishes only when torsion vanishes. To calculate the generalized Ricci tensor $R(\Omega)_{mn}$ we need the spin connection Ω with $SU(3)$ holonomy. The spin connection is most conveniently expressed in the $SU(3)$ basis.

$$\Omega_i^{\alpha}{}_{\beta} = e^{\alpha\bar{j}}\partial_i e_{\bar{j}\beta} + \frac{1}{2}\, e^{\alpha\bar{j}}(\partial_i g_{\bar{j}\ell} - \partial_\ell g_{\bar{j}i})\, e^{\ell}{}_{\beta} \ . \tag{2.18}$$

All other nonzero components of the spin connection can be obtained by complex conjugation. In Eq. (2.18) we have introduced the complex 3-bein $e^{\alpha}{}_i$ and its complex conjugate $(e^{\alpha}{}_{i})^{*} = e_{\bar{i}\alpha}$. In the complex basis the only non-zero components of the torsion are

$$
\begin{aligned}
T_{ij\bar{k}} &= \frac{1}{2}\partial_{[i}g_{j]\bar{k}} \\
T_{\bar{i}\bar{j}k} &\equiv (T_{ij\bar{k}})^{*} \; .
\end{aligned}
\tag{2.19}
$$

From Eq. (2.19) and (2.17) it is clear that on a Kahler manifold the torsion vanishes. Now it is straightforward to calculate the generalized Ricci tensor. Using (2.18) and (2.13)

$$
R(\Omega)_{i\bar{j}} = -g^{m\bar{s}}(\partial_m\partial_{[\bar{s}}g_{\bar{j}]i} - \partial_i\partial_{[\bar{s}}g_{\bar{j}]m})
\tag{2.20}
$$

where the bracket indicates antisymmetrization of the indices. This antisymmetrization makes the generalized Ricci tensor depend only on the curl of g. Hence the Ricci tensor vanishes only on Kahler manifolds. Therefore on manifold with torsion the generalized Ricci tensor does not vanish. This is the same result as obtained above in Eqs. (2.10-2.12).

We emphasize that the conclusion of a vanishing torsion is based on the requirement that the Ricci tensor vanishes. This is not a necessary conclusion in the effective field theory analysis alone. The condition that the Ricci tensor vanishes is based on the sigma model analysis for models that include just the metric and torsion. For more general sigma model with more background fields this condition may be relaxed and torsion need not vanish.

In the previous analysis we set the external gauge field to zero. We conclude our analysis of the effective field theory approach by studying constraints on the torsion and the compact manifold involving the gauge fields of $E_8 \times E_8$ or $SO(32)$. In this case the totally antisymmetric tensor H_{mnp} is the field strength associated with the antisymmetric field B_{mn} modified by the Chern-Simons terms. In form notation we have [14]

$$dH = R^2 - \frac{1}{30} F^2 . \tag{2.21}$$

Here we must consider the size L of the compact manifold relative to the Planck length ℓ. Derivatives in the compact manifold are expected to be of order $1/L$. As we mentioned earlier to be able to consider reliably an effective ten-dimensional field theory we must take $L \gg \ell$. The left and right sides of Eq. (2.21) scale differently. The compensating scale factor is the Planck length ℓ

$$\ell^{-2} dH = R^2 - \frac{1}{30} F^2 .$$

Naively, it would appear that the left hand side is of order $\ell^{-2}L^{-2}$ while the right hand side is of order L^{-4} so that $L = \ell$ is expected. If $L \approx \ell$, Eqs. (2.1) may need modification from higher derivative terms in the effective ten-dimensional field theory, so that the entire analysis becomes suspect for such a vacuum solution. Of course, if $H = 0$ this difficulty does not arise and we may concentrate on compact manifolds with $L \gg \ell$. Another way of avoiding the problem is to look for solutions in which dH is smaller than expected on naive grounds. That is, there may be vacuum metrics for which the particular combination of curls symbolized by dH are of order ℓ^2/L^4 instead of $1/L^2$, while a typical derivative is still of order $1/L$. It is this type of manifold that can be reliably used for the physics of compactified effective string theory if torsion is not zero. We have checked, for simpler solvable nonlinear partial differential equations, that solutions of this type do exist. The more difficult question of whether such a solution, or for that matter $H = 0$ solution, is preferred by the theory as an absolute minimum solution cannot be answered with the considerations presented so far.

2.2 STRING CONSIDERATIONS

So far we have given two proofs that the torsion on the compact manifold must vanish when the Ricci tensor vanishes. Since not very much is known about string theories it is instructive to derive the same equations from different points of view. Below we show how some of the equations of the effective field theory approach can be derived directly from the string theory. In our previous analysis of the effective field theory with torsion Eqs. (2.6) and (2.10-2.11) played a crucial role. Eq (2.11) followed from the demand that the generalized Ricci tensor vanishes. For sigma models with torsion it seems to be a necessary condition to ensure perturbatively conformal invariance. The Ricci tensor appears already as a counter term at one-loop level. On the other hand Eq. (2.6) had to be satisfied for the theory to be supersymmetric at the compactification scale. In the string theory this equation arises from the demand that the central charge in the Virasoro algebra must have the same value as in a ten-dimensional supersymmetric string theory in flat space. If the central charge is changed, then the critical dimension of the theory is changed. Friedan and Shenker [6] have shown that if the critical dimension is changed, there are no zero mass fermions in the theory and therefore supersymmetry is broken.

In recent papers, [15],[16] the computation of the shift in the critical dimensionality was carried out for strings compactified on group manifolds with torsion. The torsion corresponds to a Wess-Zumino term which must be included in order to preserve conformal invariance of the theory [17]. For purely bosonic strings the critical dimension on $SU(N)$ is given by [15]

$$D = 26 - d_c = \frac{(N^2-1)k}{N+k} = d_G - \frac{N(N^2-1)}{k} + O\left(\frac{1}{k^2}\right) \qquad (2.22)$$

where $d_G = N^2 - 1$ is the dimension of $SU(N)$ and k is the integer coefficient of the Wess-Zumino term. It is useful to rewrite Eq. (2.22) in the form

$$26 - (d_c + d_G) = -\frac{N(N^2-1)}{k} + O\left(\frac{1}{k^2}\right) . \qquad (2.23)$$

To make contact with our previous analysis we need the relation between the string tension and the integer k [15]

$$\alpha' = \frac{2}{k} \ . \tag{2.24}$$

From Eq. (2.24) it follows that the first term on the right hand side of Eq. (2.23) is of order α' relative to the left hand side

$$26 - (d_c + d_G) = -\frac{N(N^2-1)}{2}\,\alpha' + O(\alpha'^2) \ . \tag{2.25}$$

From the analysis of the effective field theory we expect that the order α' term is $3R - T^2$. If $R = \frac{1}{3}T^2$ the critical dimension is unchanged. Since the shift of the critical dimension in Eq. (2.22) has been calculated at the fixed point of the sigma model, we can only verify that order α' terms in Eq. (2.25) are consistent with $3R - T^2$.

