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FERMIONIC MASSES IN KALUZA-KLEIN RELATIVITY*

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ABSTRACT

Kaluza-Klein relativity is an exotic version of Kaluza-Klein theory in which the four dimensional space-times are submanifolds embedded in the high dimensional space. Dimensional reduction is obtained by reversing the embedding procedure. That is, given a space with dimension $D > 4$, find the largest dimension and geometries of its submanifolds obeying a dynamical principle. Using gravitational Casimir-like effect, these submanifolds would collapse into a single one. However this collapse is prevented by the presence of large Dirac fermionic masses associated with the embedding symmetry group. These masses reduce to the usual (small) masses in the limit of vanishing gravitation.

1. Kaluza-Klein Relativity

It has been suggested on several occasions and under different contexts that a high dimensional theory can be derived from the assumption that any four dimensional space-time is a subspace locally and isometrically embedded into another space. The extra dimensions required by the embedding would account for the gauge degrees of freedom.[1–3] Such hypothesis has been investigated to some extent, particularly in the case where the embedding space is a $4+n$ dimensional flat space M_{4+n}. The result, called Kaluza-Klein (special) relativity resembles but is distinct in many respects form the standard Kaluza-Klein theory.[4]

The use of a flat space greatly simplifies the embedding problem due to the vast number of known results. The maximum dimensionality required to differentially embed a space-time is 14 but this limit reduces to 10 when the embedding is given by analytic functions. These embeddings are determined up to a isometry $SO(p,q)$ of M_{4+n} (called the embedding symmetry). The purpose of this note is to investigate the mass spectrum associated with this symmetry and its influence on the problem of dimensional reduction. Any chirality related questions are left to a subsequent paper.

While the embedding procedure enables us to derive extra dimensions out of four, the following dimensional reduction scheme works like an inverse embedding procedure: we assume that the world started with a $D>4$ dimensional space with metric signature (p,q) and look for the physical conditions leading to the formation of submanifolds with the highest possible dimension d.

In the case of analytical embedding d relates to D as $d(d+1)=2D$ and in the case of differentiable embeddings the relation is $d(d+3)=2D$.[4] We may set $D=d+n$ and use D or n as phenomenological inputs. For example we may take $D = 10$ from superstrings or $n = 7$ from the standard $SU(3) \times SU(2) \times U(1)$ model to obtain analytically embedded manifolds with $d = 4$. Those interested in a $SO(10)$ GUT may want to consider differentiable embeddings with $n = 10$, again obtaining $d = 4$.

Once obtained the dimensionality of these manifolds, denoted by $\overline{V}_4$, we may ask about their geometries. In an isometric embedding the metric of $\overline{V}_4$ is induced by that of M_D. This is sufficient to establish a set of constraints on $\overline{V}_4$, the Gauss-Codazzi-Ricci equations, involving three sets of tensor fields: the metric $\bar{g}_{ij}$, the second quadratic form $\bar{b}_{ij}$, and the torsion vector $\overline{A}_i$ of $\overline{V}_4$. However, after eliminating all interdependencies between these equations resulting from the Bianchi identities, we end up with less equations than variables so that an additional dynamical principle is required. Following the Kaluza-Klein idea, that dynamical principle uses the Einstein-Hilbert action imposed on the high dimensional space itself, rather than directly on the submanifolds (as it is done in string theory). Thus, the flat space M_D is regarded as a particular solution of the D dimensional Einstein's equations resulting from that principle, representing the ground state of the theory.

Setting $D = 4+n$ it follows from that dynamical principle that each submanifold $\overline{V}_4$ generate a n-parameter family of manifolds orthogonal to the n extra dimensions, each member of which being a solution of Einstein-Yang-Mills equations with gauge group $SO(p-3, q-1)$, called a space-time section V_4.[4] Consequently, the physical phenomena in M_D associated with that dynamical principle is not restricted to the single 4-dimensional manifold $\overline{V}_4$ but it takes place in a region R_D of M_D covered by the family of space-time sections. A D-world line of a particle in R_D would consist of a continuous sequence of points located at each space-time section. This picture is similar to the usual decomposition of space-time in local time and space-like sections (as in the Arnowitt-Deser-Misner formulation of General Relativity). The difference is that here local time is replaced by the n extra dimensions and the space-like sections are replaced by the space-time sections.

As it happens, the region R_D does not extend to infinity along the n-extra dimensions but it is bounded by $\overline{V}_4$ and by another 4-manifold $\overline{V}'_4$, the locus of curvature centers of $\overline{V}_4$, where the metrics of the space-time sections become singular. Therefore the physical region of M_D relative to the said dynamical principle and to $\overline{V}_4$ is a $4+n$ dimensional "sheet" bounded by two 4-dimensional manifolds and with a local thickness $a(\rho)$ function of the curvature radius ρ of $\overline{V}_4$.

