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GOLDSTONE REALIZATION OF LORENTZ-SYMMETRY AND SUPERSYMMETRY AT FINITE TEMPERATURE*

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ABSTRACT

The Ward-Takahashi identities of Lorentz symmetry are examined at finite temperature. It is shown that the Lorentz symmetry is broken with the Goldstone realization. The analogy with finite-temperature supersymmetry-breaking in an R-invariant model is explained.

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The supersymmetry has been known to be broken at finite temperature for some time.[1,2] Recently,[3,4] it was found that the supersymmetry is realized in the Goldstone fashion. Namely, at finite temperature the supersymmetry is broken so that the Ward-Takahashi identities are saturated by zero-momentum singularities of the field operators, which couple to the supercurrent. The purpose of this letter is to illustrate that an analogous phenomena occurs for Lorentz symmetry in a usual field theory, and thereby advance the understanding of the nature of the symmetry breaking at finite temperature.

The formalism to treat field theories at finite temperature is well established. In particular, the real-time formalism (Thermo Field Dynamics)[6] gives a sound basis for dealing with real-time Green's functions which are physical correlation functions in real space. In ref. 3, the real-time fermion propagator was calculated to the one-loop order in a (R-noninvariant) supersymmetric model, and it was shown to acquire a zero-momentum singularity, implying the Goldstone realization of supersymmetry. A similar calculation[7] in the $1/N$-expansion yielded the same conclusion. These works motivated the systematic investigation of the Ward-Takahashi identities for the real-time Green's function for the supersymmetric theories at the tree and one-loop levels.[4,5] Ref. 4 has also investigated the R-invariant model, in which the above phenomena does not occur. We have found that both at the tree and one-loop levels at finite temperature there always are some field-operators (even for the R-invariant model) that couple to the supercharge at zero momentum (even for a massive theory), thus confirming the Goldstone realization of the supersymmetry. Correspondingly, the Green's functions of those operators have been demonstrated to have singularities at zero momentum.†

In the following, we shall see that essentially the same phenomena is found for the Lorentz symmetry in field theories. For the purpose of the illustration, we take a real scalar field ϕ of mass m and a cubic coupling $\lambda\phi^3$ and observe

† These singularity were called the "Goldstone mode"[4] or the "thermal superpair".[5]

how the Ward-Takahashi identities of Lorentz symmetry are satisfied at finite temperature. (The ϕ^4 coupling necessary for the renormalizability and the stability of the vacuum is irrelevant for our discussion.) Other cases can be worked out similarly. The relevant Ward-Takahashi identities are obtained by taking the functional derivatives of the following with respect to the source $J(x)$ and then setting $J(x) \equiv 0$,*

$$\int d^4z\, \partial^\rho \langle T\{\mathcal{M}_{\rho\mu\nu}(z)\}\rangle_J = \int d^4z J(z)(z_\mu\partial_\nu - z_\nu\partial_\mu)\langle\phi(z)\rangle_J\,, \tag{1}$$

where $\mathcal{M}_{\rho\mu\nu}(z)$ is the usual angular-momentum current of the theory written in terms of the field ϕ. The single derivative (and J=0) yields the following;

$$\langle\int d^4z\, \partial^\rho \langle T\{\mathcal{M}_{\rho\mu\nu}(z)\phi(x)\}\rangle = -i(x_\mu\partial^x_\nu - x_\nu\partial^x_\mu)\langle\phi(x)\rangle\,. \tag{2}$$

This, however, is trivially satisfied (both sides being zero) at any order of perturbation theory because of the translational invariance of the system. From the second derivative, we obtain the following,

$$\langle\int d^4z\, \partial^\rho \langle T\{\mathcal{M}_{\rho\mu\nu}(z)\phi(x)\phi(y)\}\rangle = -i(X_\mu\partial^X_\nu - X_\nu\partial^X_\mu)\langle T\{\phi(x)\phi(y)\}\rangle\,, \tag{3}$$

where $X \equiv x - y$. Note that the first term is an integration of a total derivative and therefore vanishes unless there is a zero-momentum singularity that supplies a boundary term.