On group manifolds the torsion is given by the Wess-Zumino term

$$T_{abc} = \frac{1}{2}\,f_{abc} \ . \tag{2.26}$$

To calculate the term $3R - T^2$ we also need the scalar curvature.

$$R = \frac{1}{4}\,f_{abc}f^{abc} = \frac{1}{4}\,N(N^2-1) \ . \tag{2.27}$$

Hence

$$3R - T^2 = \frac{1}{2}\,N(N^2-1) \tag{2.28}$$

and therefore Eq. (2.25) can be rewritten as

$$26 - (d_c + d_G) = \alpha'(-3R + T^2) + O(\alpha'^2) \ . \tag{2.29}$$

From Eq. (2.29) it follows that the critical dimension remains unchanged if $R = \frac{1}{3}T^2$.

A similar analysis can be performed for the supersymmetric case using the computation of the critical dimensionality for this case [12]. Again we find that $R = \frac{1}{3}T^2$ is needed to ensure that the critical dimension does not shift.

Recently Callan, Martinec, Perry and Friedan [18] have actually proven Eq. (2.29) by coupling the non-linear sigma model to an external dilaton as well as an antisymmetric tensor field. They have computed the appropriate β-function to one loop. Equation (2.29) results from equating this β-function to zero and neglecting the external dilaton field. In view of this result the full β-function can be written in terms of the scalar curvature and torsion. Using Eqs. (2.22) and (2.28) we find that on $SU(N)$

$$26 - (d_C + d_G) = -\frac{(3R - T^2)\alpha' d_G}{d_G + \alpha'(3R - T^2)} \;. \tag{2.30}$$

From this it follows that the condition $3R - T^2$ must be satisfied to all orders in perturbation theory. A similar suppression holds for the supersymmetric case [15]. Therefore, if the dilaton is set to zero, the only way to preserve supersymmetry at the compactification scale is to have zero torsion.

3. Strings in the Light Cone

In this section we construct a string theory on a background manifold with metric and torsion. As discussed in the previous section the transformation laws of Eq. (2.1) and the requirement that the resulting four dimensional effective field theory has an $N = 1$ supersymmetry determine a lot of properties of the compact manifold. In particular the manifold must admit a covariantly constant spinor with respect to the connection with torsion Ω and consequently a covariantly constant complex structure. Furthermore, the metric must be hermitian. When the torsion vanishes the relevant manifolds are Ricci flat Kahler manifolds [5]. When torsion is included one should consider hermitian manifolds satisfying Eqs. (2.13)-(2.14).

Recently two dimensional supersymmetric non-linear sigma models with torsion have been analyzed [7]. It has been shown that the sigma model has an $N=2$ supersymmetry if the manifold is hermitian and if the complex structure is covariantly constant relative to the connection that includes torsion.

The general structure of the action after elimination of the auxiliary fields is

$$I(\phi,\lambda)=\frac{1}{2}\int d^2x\Big[g_{mn}\partial_\mu X^m\partial^\mu X^n+\frac{3}{2}\,B_{mn}e^{\mu\nu}\partial_\mu X^m\partial_\nu X^n$$

$$+ig_{mn}\overline{\lambda_+^m}\not{D}^+\lambda_+^n+ig_{mn}\overline{\lambda_-^m}\not{D}^-\lambda_-^n \tag{3.1}$$

$$+\frac{1}{4}\,R^+_{mnpq}(\overline{\lambda_+^m}\rho_\mu\lambda_+^n)(\overline{\lambda_-^p}\rho^\mu\lambda_-^q)\Big]$$

where the ρ_μ's are the two-dimensional Dirac matrices and

$$\lambda^n=\lambda_+^n+\lambda_-^n\qquad\lambda_\pm^n=\frac{1}{2}\,(1\pm\gamma_5)\lambda^n\;. \tag{3.2}$$

The $\lambda_\pm^n$ are Majorana-Weyl spinors. The connection associated with the right (left) handed fermions is:

$$\Gamma_\pm{}^n{}_{pm}=\left\{\begin{matrix}n\\mp\end{matrix}\right\}\pm T^n{}_{mp} \tag{3.3}$$

where $\left\{\begin{matrix}n\\mp\end{matrix}\right\}$ is the Christoffel connection. In Eq. (3.1) $D^\pm$ is the corresponding covariant derivative. The tensor $R^\pm_{mnpq}$ is the generalized Riemann tensor constructed from the connection $\Gamma^n_{\pm mp}$, and B_{mn} is the potential associated with the torsion T_{mnp}

$$T_{mnp}=-B_{[mn,p]}\;. \tag{3.4}$$

Note that this definition of torsion does not include the Chern-Simons terms. It differs from the torsion that appears in the effective field theory $T=dB-\frac{1}{30}\,\omega_Y+$

ω_L. As discussed in Section 2, the extra terms involve a compensating dimensionful parameter. Such a parameter is the slope parameter $\alpha' \sim (\ell_{Planck})^{-2}$. In fact the Chern-Simons terms appear in the next order of the loop expansion in the sigma model with the coefficient α' [18],[19]. This is a necessary consequence of Lorentz and gauge invariance as discussed by Green and Schwarz [4].

The action (3.1) is invariant under the supersymmetry transformation

$$\begin{aligned} \delta X^n &= \delta_+ X^n + \delta_- X^n = \bar{\epsilon}_+ \lambda^n_- + \bar{\epsilon}_- \lambda^n_+ \\ \delta \lambda^n_\pm &= -i\partial\!\!\!/ X^n \epsilon_\mp - \Gamma^n_{\pm mp} \lambda^m_\pm \delta_\pm X^p \end{aligned} \tag{3.5}$$

