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ON THE APPLICATION OF CONFORMAL SYMMETRY TO QUANTUM FIELD THEORY*

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ABSTRACT

To leading order in $\alpha_s(Q^2)$, conformal symmetry specifies the eigensolutions of the evolution equation for meson distribution amplitudes, the wavefunctions which control large-momentum-transfer exclusive mesonic processes in QCD. We find that at next to leading order, the eigensolutions in various field theories depend on the regularization scheme, even for zero β-function. This is contrary to the expectations of conformal symmetry.

Conformal symmetry expresses the extended invariances of the Lagrangian of a renormalizable Lorentz-invariant theory which has no intrinsic length scale [1]. Classical relativistic field theories which are scale invariant and have a renormalizable renormalizable Lagrangian are also conformal symmetric; i.e., invariant under the conformal group, which consists of the translations, boosts and rotations of the Poincaré group, together with dilatations ($x^\mu \to \lambda x^\mu$) and conformal transformations; i.e., $inversion(x^\mu \to -x^\mu/x^2) \times translation \times inversion.$ Scale invariance, and therefore conformal symmetry, is destroyed in quantum field theory by masses, and by the renormalization procedure, which inevitably introduces a renormalization scale. In a field theory with zero β-function (e.g. QCD to one loop-order with $N_f = \frac{11}{2} N_c$), the coupling constant cannot introduce any scale dependence, so one expects conformal symmetry (with possibly anomalous dimensions for the fields) to be valid at short distances where mass effects are negligible. A general, all orders, proof that conformal symmetry is satisfied asymptotically in renormalizable field theories with zero β-function has in fact been given by Parisi [2]. This result, however, may only be true for specific ultraviolet regulators(see below).

In this letter we consider the application of conformal symmetry to the operator product expansion of two fields at short distances,

$$\psi(\tfrac{z}{2})\,\overline{\psi}(-\tfrac{z}{2}) \sim \sum_{n}^{\infty} \widetilde{C}_n(z^2 - i\epsilon z_0) \sum_{m=n}^{\infty} \Gamma^{(i)}_\alpha z_{\alpha_1} \cdots z_{\alpha_m}\, 0^{(n)\alpha_1 \cdots \alpha_m \alpha}(0) \qquad (1)$$

where $i = 1, 2$ with

$$\Gamma^{(i)}_\alpha = \begin{cases} \gamma_\alpha & i = 1 \\ \gamma_\alpha \gamma_5 & i = 2 \end{cases} \qquad (2)$$

$$0^{(n)\alpha_1\cdots\alpha_m\alpha}_{(i)}(0) = \sum_{k=0}^{n} d_{mnk}\partial^{\alpha_{k+1}}\cdots\partial^{\alpha_m}\overline{\psi}(0)\,\Gamma^{\alpha}_{(i)}\overleftrightarrow{D}^{\alpha_1}\cdots\overleftrightarrow{D}^{\alpha_k}\psi(0)\ . \qquad (3)$$

Here D is the covariant derivative and the $\widetilde{C}_n(x^2 - i\epsilon z_0)$ are singular functions of well-defined dimension (powers of logarithms in QCD) [3,4].

The diagonal matrix elements of this expansion control the scaling-violations of deep inelastic structure functions. The off-diagonal matrix element between the vacuum and a meson state defines the distribution amplitude [5]:

$$\phi(x_i, Q) = \int \frac{dz^-}{2\pi}\, e^{i(x_1-x_2)z^-/2} \left\langle 0 \left| \overline{\psi}(-z)\, \frac{\gamma^+\gamma_5}{2\sqrt{2}}\, \psi(z) \right| \pi \right\rangle^{(Q)} \Bigg|_{z^+=z_\perp=0} \qquad (4)$$

in $A^+ = 0$ gauge. [In other gauges there is a path-ordered factor $\exp(ig\int_{-1}^{1} ds\, A^+(zs)z^-/2)$ between the $\overline{\psi}$ and ψ, making ϕ gauge invariant to leading twist.] The pion momentum is chosen as $p_\perp = 0, p^+ = 1$. The distribution amplitude contains all of the non-perturbative input which enters large momentum transfer exclusive process amplitudes such as $F_\pi(q^2)$, $\mathcal{M}(\gamma\gamma \to \pi^+\pi^-)$, etc., but it is itself process-independent.