The boundary $\overline{V}_4$ is a particular member of the family of space-time sections corresponding to the vanishing of the n parameters. If we apply this condition to the Einstein-Yang-Mills equations then the gauge term disappears so that $\overline{V}_4$ itself is not a solution of those equations but it is a solution of a compatible set of equations $\overline{G}_{ij} = t_{ij}(\bar{b})$ where $t_{ij}(\bar{b})$ depends only on the second quadratic form $\bar{b}_{ij}$ of $\overline{V}_4$. If we assume that $\overline{V}_4$ is a solution of Einstein's equations $\overline{G}_{ij} = t_{ij}$ (source) and take it to be the physical space for low energy physics (say as compared to 10^{19} GeVs), then $\bar{b}_{ij}$ may be related to the source. Furthermore an observer in $\overline{V}_4$ may agree that R_D is a physical space if points located at different space-time sections are identified with a single point in $\overline{V}_4$. Then, this observer may look at the parameter space as an internal space which will appear to him as a bounded and closed space. Therefore the compactification of this space is an extrinsic rather than intrinsic property of itself.

The only condition imposed by the embedding on $a(\rho)$ is that it should be smaller than the smallest curvature radius of $\overline{V}_4$. Therefore $a(\rho)$ should be small under strong gravitational field but it appears to be impossible to derive a quantitative description of $a(\rho)$ by geometric means only. However it has been suggested that quantum effects of strong gravitational fields in M_D produce a Casimir-like effect acting on the boundaries $\overline{V}_4$ and $\overline{V}'_4$, making $a(\rho)$ shrink to zero.[5] In such a case we can say roughly that $a(\rho)$ vary with negative powers of ρ.

If quantum effects of gravitation produce $a(\rho) \to 0$ we would obtain a full dimensional reduction of R_D to $\overline{V}_4$, a process which could be called space-time morphogenesis. In reality this may never happen because at some stage fermionic matter must appear (so far we have used only bosonic fields). It has been shown that the presence of large fermionic masses near the beginning of the universe may intervene in the right proportion to halt the collapse of $a(\rho)$ to zero.[6]

2. Dirac's Equation in M_{4+n}

The generalization of Dirac's equation for the de Sitter space-time was given by Dirac in 1935:

$$(\gamma^\alpha \partial_\alpha - M)\,\psi = 0 \qquad \alpha = 1, \ldots, 5 \tag{1}$$

where the de Sitter space-time was embedded in a 5-dimensional space M_5 with metric $\eta_{\alpha\beta}$ with signature $(p,q) = (4,1)$ (or (3,2) for the anti de Sitter space-time).[7] In this case the 4×4 matrices are representations of the Clifford algebra $E^{(\alpha}E^{\beta)} = \eta^{\alpha\beta}$ in M_5. The mass operator is proportional to the second order Casimir operator of the embedding symmetry $SU(4,1)$ (or $SO(3,2)$):

$$M^2 = \frac{1}{R^2}\, L_{\alpha\beta} L^{\alpha\beta} \qquad \alpha, \beta = 1, \ldots, 5 \tag{2}$$

where $L_{\alpha\beta}$ denote the Lie algebra generator of that group and R is the curvature radius of the universe. This generalization is a consequence of the well-known property that the de Sitter group contracts into the Poincarè group as the contracting factor $1/R$ tend to zero (which means the flat limit of space-time). Simultaneously the mass operator M transforms into the Poincarè mass operator in that limit. Such property has led to a great deal of interest in the de Sitter group as a cosmological replacement for the Poincarè group. The important consequence of this replacement if it is taken seriously is that gravitation, even at the extreme weak level of a cosmological model, would play a role in the particle structure, notably in what concern their masses and mass splittings.[8,9] As it has been noted[8] this could mean that local and strong gravitational fields such as the one existing moments after the Big Bang cannot be neglected in the analysis of fermionic masses.

The contraction property of the de Sitter group can be generalized to the case of a local gravitational field in a straightforward manner. Instead of M_5 we take the local embedding space M_{4+n} with a group of isometries $SO(p,q)$. Instead of R we take the n local curvature radii ρ^A corresponding to each extra dimension so that $\rho^2 = \Sigma\epsilon_A\rho^{A^2}$, $A = 5\ldots 4+n$, $\epsilon_A = \pm 1$ depending on the signature (p,q). Then using $1/\rho^A$ as contracting factors, as $\rho^A \to \infty$ we obtain the contraction[4,9]

$$SO(p,q) \to P_4 \times SO(p-3,q-1) \ .$$

Therefore it is possible to generalize (2) to a local gravitational field using the $SO(p,q)$ covariance. Constructing the Clifford algebra over M_{4+n} and finding its representations we obtain a set of matrices Γ^α and $2^{[4+n]/2}$ component spinors.