The sigma model has another supersymmetry [5]

$$\begin{aligned} \delta X^n &= \delta_+ X^n + \delta_- X^n = f^n_{-m} \bar{\epsilon}_+ \lambda^m_- + f^n_{+m} \bar{\epsilon}_- \lambda^m_+ \\ \delta(f^n_{\pm m} \lambda^m_\pm) &= -i\partial\!\!\!/ X^n \epsilon_\pm - \Gamma^n_{\pm mp} f^m_{\pm \ell} \lambda^\ell_\pm \delta_\pm X^p \end{aligned} \tag{3.6}$$

provided the complex structure $f^n_{\pm m}$ satisfies the following conditions

$$g_{nm} f^n_{\pm p} f^m_{\pm \ell} = g_{p\ell} \tag{3.7a}$$

$$\begin{aligned} \nabla^\pm_\ell f^n_{\pm m} &\equiv \partial_\ell f^n_{\pm m} + \Gamma^n_{\pm p\ell} f^p_{\pm m} - f^n_{\pm p} \Gamma^p_{\mp m\ell} \\ &\equiv \nabla_\ell f^n_{\pm m} \pm T^n{}_{p\ell} f^p_{\pm m} \mp f^n_{\pm p} T^p_{m\ell} = 0 \end{aligned} \tag{3.7b}$$

$$R^+_{mnp\ell} f^m_{+q} f^n_{+r} f^p_{-s} f^\ell_{-t} = R^+_{qrst} \; . \tag{3.7c}$$

Equation (3.7a) implies that the complex structure is covariantly constant with respect to the connection with torsion.

For the analysis below it is important to note that conformally invariant nonlinear sigma models possess another type of supersymmetry. This supersymmetry is the partner of the local Kac-Moody transformation and has the form

$$\delta X^i = 0 \qquad \delta\lambda^i_\pm = \delta^i_\pm \,. \tag{3.8}$$

An example of a theory where this symmetry can be easily verified [20] is the supersymmetric non-linear sigma model with Wess-Zumino term. When the β-function vanishes this sigma model is invariant under transformation (3.8).

The Lagrangian of Eq. (3.1) can be easily generalized to an arbitrary number p of left moving fermions and q right moving fermions. This leads to a supersymmetric model of type (p, q) [21]. Although we shall concentrate on the $(1, 1)$ model of Eq. (3.1) our results will generalize to any (p, q) type of model. Note that the $(2, 2)$ model corresponds to a Green-Schwarz string of type I [2] while the $(1, 0)$ model corresponds to the heterotic string [22]. The (1,0) model is obtained from Eq. (3.1) by deleting the λ_+ field.

The renormalization of non-linear σ-models on manifolds with torsion is discussed at length in Ref. [8]. Using the background field method, it is straightforward to show that both the metric and the potential of the torsion acquire one loop counter terms

$$\begin{aligned} g^{(1)}_{mn} &= \frac{1}{4\pi\,(d-2)}\,(R_{mn} + R_{nm}) \\ B^{(1)}_{mn} &= \frac{1}{4\pi\,(d-2)}\,(R_{mn} - R_{nm}) \end{aligned} \tag{3.9}$$

where R_{mn} is the generalized Ricci tensor with torsion. From Eq. (3.9) it is clear that in order to have a zero β-function R_{mn} must vanish. Therefore the string theory is conformally invariant only if the torsion parallelizes the Ricci tensor.

Next we would like to elevate the two dimensional supersymmetry to a space-time supersymmetry. We will work in ten dimensions. When the compact manifold is flat this amounts to going from the Neveu-Ramond-Schwarz version of the string theory to the Green-Schwarz superstring. Below we investigate the relation of these two theories in the light cone gauge, with a nontrivial curved background and torsion. In the Appendix we discuss some aspect of the covariant approach. We demonstrate how the light cone supersymmetries ϵ and δ arise from the rigid and local supersymmetries of the covariant action [23]. As discussed in Ref. [5], the ϵ and δ supersymmetries are necessary to guarantee an $N = 1$ supersymmetry from the four-dimensional point of view.

The generalization of the Green-Schwarz superstring to curved space turns out to be tricky. The full ten-dimensional $N = 1$ ϵ supersymmetry does not in general exist in the light cone. Only when the manifold is parallelized does the full ten-dimensional supersymmetry exist. There are two ways this can happen. Either the manifold is flat or the torsion parallalizes the full Riemann tensor. This is very similar to the covariant form of the superstring action in flat space. The string can be viewed as a sigma model on a supermanifold with parallelizing torsion [24]. If the torsion does not parallelize the Riemann tensor the action is only invariant under a subset of the eight component ϵ supersymmetry. The number of supersymmetries left over is determined by the holonomy group.

The light cone action for a Green-Schwarz superstring with zero torsion in curved space has been considered by Candelas, Horowitz, Strominger and Witten. [5] We discuss here the generalization to manifolds with torsion. In this case the action is given by

$$\begin{aligned} I = \int d^2\sigma \Bigg[& \frac{1}{2} g_{ij} \partial_\alpha X^i \partial^\alpha X^j + \tfrac{1}{2}\, \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j \\ & + \frac{i}{4}\, \overline{S}_+ \gamma_+ \rho^\alpha D^+_\alpha S_+ + \frac{i}{4}\, \overline{S}_- \gamma_+ \rho^\alpha D^-_\alpha S_- \\ & + \frac{1}{4}\, R^+_{ijk\ell} \overline{S}_+ \gamma_+ \gamma^i \gamma^j \rho^\alpha S_+ \overline{S}_- \gamma_+ \gamma^k \gamma^\ell \rho_\alpha S_- \Bigg] \end{aligned} \tag{3.10}$$

where $S = S_+ + S_-$ and $S_+(S_-)$ is a right (left)-moving fermion. The tensor g_{ij} is the metric in the transverse space. The covariant derivative in Eq. (3.10) is $D_\alpha = \partial_\alpha + \frac{\partial x^i}{\partial \sigma^\alpha} \Omega^\pm_i$ and $\Omega^\pm_i = \omega_i \pm T_i$ is the connection with torsion in the transverse direction. The ρ_α's are the two-dimensional Dirac matrices. We first consider only the quadratic terms of the action (3.10) ignoring for the moment the $O(S^4)$ terms. We return to them later. In flat space the action (3.10) has two eight component supersymmetries [5].

$$\delta X^i = (p^+)^{-1/2}\sqrt{2}\,\bar{\epsilon}\gamma^i S \tag{3.11a}$$

$$\delta S^\gamma = i\frac{(p^+)^{-1/2}}{\sqrt{2}}\,(\gamma_- \gamma_M(\rho\cdot\partial x^M)\epsilon)^\gamma \tag{3.11b}$$

and

$$\delta x^i = 0 \qquad \delta S^\alpha = \delta^\alpha \tag{3.12}$$

where ϵ and δ are eight component real spinors of $O(8)$. In curved space it is straightforward to see that the kinetic part of the action in Eq. (3.19) is invariant under the δ-supersymmetry of Eq. (3.12) provided that δ is covariantly constant. Later we shall see that the four-fermi term and hence the full Lagrangian is invariant as well. In curved space the transformation (3.11b) gets modified by terms of the form $\omega_i\gamma_- S\delta X^i$. One cannot usually implement the full eight components ϵ supersymmetry. To see this let us consider the variation of the action to linear order in the fermion fields. The variation of the fermionic kinetic term produces among others a term of the form

$$\delta I_f^{\text{kinet.}} = i\,\overline{S}\rho^\alpha\rho^\beta\,\frac{\partial X^i}{\partial\sigma^\alpha}\,\frac{\partial X^j}{\partial\sigma^\beta}\,\gamma_i\,\Omega_j\epsilon\ . \tag{3.13}$$