Generally exclusive amplitudes involving large momentum transfer can be factored into a convolution of distribution amplitudes $\phi(x_i, Q)$, one for each hadron, with a hard-scattering amplitude T_H. The pion's electromagnetic form factor, for example, can be written as [5]

$$Q^2 F_\pi(Q) = \int_0^1 [dx] \int_0^1 [dy] \phi^*(x_i, Q)\, T_H(x_i, y_i, Q)\, \phi(y_i, Q) \left\{ 1 + 0\left(\frac{1}{Q}\right) \right\} \qquad (5)$$

where $[dy] = dy_1 dy_2 \delta(1 - \sum_i y_i)$ and $Q^2 = -q^2$ is large. Here $\phi(y_i, Q)$ is the probability amplitude for finding the valence $q\overline{q}$ Fock state in the initial pion, with constituents carrying longitudinal momentum $y_1 p_\pi$ and $y_2 p_\pi$, respectively; T_H is the amplitude for scattering the $q\overline{q}$ state from the initial to the final direction, and ϕ^* is the amplitude for the final-state $q\overline{q}$ to fuse back into a pion.

The matrix element in eq. (4) contains an ultraviolet divergence coming from the light-cone singularity at $z^2 = 0$. This divergence is regulated by introducing a momentum cut-off or other renormalization scale equal to Q. Any regulator that is both Lorentz invariant and gauge invariant can be used, provided a consistent scheme is used when computing both the perturbative hard-scattering amplitude T_H and the distribution amplitudes.

Using the expansion (3), $\phi_\pi(x, Q)$ at large Q^2 has the form[3]

$$\phi(x_i, Q) = x_1 x_2 \sum_{n=0}^{\infty} P_n(x_1 - x_2, \alpha_s(Q)) \widetilde{M}_n(Q) \tag{6}$$

where

$$\widetilde{M}_n(Q) = \widetilde{M}_n(Q_0) \exp \left[- \int_{Q_0}^{Q} \frac{d\widetilde{Q}}{\widetilde{Q}} \, \gamma^{(n)}(\alpha_s(\widetilde{Q})) \right] . \tag{7}$$

The functions P_n satisfy

$$\begin{aligned} Q^2 \, \frac{\partial}{\partial Q^2} \, x_1 x_2 P_n(x_1 - x_2, \alpha_s(Q)) &= \tfrac{1}{2} \gamma^{(n)}(\alpha_s) \, x_1 x_2 P_n(x_1 - x_2, \alpha_s) \\ &\quad + \int [dy] \, V(x_i, y_i, \alpha_s) \, P_n(y_1 - y_2, \alpha_s) \, . \end{aligned} \tag{8}$$

If $\beta = 0$ the left-hand side of eq. (8) is zero. The kernel V can be defined systematically order by order in perturbation theory:

$$V(x_i, y_i, \alpha_s(Q)) = \frac{\alpha_s(Q)}{4\pi} \, V_1(x_i, y_i) + \left(\frac{\alpha_s(Q)}{4\pi} \right)^2 V_2(x_i, y_i) + \cdots \tag{9}$$

Clearly $\phi(x, Q)$ is only logarithmically dependent on Q; the main Q-dependence of an exclusive process is due to T_H. A detailed procedure for computing V is given in ref. [3]. The anomalous dimensions $\gamma^{(n)}(\alpha_s)$ for the operators $O^{(n)}_{(i)}$ have already been determined through two loops in the analysis of moments in deep inelastic scattering. Thus the diagonal matrix elements of V_2 are known.

If conformal symmetry is applicable, then the off-diagonal matrix elements of V and the P_n are kinematically determined by the anomalous dimensions $\gamma^{(n)}(\alpha_s)$, in a way which is analogous to the partial wave expansion for the rotation group [4]. The general result for operators $O^{(n)}$ bilinear in spin zero fields in scalar field theory is

$$P_n(x) \propto \frac{1}{(1-x^2)} \frac{d^n}{dx^n} (1-x^2)^{[n+(d-1)+\frac{1}{2}\gamma_n(\alpha_s)]} \tag{10}$$

where d is the canonical dimension of ϕ ($d = 1$ in 4-dimensions, $d = 2$ in six dimensions). For spin $\frac{1}{2}$ fields, with $O^{(n)}$ as defined in eq. (3) conformal symmetry predicts

$$P_n(x) \propto \frac{1}{(1-x^2)} \frac{d^n}{dx^n} (1-x^2)^{[n+(d-\frac{1}{2})+\frac{1}{2}\gamma_n]} \tag{11}$$