Let us first examine how the Ward-Takahashi identity (3) is satisfied at the tree level. The free propagator at temperature $1/\beta$ is

$$G_\beta(p) = \frac{i}{p^2 - m^2 + i\epsilon} - u_B(p)\delta(p^2 - m^2)\,, \tag{4}$$

$$u_B(p) \equiv \frac{2\pi}{e^{\beta|p\cdot n|} - 1}\,, \tag{5}$$

where n is the four-velocity of the system ($n^2 = 1$, usually chosen to be $(1, \vec{0})$).

* In TFD, corresponding to each field in the zero-temperature theory, there is a ghost-like partner field. However, in the order of perturbation used in this paper, the extra field is irrelevant.

In the momentum representation, the left hand side of (3) is

$$\lim_{q\to 0} q^\rho \left[\frac{\partial}{\partial q^\nu} K_{\rho\mu} - \frac{\partial}{\partial q^\mu} K_{\rho\nu}\right] G_\beta\left(p+\frac{q}{2}\right) G_\beta\left(p-\frac{q}{2}\right), \tag{6}$$

where

$$K_{\rho\mu} \equiv \delta_{\rho\mu}\left(p^2 - \frac{q^2}{4} - m^2\right) - 2\left(p_\rho p_\mu - \frac{1}{4} q_\rho q_\mu\right). \tag{7}$$

Using the property,

$$q^\rho K_{\rho\mu} = \left(p+\frac{q}{2}\right)_\mu \left[\left(p-\frac{q}{2}\right)^2 - m^2\right] - \left(p-\frac{q}{2}\right)_\mu \left[\left(p+\frac{q}{2}\right)^2 - m^2\right], \tag{8}$$

we find that (6) reduces to

$$iD_{\mu\nu}(p) G_\beta(p) = -i(p_\mu n_\nu - p_\nu n_\mu) \frac{du_B}{d(p\cdot n)} \delta(p^2 - m^2), \tag{9}$$

where

$$D_{\mu\nu}(p) \equiv p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}, \tag{10}$$

We find that (9) is exactly equal to the right hand side of (3).

At the one-loop level, we shall examine the truncated version of (3) for simplicity. The relevant Feynman diagrams are illustrated in figs. 1 and 2. For the right hand side (fig. 1), by using the tree level equality proved above, we obtain the following $q \to 0$ limit,

$$\begin{aligned} &-\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} D_{\mu\nu}(p+k) u_B(p+k) \delta((p+k)^2 - m^2) \\ &= -\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} (D_{\mu\nu}(p) + D_{\mu\nu}(k)) u_B(p+k) \delta((p+k)^2 - m^2). \end{aligned} \tag{11}$$

The $D_{\mu\nu}(k)$ term gives no contribution to (11). One can prove this by using

$$k_\mu \frac{i}{k^2 - m^2} = i \frac{\partial}{\partial k^\mu} \ln(k^2 - m^2), \tag{12}$$

and doing the partial integration to reduce $D_{\mu\nu}(k)$ to $\left[\frac{\partial}{\partial k_\mu}, \frac{\partial}{\partial k_\nu}\right] = 0$. The remaining $D_{\mu\nu}(p)$ term of (11) is equal to the contribution of the self-energy correction (fig. 2) on the right hand side. Thus we confirm that the Ward-Takahashi identity (3) is nontrivially satisfied to one-loop order.

From the fact that (9) and (11) are nonzero at finite temperature, we conclude that at finite temperature there is a zero-momentum singularity in the propagator of the bilinear operator $\phi\phi$. Otherwise, (3) should vanish since the left hand side is an integration of a total derivative.