In two dimensions the gamma matrices satisfy $\rho^\alpha\rho^\beta = \rho_5\epsilon^{\alpha\beta} + g^{\alpha\beta}$ and therefore Eq. (3.13) contains a term proportional to $\epsilon^{\alpha\beta}$. This term can certainly not be cancelled by the variation of any bosonic term unless the torsion term is included. In general the variation of torsion only cancels part of the $\epsilon^{\alpha\beta}$. Furthermore since the spin connection Ω_i contains two gamma matrices the term exhibited in Eq. (3.13) involves a multiplication of three γ matrices without any explicit symmetrization between them. Such a term cannot be cancelled by other terms appearing in the variation of the action (3.10). These troublesome terms do not appear in the flat space case since in this case the spin connection Ω vanishes.

In order to understand the fate of the ϵ supersymmetry we study the relation between the Neveu-Schwarz-Ramond string and the Green-Schwarz string. In the Neveu-Schwarz-Ramond version of the theory the fermion field is a vector under $O(8)$ while in the Green-Schwarz version it is a spinor. A prerequisite for the two versions to describe the same theory, is that both spinor and vector representations have the same dimensionality. This property is unique for $O(8)$. Compactifying the theory on a six-dimensional manifold amounts to considering the subgroup $O(6) \times O(2)$ of $O(8)$. Now it is clear that in order for the construction to make sense there must be a subgroup $H \subset O(6)$ such that the decomposition of the vector and spinor representation under H is the same [5]. This is the case for $H = SU(3)$ which is the holonomy group of the six-dimensional manifold in the absence of torsion. With torsion the holonomy group can be a subgroup of $SU(3)$. If the holonomy group is $SU(3)$ or a subgroup of $SU(3)$ we show below the equivalence of the Green-Schwarz and the Neveu-Schwarz-Ramond versions of the theory. If we accept this statement it is clear that the original Green-Schwarz action without torsion cannot have an eight component supersymmetry. This would have implied that the Neveu-Schwarz-Ramond action on the compact manifold has $N = 6$ supersymmetry. However, it is a well known result that supersymmetric non-linear sigma models on Kahler manifold have at most $N = 2$ supersymmetry.

To prove the equivalence of the Green Schwarz and Neveu-Schwarz-Ramond superstring action in the light cone gauge let us consider the action (3.10). From the discussion above it is clear that it is useful to work in the $SU(3)$ basis. The fermions in this basis have the form

$$S^A = \begin{pmatrix} \psi^\alpha \\ \chi_1 \\ \psi_\alpha \\ \chi_2 \end{pmatrix} \qquad \begin{array}{l} A = 1, \ldots, 8 \\ \\ \alpha = 1, 2, 3 \end{array} \tag{3.14}$$

This corresponds to the following decomposition of the spinor representation under $SO(8) \supset SO(6) \supset SU(3)$

$$8(\text{spinor}) \rightarrow 4 + \overline{4} \rightarrow 3 + 1 + \overline{3} + 1\ . \tag{3.15}$$

The spinors ψ^α (ψ_α) are triplets (antitriplets) and the spinors χ_1 and χ_2 are singlets of $SU(3)$. The Majorana condition requires that $\psi_\alpha = (\psi^\alpha)^*$ and $\chi_2 = \chi_1^*$. Since the $SU(3)$ indices in Eq. (3.14) are tangent space indices we need a vielbein to transform the tangent space indices to world indices. In the $SU(3)$ basis the vielbein is defined to be [13]

$$\begin{aligned} e^\alpha{}_m &= E^\alpha{}_m - iE^{\alpha+3}{}_m \\ \overline{e}_{\alpha m} &= E^\alpha{}_m + iE^{\alpha+3}{}_m = (e^\alpha{}_m)^* \end{aligned} \tag{3.16}$$

where $E^n{}_m$ is the six-bein in the real basis. To write the action in the $SU(3)$ basis it is convenient to use the $SU(3)$ covariant γ matrices

$$\begin{aligned} \gamma^\alpha &= \frac{1}{2}\,(\Gamma_\alpha + i\Gamma_{\alpha+3}) \\ \overline{\gamma}_\alpha &= \frac{1}{2}\,(\Gamma_\alpha - i\Gamma_{\alpha+3}) = (\gamma^\alpha)^+ \end{aligned} \tag{3.17}$$

$$\{\gamma^\alpha, \gamma^\beta\} = \{\overline{\gamma}_\alpha, \overline{\gamma}_\beta\} = 0 \qquad \{\gamma^\alpha, \overline{\gamma}_\beta\} = \delta^\alpha{}_\beta$$

where the Γ_I's are the 8×8 hermitian Dirac matrices

$$\begin{array}{lll} \Gamma_1 = -\sigma_1 \times \sigma_1 \times \sigma_2 & \Gamma_2 = -\sigma_1 \times \sigma_2 \times 1 & \Gamma_3 = \sigma_1 \times \sigma_3 \times \sigma_2 \\ \Gamma_4 = \sigma_2 \times \sigma_2 \times \sigma_1 & \Gamma_5 = -\sigma_2 \times \sigma_2 \times \sigma_3 & \Gamma_6 = \sigma_2 \times 1 \times \sigma_2\ . \end{array} \tag{3.18}$$

We now return to the fermionic kinetic term and rewrite it in the $SU(3)$ basis. The covariant derivative has two pieces: the ordinary derivative and the connection part. In the $SU(3)$ basis the ordinary derivative part for the right moving fermion has the form

$$
\begin{aligned}
\mathcal{L}_1^+ = \frac{i}{4}\,\overline{S}_+\gamma_+\partial\!\!\!/ S_+ &= \frac{i}{2}\left\{\overline{\psi}_{+\alpha}\partial\!\!\!/\psi_+^\alpha + \overline{\chi}_{+2}\partial\!\!\!/\chi_{+1}\right. \\
&= \frac{i}{2}\left\{\overline{\psi}_+^{\bar{j}} e_{\alpha\bar{j}}\partial\!\!\!/(e_i^\alpha\psi_+^i) + \overline{\chi}_{+2}\partial\!\!\!/\chi_{+1}\right\} \\
&= \frac{i}{2}\left\{g_{i\bar{j}}\overline{\psi}_+^{\bar{j}}\partial\!\!\!/\psi_+^i + \overline{\psi}_+^{\bar{j}}(e_{\alpha\bar{j}}\partial\!\!\!/ e_i^\alpha)\psi_+^i + \overline{\chi}_{+2}\partial\!\!\!/\chi_{+1}\right\} .
\end{aligned}
\tag{3.19}
$$