Then the equation corresponding to (2), covariant under $SO(p,q)$ is

$$(\Gamma^\alpha \partial_\alpha - M(\rho))\,\psi = 0 \qquad \alpha = 1,\ldots,4+n\ , \tag{3}$$

where now $M(\rho)$ is proportional to the second order Casimir operator of that group:

$$M^2(\rho) = K^2(\rho)\ \frac{L_{\alpha\beta}L^{\alpha\beta}}{\rho^2}\ , \quad \alpha,\beta = 1,\ldots,4+n \tag{4}$$

where $L_{\alpha\beta}$ is a Lie algebra generator of $SO(p,q)$. The factor $K(\rho)$ was included to eliminate mass divergence at extreme gravitational field. Therefore we require that $\lim_{\rho\to 0} K(\rho)/\rho = 0$ and to avoid interference with the group contraction process we also require that $\lim_{\rho\to\infty} K(\rho) = 1$. Again as $SO(p,q)$ contracts in the flat limit of the space-time $\overline{V}_4$, $M(\rho)$ tend to the Poincarè mass operator.

For an observer sitting in $\overline{V}_4$ Eq. (3) appears as if projected (that is restricted) to $\overline{V}_4$:

$$(\Gamma^i\partial_i + \Gamma^A\partial_A - M)\,\psi|_{\overline{V}_4} = 0\ , \quad i = 1,\ldots,4\ , \quad A = 5,\ldots,4+n\ . \tag{5}$$

where as in conventional Kaluza-Klein theory, $\Gamma^A\partial_A$ is interpreted by that observer as an internal mass operator. Therefore the total mass operator is $\Gamma^A\partial_A - M$. Remembering that for that observer the extra coordinates appear as periodic, ψ can be harmonically expanded in these coordinates. If the observer does not possess a high energy probe he can detect only the zero mode $\psi^{(0)}$ of that expansion so that at low energies (5) reduces to

$$(\Gamma^i\partial_i - M)\,\psi^{(0)} = 0\ , \quad i = 1,\ldots,4\ .$$

In the flat limit of $\overline{V}_4$ this equation reduces to the usual Dirac equation (in a high dimensional spinor representation) with the usual mass spectrum.

It is also possible that the observer while still in $\overline{V}_4$ can use a high energy probe so that a dependence of ψ on the extra coordinates is detected. In this case the two mass terms in (5) contribute to the total mass (nonzero modes being considered). The internal mass term $\gamma^A \partial_A \psi|_{V_4}$ is the same as the conventional Kaluza-Klein theory, proportional to $1/a(\rho)$. If we take that at the Big Bang $a(\rho) \to \infty$ this term does not contribute to mass at that moment. However soon after that, quantum gravitation imposes that $a(\rho)$ becomes small to the order of Planck's length κ_0, so that this term contributes to large fermionic masses. The second contribution to mass given by M is zero at the Big Bang itself $\rho^A \to 0$ but again, as $a(\rho) \to \kappa_0$ (4) indicates that this term contributes to large masses. It may sound strange to have two mass terms contributing to large masses at the beginning of the universe. In fact we have only one mass operator and the division in two terms is only a particular view of the four dimensional observer. Notice that in (4) $L_{\alpha\beta}L^{\alpha\beta}$ also contain internal operators.

As the universe expands and gravitation becomes weak ($\rho^A \to \infty$) the contribution from $\Gamma^A \partial_A$ vanishes (because only $\psi^{(0)}$ appears) while the contribution from M gives masses approaching the observed values.

In conclusion we can say that considerations on the symmetry $SO(p,q)$ and the corresponding covariant Dirac equation makes it possible to halt the gravitational Casimir effect soon after the beginning of the universe and still end up with the usual fermionic masses when the universe expands.

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REFERENCES

1. P. G. Bergmann, Unitary Field Theories, article in Unified Field Theories of More than 4-Dimensions. V. De Sabbata and E. Schmutzer (Editors), World Scientific, Singapore (1983), page 1. Also see H. Stephani, Embedding, ibid, page 299.
2. T. Regge and C. Teitelboin, General Relativity a la String. Proc. II Marcel Grossman Meeting, Trieste (1975). See also, S. Deser, F.A.E. Pirani, D. C. Robinson, Phys. Rev. D14, 3301 (1976).
3. B. Holdom, The Cosmological Constant and the Embedded Universe, Stanford preprint ITP-744 (1983).
4. M. D. Maia, Phys. Rev. D31, 262 and 268 (1985). See also "On Kaluza-Klein Relativity", University of Washington preprint 40048-26 P4 (1985).
5. T. Appelquist, A. Chodos and E. Myers, Phys. Lett. 127B, 51 (1983).
6. M. Rubin and B. D. Roth, Phys. Lett. 127B, 55 (1983).
7. P.A.M. Dirac, Ann. Math. 36, 657 (1935).
8. P. Roman, Nuovo Cimento A45, 268 (1866). See also article and comments in Proceedings of the Conference on Non Compact Groups in Particle Physics, Y. Cho (Editor), Benjamin, NY (1966), page 89.
9. M. D. Maia, J. Math. Phys. 25, 2090 (1984).