The broken Ward-Takahashi identity (3) may be a little unfamiliar at first sight, since the breaking occurs only at finite temperature and also the order parameter is non-linear in the field operator. However, this situation is quite analogous to the R-invariant supersymmetric model (at finite temperature) discussed in ref. [4]. This model consists of two real (scalar and pseudoscalar) field (A, B), a Majorana field (ψ), and two auxiliary field (F, G). This theory can have an R-symmetry $(A+iB \to e^{i\alpha}(A+iB)$, $\psi \to e^{-i\alpha\gamma^5/2}\psi$, $F+iG \to e^{2i\alpha}(F+iG))$. The relevant Ward-Takahashi identities of supersymmetry, which corresponds to (2) and (3), are the following,

$$\langle \int d^4z\, \partial^\rho \langle T\{S_\mu(z)\bar{\psi}(x)\}\rangle = i\langle F(x)\rangle\,, \tag{13}$$

$$\langle \int d^4z\, \partial^\rho \langle T\{S_\mu(z)\psi(x)A(y)\}\rangle = i\langle T\{\psi(x)\bar{\psi}(y)\}\rangle - \partial\!\!\!/\langle T\{A(x)A(y)\}\rangle + ...\,, \tag{14}$$

where S_μ is the supercurrent and .... are the terms with auxiliary fields.

The supersymmetry is broken at finite temperature. At tree level, it has been found[4] that (14) is satisfied trivially for finite temperature (both sides are zero) while (13) is nontrivial. Similarly to the case examined above, this implies that the propagator of the bilinear operator ψA has a zero-momentum singularity. In ref. [4], we have confirmed this explicitly and saw that the residue goes to zero as $e^{-\beta m}$ for zero temperature limit. At higher orders of perturbation theory, the breaking manifests itself differently depending on the existence of the other symmetry that is not broken at finite temperature, in this case, R-invariance.

In a theory without this invariance, the Ward-Takahashi identity (13) become trivial at the one-loop level. Instead, (14) becomes nontrivial: The auxiliary field acquires a vacuum expectation value and the (single) fermion propagator acquires a zero-momentum singularity.[8] The latter is because the tree-level singularity of ψA couples to the single ψ-channel at the one-loop order. In the R-invariant model, $\langle F \rangle$ always vanishes; the R-invariance is not broken at finite temperature. The single fermion propagator does not acquire the zero-momentum singularity. Thus, the Ward-Takahashi identity (14) is satisfied nontrivially even at one-loop level.

What we observe for the Lorentz symmetry is exactly parallel to the supersymmetry breaking explained above. In both cases, the Ward-Takahashi identities ((3) and (13)) that have bilinear operators (the propagator) as the order parameter (right hand sides) are nontrivial at finite temperature. The Ward-Takahashi identities ((2) and (14)) with a single operator order parameter are trivial due to the additional symmetries (translational and R-invariance) that are good at finite temperature.

In both (3) and (13), the nonvanishing contribution is related to the existence of the parameters that specify the property of the heat bath. For Lorentz symmetry, it is the four-velocity n. For supersymmetry, we can imagine a Grassman parameter ϵ that specifies which fields follow the bose statistics and fermi statistics. (We can give the fermionic thermal distribution to $\psi + \epsilon A + ...$ and the bosonic thermal distribution to $A + \bar{\epsilon}\psi$ etc.) A thermal equilibrium has specific values of these parameters. The heat bath would have no physical meaning unless it distinguishes between the particles that are related by the Lorentz symmetry or supersymmetry.* In fact, in the above treatments, we have taken particular values for these parameters (for supersymmetry, $\epsilon = 0$). The choice of the parameters affect the Green's function through the boundary conditions.

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* In some literature (refs. 3-6 in ref. 4, and also ref. 8), the "unphysical heat bath" in which all particles obey the same bosonic statistics has been discussed. The relation of this unphysical statistics and the decoupling of the supermupltiplet is pointed out in ref. 9.