We have defined ψ^i through $\psi^\alpha = e^\alpha{}_i\psi^i = (\psi_\alpha)^*$ and we have used the fact that $g_{i\bar{j}} = e^\alpha{}_i e_{\alpha\bar{j}}$. The two-dimensional gamma matrix ρ^0 stays intact and just gives $\overline{\psi}$. We have a similar expression for the left moving fermions. From now on we drop the singlet fields χ_1 and χ_2 and only consider the fields that carry non-trivial $SU(3)$ quantum numbers. These are the only fields which couple to the spin connection.

Next we consider the connection part

$$
\mathcal{L}_2^+ = \frac{i}{4}\,\overline{S}_+\gamma_+\rho^\alpha\frac{\partial z^i}{\partial\sigma^\alpha}\,\Omega_i^+ S_+ \tag{3.20}
$$

where

$$
\Omega_i^+ = \frac{1}{4}\,\Omega_i^{+a}{}_b[\gamma_a\gamma^b] .
$$

In the $SU(3)$ basis the spin connection Ω_i can be decomposed as follows

$$
\frac{1}{4}\,\Omega_i^{+a}{}_b[\gamma_a\gamma^b] = \frac{1}{2}\,\overline{\Omega}^+_{i\alpha\beta}\gamma^\beta\gamma^\alpha + \frac{1}{2}\,\Omega_i^{+\alpha\beta}\overline{\gamma}_\beta\overline{\gamma}_\alpha + \frac{1}{2}\,[\gamma^\beta,\overline{\gamma}_\alpha]\Omega_i^{+\alpha}{}_\beta . \tag{3.21}
$$

The different terms in Eq. (3.21) correspond to the decomposition of the adjoint of $SO(6)$ under $SU(3)$

$$
15 \to 8 + 3 + \overline{3} + 1 . \tag{3.22}
$$

Similar expressions hold for the left moving fermions. When the holonomy group is $SU(3)$ only the adjoint of $SU(3)$ survives in the decomposition of Eq.(3.21)

$$\overline{\Omega}^{\pm}_{i\alpha\beta} = \Omega^{\pm\alpha\beta} = 0\ , \quad \text{and} \quad \Omega^{\pm\alpha}_{\bar{i}}{}_{\alpha} = 0\ . \tag{3.7}$$

This form of the spin connection can also be understood by studying the covariantly constant spinors $D_p\eta = 0$. Under $SU(3)$ the spinor of $O(8)$ decomposes as $8 \to 3 + \overline{3} + 1 + 1$. From the commutator $[D_n, D_m]\eta = R_{mnpq}\gamma^p\gamma^q\eta$ it is clear that the $SU(3)$ singlets in the decomposition of the spinor correspond to the covariantly constant spinors. From Eqs. (3.14) and (3.18) it follows that the covariantly constant spinors have the form $\eta = (000a000b)$. Therefore in the basis provided by (3.18) the equation $D_p\eta = 0$ is satisfied provided the spin connection has the block diagonal form determined by Eq. (3.23). If the holonomy group is a subgroup of $SU(3)$ some of the elements of $\Omega^{\alpha}{}_{\beta}$ vanish. Now it is straightforward to rewrite $\mathcal{L}_2$ in the $SU(3)$ basis

$$\begin{aligned}
\mathcal{L}_2^+ &= \frac{i}{2}\left\{\overline{\psi}_{+\alpha}\rho\cdot\frac{\partial z^i}{\partial\sigma}\,\Omega_i^{+\alpha}{}_{\beta}\psi_+^{\beta} + \overline{\psi}{}^{\alpha}_{+}\rho\cdot\frac{\partial z^{\bar{i}}}{\partial\sigma}\,\Omega_{\bar{i}}^{\beta}{}_{\alpha}\psi_{+\beta}\right\}\\
&= \frac{i}{2}\Bigg\{\overline{\psi}_{+\alpha}\rho\cdot\frac{\partial z^i}{\partial\sigma}\,[-e^j_{\beta}\partial_i e^{\alpha}_j + e^j_{\beta}\Gamma^k_{+ij}e^{\alpha}_k]\,\psi^{\beta}_+\\
&\quad + \overline{\psi}{}^{\alpha}_{+}\rho\cdot\frac{\partial \bar{z}^{\bar{i}}}{\partial\sigma}\,[-e^j_{\beta}\partial_{\bar{i}}e^{\alpha}_j + e^j_{\beta}\,\Gamma^k_{+\bar{i}j}e^{\alpha}_k]\psi^{\beta}_+\Bigg\}\\
&= \frac{i}{2}\left\{-\,\psi^{\bar{l}}_+ e_{\alpha\bar{l}}\partial\!\!\!/ e^{\alpha}_j\psi^j_+ + g_{\bar{l}k}\overline{\psi}{}^{\bar{l}}_+\rho\cdot\frac{\partial z^i}{\partial\sigma}\,\Gamma^k_{+ij}\psi^j_+ + g_{\bar{l}k}\overline{\psi}{}^{\bar{l}}_+\rho\cdot\frac{\partial z^{\bar{i}}}{\partial\sigma}\,\Gamma^k_{+\bar{i}j}\psi^j_+\right\}
\end{aligned} \tag{3.24}$$

Combining Eqs. (3.19) and (3.24) we get

$$\mathcal{L}_{\text{kin}} = \frac{i}{2}\,g_{\bar{i}j}\left\{\overline{\psi}{}^{\bar{i}}_+ D\!\!\!\!/_+\psi^j_+ + \overline{\psi}{}^{\bar{i}}_- D\!\!\!\!/_-\psi^j_-\right\} \tag{3.25}$$

where

$$D\!\!\!\!/_\pm\psi^j_\pm = \rho\cdot\frac{\partial z^i}{\partial\sigma}\,(\partial_i\psi^j + \Gamma^{+j}_{\pm ik}\psi^k_\pm) + \rho\cdot\frac{\partial z^{\bar{i}}}{\partial\sigma}\,(\partial_{\bar{i}}\psi^j_\pm + \Gamma^j_{\bar{i}k}\psi^k_\pm)\ . \tag{3.26}$$

Equation (3.25) just gives us the correct fermionic kinetic term of the supersymmetric non-linear sigma model with fermions transforming as vectors. This is the Neveu-Schwarz-Ramond version of the string in curved background including torsion. Since the compact manifold is a complex manifold with $SU(3)$ holonomy, the non-linear sigma model has an $N = 2$ supersymmetry. As we have remarked before this already implies that no general $N = 1$ ten-dimensional (two eight components) supersymmetry is expected to exist in the covariant (light cone) approach.