where d is the canonical dimension of ψ ($d = \frac{3}{2}$ in 4-dimensions, $d = \frac{1}{2}$ in 2 dimensions). The results are true in any space-time dimension. Note that for the ϕ^4 interaction in 4-dimensions and the $(\overline{\psi}\psi)^2$ interaction in 2-dimensions the potential V_1 is a contact potential with measure $(1-x^2)^0$, thus yielding $P_n(x) = \overline{P}_n(x) =$ Legendre polynomials for leading order, in agreement with eqs. (10) and (11). (Actually only $n = 0$ appears in the potential.) In the case of ϕ^3 in 6-dimensions and gauge theory in 4-dimensions, the leading order polynomials are the $C_n^{3/2}$, as expected.

If conformal symmetry is applicable to a renormalizable theory with $\beta = 0$, then the first order corrections to the $P_n(x)$ must be regulator- and schemeindependent, since the anomalous dimensions do not depend on the ultraviolet regulator to this order. However, by explicit two-loop calculations we find that this prediction is not correct.

The analysis is particularly clear in ϕ^3 theory in six dimensions. In $[\phi^3]_6$ we find that the conformal symmetry prediction is correct using dimensional regulation but not in Pauli-Villars regularization. By definition, the ultraviolet divergence in the distribution amplitude $\phi(x,Q)$ is removed in Pauli-Villars regularization by subtracting diagrams with the gluon mass set equal to Q. The distribution amplitudes in this scheme and dimensional regularization can be related to each other through a correction to the evolution kernel beyond leading order. In ref. [3] we give a complete calculation of the distribution amplitude and the evolution kernel through two loops for $[\phi^3]_6$. We keep only the crossed-ladder and ladder contributions, so that the distribution amplitude satisfies the Callan-Symansik equation for $\beta = 0$. We then find that the functions $P_n(x_1 - x_2, \alpha_s)$, the eigensolutions of the evolution equation for the distribution amplitude, are exactly those predicted by conformal symmetry (eq. (10), with $d = 2$), but that this result holds only for dimensional regularization, *not* Pauli-Villars.

The origin of the difference between the regulators is that in dimensional regularization [$6-2\epsilon$ dimensions] the coupling constant acquires non-zero dimension. Thus the scale invariance of the theory is destroyed, and the β-function is non-zero even in leading order:

$$\frac{d}{d\ell n\mu}\,\alpha = \beta(\alpha) = -\epsilon\alpha + \cdots \neq 0\ . \tag{12}$$

Since the two-loop ladder graphs contribute at order $1/\epsilon^2$, there is a surviving $1/\epsilon$ contribution to the kernel V at order α^2 from $\beta \neq 0$. Including this contribution, one retains conformal symmetry to this order using dimensional regularization, whereas it is broken using the Pauli-Villars regulator. This effect reflects the special sensitivity to the infrared region in the two-loop calculation.

We find that also in gauge theory conformal symmetry cannot be simultaneously true in both Pauli-Villars and dimensional regularization. In fact explicit calculations [6] of the second order evolution kernel in gauge theories (Abelian QED and $SU(N_c)$ QCD) using dimensional regularization and $\beta_0 = 0$ ($N_F = (11/2)\ N_c$) do not agree with the conformal symmetry prediction. The results have been checked in both light-cone and Feynman gauges. This conflict is unresolved, and hints at an even subtler breakdown of conformal symmetry in gauge theory.

If the source of this breakdown can be identified, then conformal symmetry could still be useful as a guide to the higher order corrections to the distribution amplitude. More important, this unexpected breakdown points to new effects which control the short distance structure of gauge theory, and gives caution to the formal use of conformal symmetry results.

The proof of conformal symmetry for renormalizable field theories with $\beta = 0$ given by Parisi [2] depends on the existence of subtractions at zero four momentum; in general this subtraction procedure may not commute with the zero mass limit of the theory. In addition, the right-hand side of the Ward identity for conformal symmetry is set to zero in the limit $\beta \Rightarrow 0$. However, we notice this right-hand-side contribution is more singular in the infrared massless limit than the corresponding terms in the Ward identity for scale transformations. (Compare the r.h.s. of eqs. (2) and (4) of ref. [2].) These effects could be the origin of the fundamental inconsistencies with conformal symmetry which we have discussed here.

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