The current densities $\mathcal{M}_{\rho\mu\nu}$ and S_μ do not induce changes the parameters, n and ϵ. They concern with Green's function, which are linear responses of the system obtained in a given heat bath. In particular, the supercurrent can annihilate an (on-shell) fermion in the thermal equilibrium and create a boson of same momentum (and vice versa), which is allowed because of the supersymmetric spectrum (at zero temperature). This is the essential reason for the zero-momentum contribution.[4,5] The same argument applies for the present case of Lorentz symmetry. The current $\mathcal{M}_{\rho\mu\nu}$ takes a particle from the thermal equilibrium and replaces it with a particle with a momentum that is infinitesimally different from the initial momentum. This of course is possible since the original (zero temperature) spectrum is Lorentz-invariant.

It should be stressed that the situation for Lorentz symmetry and supersymmetry at finite temperature is exactly parallel to the usual spontaneous symmetry breaking: Imagine a elementary Goldstone model of a complex scalar ϕ. In order to specify a physical vacuum, we have to choose the phase of the background ϕ_{bg} to be a particular value. (There are unphysical choices like the superposition of all different phases or $\phi_{bg} = 0$.) This choice does not violate the symmetry explicitly, and in fact the Goldstone particle exists. The current density of a given symmetry always acts only on a fluctuation of the system and does not transform the vacuum expectation values of the relevant field operators (the order parameters). The way the Ward-Takahashi identities are satisfied for the usual Goldstone model is the same as that for Lorentz symmetry and supersymmetry. The major difference is that in case of the finite temperature theory, the notion of the "particle" is obscure. The particles are constantly created and annihilated to achieve the thermal equilibrium. Thus the existence of a singularity does not imply a existence of a particle. It simply means the existence of a long range correlation. For this reason, the usual terminology of "spontaneous breaking" and "Goldstone particle" does not illuminate the current situation. However, from the way the Ward-Takahashi identities are satisfied, it is appropriate to call them the Goldstone realization of the (Lorentz or super-) symmetry.

In conclusion, we find that there are singular zero-momentum contributions to the Ward-Takahashi identities for the Lorentz symmetry at finite temperature. The singularity is in the sector of the bilinear operator $\phi\phi$ even at the higher order of the perturbation theory because of the unbroken translational symmetry. This situation is exactly parallel to what has been previously found in the R-invariant supersymmetric theory: The symmetry is realized in a Goldstone fashion at finite temperature, but the zero-momentum singularity does not appear in a single field operator sector due to a conserved symmetry. This Goldstone realization is due to the property of the physical heat bath, in which the symmetry is broken, as is true for the usual spontaneous symmetry breaking in the vacuum.

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FIGURE CAPTIONS

1. The only Feynman diagram that contributes to the left hand side of (3). The small circle with cross represents the current term. The limit $q \to 0$ is taken at the end of the calculation. The external lines are actually truncated but shown here for clarity.

2. The self-energy correction that contributes to the right hand side of (3).

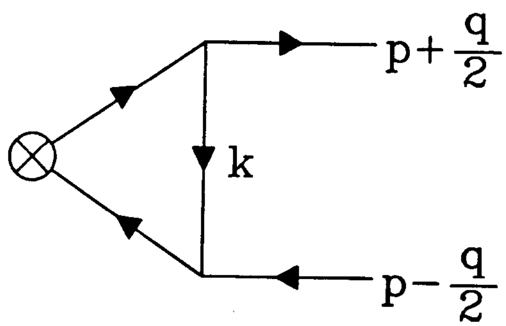

Fig.1

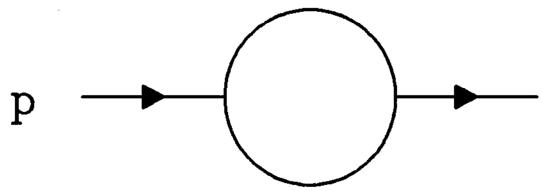

Fig.2