Above we showed that the Green-Schwarz string theory on curved background with torsion is not invariant under the full ten-dimensional ϵ supersymmetry. To see what part of the full supersymmetry survives in curved space we again use the $SU(3)$ basis and the existence of an $SU(3)$ holonomy. To keep our analysis as simple as possible we consider the supersymmetry transformation of the bosonic fields,

$$\delta X^i = (p^+)^{-1/2}\bar{\epsilon}\gamma^i S \ . \tag{3.27}$$

The transformation of the fermionic fields works in a similar way. We consider only the compact part of the background manifold. It is advantageous to work with the $SU(3)$ triplet and anti-triplet combination of the x's defined in Eq.(2.15). The spinor ϵ has a similar decomposition under $SU(3)$ as ψ given in Eq.(3.14). The action for the superstring is invariant when the triplet ϵ^α and anti triplet ϵ_α vanish. Equation (3.27) takes the form

$$\delta z^m = \bar{\epsilon}_1 e^m_\alpha \psi^\alpha_1 = \bar{\epsilon}_1 \psi^m \tag{3.28}$$

and

$$\delta z_m = \bar{\epsilon}_2 e^\alpha_m \psi_\alpha = \bar{\epsilon}_2 \psi_m \tag{3.29}$$

where ϵ_1 and ϵ_2 are the $SU(3)$ singlets and $\epsilon_2 = \epsilon_1^*$. We thus obtain two supersymmetries on the curved space associated with the singlet parameters $\epsilon_1 = \epsilon_2^*$.

This is of course a manifestation of the $SU(3)$ holonomy. Since the $SU(3)$ group is unbroken it is obvious that the $N=2$ supersymmetry parameters in the Neveu-Schwarz-Ramond version, must correspond to the $SU(3)$ singlets. As far as the flat space part is concerned we have two other supersymmetries under which the single spinors χ_1 and χ_2 of Eq. (3.14) transform. These two supersymmetries are associated with yet another ϵ spinor which again have the form $(0,0,0,\epsilon_3,0,0,\epsilon_4)$ where ϵ_3 and ϵ_4 are also $SU(3)$ singlets.*

So far we have not considered the higher order fermionic terms. In the $SU(3)$ basis the light cone Green-Schwarz string reduces to the Neveu-Schwarz-Ramond string which is just a sigma model on a curved manifold. Therefore it is clear that in the light cone the action will be invariant provided the ϵ has the form $\epsilon = (0,0,0,\epsilon_1,0,0,0,\epsilon_2)$. To check the δ-supersymmetry is equally straightforward. Since δ is covariantly constant spinor we have

$$(R^{\pm}_{mnpq}\gamma^p\gamma^q)\delta_{\pm} = 0 \ . \tag{3.30}$$

Furthermore since $R^{+}_{mnpq} = R^{-}_{pqmn}$ it follows that the superstring action (3.10) is invariant under the δ supersymmetry.

---

* Actually to work it in details we have to construct all the 32×32 gamma matrices associated with $SO(10)$. Until now we could work with the six 8×8 gamma matrices of $SO(6)$ in Eq. (3.18). These $SO(10)$ gamma matrices can be taken to be $\gamma_\mu \otimes 1_8$, $\gamma_5 \otimes \Gamma_i$ with γ_μ, γ_5 the usual four-dimensional Dirac matrices ($\mu = 1,\ldots,4$) and Γ_i the 8×8 gamma matrices of Eq. (3.18) ($i = 1,\ldots,6$). When we impose the Weyl condition we can work with 16 component spinors.

4. Compactification on $M_6 \times (S^3 \times S^1)$

To illustrate some of the general discussion of the previous sections let us consider string theory on $M_6 \times K$ where the compact manifold K is the hermitian "integer CP^2" manifold. This is a four-dimensional manifold with two complex coordinates (z_1, z_2). The points (z_1, z_2) and (nz_1, nz_2) are identified for all integers n. The natural metric for this manifold in complex coordinates is

$$dx^2 = \frac{dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2}{\bar{z}_1 z_1 + \bar{z}_2 z_2} \,. \tag{4.1}$$

The manifold defined by the metric (4.1) is $S^3 \times S^1$. To show this we introduce real coordinates $(\phi^1, \phi^2, \phi^3, \omega)$. These are related to the complex coordinates z_1, z_2 by

$$z_1 = e^{\omega}(\phi^1 + i\phi^2) \qquad z_2 = e^{\omega}\left(\phi^3 + i\sqrt{1 - \vec{\phi}\cdot\vec{\phi}}\right) \,. \tag{4.2}$$

In these real coordinates the metric has the form

$$ds^2 = (d\omega)^2 + g_{ab} d\phi^a d\phi^b \qquad g_{ab} = \delta_{ab} + \frac{\phi_a \phi_b}{(1 - |\phi|^2)} \tag{4.3}$$

and the manifold is indeed $S_3 \times S_1$. To parallelize S_3 we introduce torsion

$$T_{abc} = \frac{4a}{(1 - |\phi|^2)^{1/2}}\, \epsilon_{abc} \tag{4.4}$$

to the sigma model via the Wess-Zumino term [8]. In Eq. (4.4) a is a numerical constant. It will be determined by the requirement that the theory is both conformal invariant and has a two-dimensional $N = 2$ supersymmetry. In complex coordinates the nonvanishing components of the torsion are

$$\begin{aligned} T_{ij\bar{i}} &= \frac{a\bar{z}_j}{(|z_1|^2 + |z_2|^2)^2} \\ & \qquad\qquad\qquad\qquad i \neq j \quad i,j = 1,2 \,. \\ T_{\bar{i}\bar{j}i} &= \frac{-az_j}{(|z_1|^2 + |z_2|^2)^2} \end{aligned} \tag{4.5}$$

The non-zero components of the connection

$$\Gamma^i_{jk} = \left\{ {i \atop jk} \right\} + \frac{1}{2}\left(T_{jk}{}^i - T^i{}_{jk} - T^i{}_{kj}\right) . \tag{4.6}$$

are proportional to either $a-1$ or $a+1$. In Eq (4.6) $\left\{ {i \atop jk} \right\}$ is the Christoffel connection. The generalized curvature vanishes whenever $a = \pm 1$. This certainly guarantees that the β-function vanishes. This also guarantees that the conditions of Eq. (3.7) hold. The sigma model has an $N=2$ supersymmetry if $a = \pm 1$. To write down the torsion and curvature it is convenient to introduce new real coordinates $x_i \quad i = 1 \ldots 4$ defined by

$$z_1 = x_1 + ix_2 \qquad z_2 = x_3 + ix_4 . \tag{4.7}$$

In these coordinates, the torsion turns out to be

$$T_{mnp} = a\,\frac{\epsilon_{mnpq}x^q}{(|x|^2)^2} \qquad |x|^2 = \sum x_i^2 \tag{4.8}$$

The generalized curvature is given by

$$\begin{aligned} R_{mnpq} = \frac{2(1-a^2)}{(|x|^2)^3}\Big\{ & \delta_{mq}x^n x^p - \delta_{qn}x^m x^p \\ & - \delta_{mp}x^n x^q + \delta_{np}x^m x^q + \delta_{mp}\delta_{nq}|x|^2 - \delta_{np}\delta_{mq}|x|^2 \Big\} . \end{aligned} \tag{4.9}$$

and the spin connection including torsion is

$$\Omega^a{}_{bm} = -\frac{1}{|x|^2}\left(\delta_{am}x^b - \delta_{mb}x^a\right) + a\,\frac{\epsilon_{amb\sigma}x^\sigma}{|x|^2} . \tag{4.10}$$

Note that the spin connection for $a = \pm 1$ is either self dual or anti self dual. This means that in the Lagrangian of Eq. (3.1) the kinetic term for the left or right handed fermions involves only simple derivative. Since the generalized curvature is zero for $a = \pm 1$ (Eq. (4.9)) the connection is "a pure gauge". Therefore we can by a change of basis move the connection from the right handed to the left hand fermions and vice versa. There are four covariantly constant spinors: two left handed and two right handed and therefore the sigma model has $N = 4$ supersymmetry. As we remarked earlier the compact manifold should admit a covariantly constant complex structure. For $a = \pm 1$ Eq. (3.7) has a solution. We find

$$f^n{}_m = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix} . \tag{4.11}$$

It is interesting to note that the condition $a = \pm 1$ arises from both the conformal and supersymmetry invariance. The only condition of Section 2 which is not met by our example is that of Eq. (2.2). An explicit calculation shows that

$$H\epsilon_{\pm} = i\sqrt{K}\epsilon_{\mp} . \tag{4.12}$$

As we mentioned in the Introduction, K is associated with the cosmological term. An indication to this statement can be obtained by looking at the gravitino transformation law, comparing it to the transformation law in four dimension with cosmological term [5].

Similar analysis can be repeated for the manifold $M_4 \times S^3 \times S^3$. Again torsion appears as the Wess-Zumino terms on each of the S^3 spheres. The torsion parallelizes the compact manifold, hence the associated β-function vanishes.

In the analysis so far we investigated the compact part K. The actual manifold is $M \times K$. Form the nonlinear sigma model point of view the β-function should vanish on M as well as on K. Compactifications which lead to cosmological constants are inconsistent if M is maximally symmetric. On an (anti) de-Sitter space the Ricci tensor does not vanish hence the β-function is non-zero.

A priori one could contemplate having other background fields on M. As long as M is maximally symmetric the dilaton cannot appear as a background field. It is also straightforward to show that torsion cannot be introduced to parallelize the generalized Ricci tensor on an (anti) de-Sitter space. This conclusion is in accordance with a recent argument by Witten [25] who has shown using scaling arguments that solutions to the ten dimensional supergravity equations of motion do not allow cosmological constant. Compactifications on M which is not maximally symmetric and which lead to cosmological constant, may turn out to be of interest as cosmological vacuum configurations. It would be interesting to find out whether in the presence of metric and/or torsion and/or dilaton background fields such solutions exist.

5. Conclusions and Remarks

We have considered string theories with metric and torsion as background fields. We have shown that in this case the string theory has a four-dimensional $N = 1$ supersymmetry in four dimensions only if torsion vanishes on the compact manifold. In our proofs we have invoked both string considerations and effective field theory considerations based on the transformation laws of the modified Chaplin-Manton ten-dimensional supergravity. For the effective field theory approach we were limited to the case in which the radius of compactification was much larger than the Planck length and therefore the higher derivative terms in the effective supergravity theory could be neglected. If this condition is not satisfied [9],[10] one cannot rely on our analysis of the effective field theory approach. However, the string approach based on the sigma model is more general and does not rely on truncating the higher order modes.

For the sigma model approach the conclusion that the torsion must vanish relied heavily on the fact that the generalized Ricci tensor had to vanish. We must emphasize that Ricci flatness was demanded on the basis of one loop finiteness. This may not be true in all orders. One is lead then to consider a situation in which the β-function has a nontrivial fixed point rather than being identically zero. Furthermore we have not investigated the possibility of more general representations of the compactified string theory that may include additional background fields such as the dilaton and gauge fields. Therefore the conclusion of zero torsion which followed from the vanishing of the Ricci tensor may be relaxed in a more general theory.

Recently it has been argued [26] that the β-function vanishes when the generalized Ricci tensor has the form $\nabla^i\Lambda^j$ where Λ^j is some globally defined vector field. In this case the one loop counterterms in the sigma model can be absorbed into the redefinition of the coordinates $x^i \to x^i + \Lambda^i$ since $\mathcal{L}(g^{ij} + B^{ij}) = \nabla^i\Lambda^j$ where $\mathcal{L}$ is the Lie derivative and ∇^i is the covariant derivative with torsion. Clearly a counterterm of the form $\mathcal{L}_{c.t.} = \int d^2\sigma(\nabla_{(i}\Lambda_{j)}\partial_\alpha X^i\partial_\alpha X^i + \nabla_{[i}\Lambda_{j]}\partial_\alpha X^i\partial_\beta X^j\epsilon^{\alpha\beta})$ vanishes on shell when one integrates by parts. To decide if the theory is conformally invariant one has to calculate the trace of the energy momentum tensor $\langle\theta^\alpha_\alpha\rangle$. This calculation is most easily done by introducing the dilaton field ϕ and coupling it to θ^α_α. In $2+\epsilon$ dimensions this amounts to considering the Lagrangian

$$\mathcal{L} = \int d^{2+\epsilon}\sigma\, e^{\epsilon\phi(x)}(g_{ij}\partial_\alpha X^i\partial^\alpha X^j + B_{ij}\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j)\ .$$

The trace of the energy momentum tensor is now given by the derivative of the effective action with respect to the dilaton field ϕ at $\phi = 0$. Since $\langle\theta^\alpha_\alpha\rangle$ is local we cannot integrate by parts as above in the effective action to get rid of terms of the form $\nabla_i\Lambda_j$.*

The torsion in the sigma model action is a density only in two dimensions. Therefore we have to introduce the factor $e^{\epsilon\sigma}$ in front of this term once we go to $2+\epsilon$ dimensions. This factor guarantees that the classical action is conformally invariant. However, in the presence of this factor we lose gauge invariance. To

* The preceding argument was developed in conversation with D. Freedman.

phrase it differently, the equation of motion for X includes now the gauge field B_{ij} explicitly, rather than the field strength T_{ijk}. The condition for conformal invariance was analyzed in detail by Callan, Perry, Martinec and Friedan [18]. It is important to realize that the result of their calculation is gauge invariant and therefore gives us some confidence that it is the correct result. Nevertheless a regularization scheme which preserves conformal invariance, gauge invariance and supersymmetry (for supersymmetric models) is clearly needed.

There still remains the possibility that for some Λ^j one can define a new symmetric traceless and conserved energy momentum tensor. When $\Lambda^i = \nabla^i\phi$ the freedom of performing coordinate transformations amounts to introducing a background dilaton field. The one loop contribution of a dilaton to the β-function is indeed of the form $\nabla^i\Lambda^j$ with $\Lambda^i = \nabla^i\phi$. Therefore we expect that in this case the modified energy momentum tensor is the same as for the theory with the dilaton ϕ as a background field.

In this paper we have also constructed string theories on compact manifold with torsion in the light cone gauge. In order for the corresponding sigma models to have $N = 2$ supersymmetry we had to restrict ourselves to hermitian manifolds which admit covariantly constant complex structure. Furthermore, in order to have vanishing β-function torsion must parallelize the Ricci tensor.

The starting point of our construction of the light cone string theory in curved space with torsion was based on the Neveu-Schwarz-Ramond type of string theory because in this string theory the fermions are vectors of $0(d-2)$. In ten dimension we used this theory as a guideline to construct a string theory of the Green-Schwarz type in the light-cone gauge. Using the $SU(3)$ holonomy of the compact space we were able to show the equivalence of the two theories. This analysis was done in the presence of torsion.

The restriction to an holonomy group $H \subseteq SU(3)$ is crucial to our approach. It restricts the fermionic interaction term in the light cone Lagrangian to a four-fermion coupling. In the covariant approach it gives rise to fermionic terms only up to eight-fermion couplings [27].

We thank M. Peskin for bringing to our attention the revised version of Ref. 5 where one can find arguments that the torsion must vanish on the compact manifold. The construction of the Green-Schwarz light cone action in curved space with $SU(3)$ holonomy is discussed also in recent papers by Fradkin and Tseytlin [28].

APPENDIX

In this appendix we discuss the relationship between the supersymmetry transformation of the covariant Green-Schwarz action and the light cone action. In flat background the Green-Schwarz superstring is invariant under the following supersymmetry transformation [23]

$$\delta_\eta S^A = \eta^A \qquad A = 1,2 \tag{A.1}$$

$$\delta_\eta X^\mu = i\overline{\eta}^A \gamma^\mu S^A \qquad A = 1,2 \tag{A.2}$$

In addition the green-Schwarz superstring action is invariant under a local fermionic symmetry [23]

$$\delta_\kappa S^A = 2i\gamma \cdot \pi_\alpha \kappa^{A\alpha} = 2i\gamma_\mu(\partial_\alpha X^\mu - i\overline{S}^B \gamma^\mu \partial_\alpha S^B)\kappa^{A\alpha} \tag{A.3}$$

$$\delta_\kappa X^\mu = i\overline{S}^A \gamma^\mu \delta_\kappa S^A \ . \tag{A.4}$$

This fermionic symmetry allows one to eliminate the unphysical fermionic degrees of freedom. In the light cone the fermions satisfy the condition $\gamma^+ S = 0$. From the transformation (A.2) and (A.4) it immediately follows that

$$\delta_\kappa X^+ = \delta_\eta X^+ = 0 \tag{A.4}$$

when $\gamma^+ S = 0$.

Unlike S, η and κ need not obey the light cone condition. We show below that if one imposes this condition on $\eta(\kappa)$ we recover the $\delta(\epsilon)$ supersymmetry of the light cone action. When η satisfies the light cone condition $\gamma^+\eta = 0$ Eqs. (A.1) and (A.2) take the simple form

$$\delta_\eta S^A = \eta^A \tag{A.6}$$

$$\delta_\eta X^i = 0\ . \tag{A.7}$$

This is precisely the δ-supersymmetry of the light cone action. The ϵ supersymmetry of the light cone action arises when one imposes the condition $\gamma^+\kappa = 0$. In this case the transformation of Eqs. (A1-A4) have the form

$$\delta_\eta S^A = \eta^A \tag{A.8}$$

$$\delta_\eta X^i = i\overline{\eta}^A \gamma^i S^A \tag{A.9}$$

$$\delta_\kappa X^i = 0 \tag{A.10}$$

$$\delta_\kappa S^A = 2i\gamma_+ \rho^+ \kappa^{\tau A} + 2i\gamma_i \partial_\alpha X^i \kappa^{\alpha A} \tag{A.11}$$

Adding Eqs. (A.8)-(A.11) we find

$$\delta_\epsilon X^i = (\delta_\eta + \delta_\kappa) X^i = (p^+)^{-1/2} \overline{\epsilon} \gamma^i S$$

$$\delta_\epsilon S = (\delta_\eta + \delta_\kappa) S = 2i(p^+)^{1/2} \gamma_- \gamma_i (\rho \cdot \partial X^i) \epsilon$$

where $\kappa^\alpha = -\frac{1}{2}(p^+)^{-1/2}\gamma_- \rho^\alpha \epsilon$ and $\eta = i(p^+)^{-1/2}\gamma_+\gamma_-\rho^0\epsilon$. Thus the ϵ supersymmetry of the light cone theory appears as a linear combination of η and κ transformations.

We expect that the above consideration can be repeated in curved space. However, in this case δ must be covariantly constant. Therefore η must also be covariantly constant. From this it immediately follows that ϵ in the light cone must be covariantly constant. This agrees with our considerations of Section 3.

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