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Angular Momentum and Spin Within a Self-Consistent, Poincaré Invariant and Unitary Three-Particle Scattering Theory*

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ABSTRACT

The self-consistent, Poincaré invariant and unitary three-particle scattering theory developed in a previous paper is extended to include angular momentum conservation and individual particle spin. The treatment closely follows that of the scalar case, with the complete set of angular momentum states for three free particles developed by Wick used in place of scalar plane wave states.

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1. Introduction

In a previous paper[1] we presented a self-consistent, Poincaré invariant and unitary scattering theory for three distinguishable scalar particles of finite mass. The goal of this paper is to extend the treatment to particles of arbitrary spin and to include the effects of angular momentum conservation.

Two concepts crucial to the development of a relativistic three-body scattering theory are introduced in Ref. 1. The first is the use of velocity conservation in place of momentum conservation in order to separate Lorentz invariance from the off-shell continuation in energy. The second is the introduction of factors independent of intermediate state integrations into the relation between the two- and three-body off-shell variables. Both ideas, as well as the general operator form of the scattering theory, are used here without further comment. The only differences are in the definitions of particle states and operator matrix elements.

A complete orthonormal set of angular momentum states for three free particles is developed by Wick.[2] Single particle helicity states, from which the angular momentum states are constructed, are defined by the action of a Lorentz boost in the $\hat{z}$ direction onto a particle at rest, followed by a rotation. In the spin zero case, this is equivalent to LM(2.5). Throughout our discussion we adopt Wick's state definitions, normalization, phase conventions, and notation. For details, the reader is referred to Ref. 2.

Chapter 2 extends Wick's treatment to define another complete three-particle basis, which is used in Chapter 3 to develop the two- to three-body connection. The resulting angular momentum conserving integral equations are presented in Chapter 4. Chapter 5 relates the solutions of these equations to the physical probability amplitude. Chapter 6 summarizes the results.

2. Three-Body States

A proceedure similar to that used to obtain W(17) is followed in order to obtain the matrix elements between states in the plane wave basis and states in the three-body angular momentum basis. A Lorentz transformation $h(P)$, satisfying

$$h(P)\ P^0 = P\ ,$$

is applied to W(24).[3] Then the operator acting on $|q_1\,\nu_1, q_2\,\nu_2, q_3\,\lambda_3\rangle$ is

$$L = H(P)\ S\ .$$

The rotation s is specified by p_1, p_2, and p_3 through

$$s = h^{-1}(P)\ l$$

and

$$p_1 = l\,q_1$$

$$p_2 = l\,q_2$$

$$p_3 = l\,q_3\ ,$$

where p_1, p_2, and p_3 are restricted to obey

$$p_1 + p_2 + p_3 = P$$

$$p_1^2 = m_1^2 \qquad p_2^2 = m_2^2 \qquad p_3^2 = m_3^2$$

$$(p_1 + p_2)^2 = w^2 \ .$$

An integration variable change gives

$$\tfrac{1}{16} \left[pq/wW\right] \ \sin\theta \ d\theta \ dU_s$$

$$= \widetilde{d}p_1 \ \widetilde{d}p_2 \ \widetilde{d}p_3 \ \delta((p_1 + p_2)^2 - w^2) \ \delta^4(p_1 + p_2 + p_3 - P) \ .$$

With

$$L \left| q_1 \, \nu_1, q_2 \, \nu_2, q_3 \, \lambda_3 \right\rangle$$

$$= \sum_{\mu_1 \, \mu_2 \, \mu_3} U^{(1)}_{\mu_1 \nu_1}(p_1; l) \ U^{(2)}_{\mu_2 \nu_2}(p_2; l) \ U^{(3)}_{\mu_3 \lambda_3}(p_3; l) \ \left| p_1 \, \mu_1, p_2 \, \mu_2, p_3 \, \mu_3 \right\rangle$$

and

$$D^J_{\Lambda M}(s^{-1}) = U^{(J)}_{\Lambda M}(l^{-1}; P) \ ,$$

we find

$$\left\langle p_1\,\mu_1, p_2\,\mu_2, p_3\,\mu_3 \middle| P'J'M'; w'j'm'\lambda_1\,\lambda_2; \lambda_3 \right\rangle \tag{2.1}$$

$$= 4\,\left[wW/pq\right]^{\frac{1}{2}}\,\mathcal{N}_{J'}\,\mathcal{N}_{j'}\,e^{-i\pi s_2}\,\delta^4(P-P')\,\delta(w^2-w'^2)$$

$$\times\, d^{j'}_{m'\lambda'}(\theta)\sum_{\nu_1\nu_2} d^{s_1}_{\nu_1\lambda_1}(\beta_1)\,d^{s_2}_{\nu_2\lambda_2}(\beta_2)\,U^{(J')}_{\Lambda'M'}(l^{-1};P)$$

$$\times\, U^{(1)}_{\mu_1\nu_1}(p_1;l)\,U^{(2)}_{\mu_2\nu_2}(p_2;l)\,U^{(3)}_{\mu_3\lambda_3}(p_3;l)\ ,$$

where

$$\lambda' = \lambda_1 - \lambda_2$$

$$\Lambda' = m' - \lambda_3$$

$$\mathcal{N}_j = \left[(2j+1)/4\pi\right]^{\frac{1}{2}}\ ,$$

and l is specified by the condition that l^{-1} transform P to rest, $\mathbf{p}_3$ to a vector in the direction of the negative z-axis, and $\mathbf{p}_1$ to a vector in the $xz\,(x>0)$ half-plane. The angles θ, β_1, and β_2 are functions of W, w, and $(p_2+p_3)^2$, as indicated in W(Fig. 1).

In the three-body angular momentum basis completeness involves an integration over

$$d^4P\;d(w^2) = \left[W^3/u^0\right]\,\left[\xi(W,v^0)\right]^{-1}\,dW\;d^3u\;dv^0\ , \tag{2.2}$$

where

$$\xi(W, v^0) = \frac{W v^0 - w}{2W w^2}$$

$$w = \omega(W, v^0, m_3^2) \ .$$

In order to streamline the forthcoming equations, we adopt a matrix notaton. Underlined symbols represent elements of $(2s_1+1)(2s_2+1)(2s_3+1)$ dimensional square matrices. The particular elements under consideration will be obvious from the context. Thus

$$\left\langle p_1\, \mu_1, p_2\, \mu_2, p_3\, \mu_3 \middle| P' J' M'; w' j' m' \lambda_1\, \lambda_2; \lambda_3 \right\rangle \tag{2.3}$$

$$= \left[u^0/W^3\right] \left[v^{0^2}\left(v^{0^2} - 1\right)\right]^{-\frac{1}{4}} \left[\rho(W, v^0)\ \xi(W, v^0)\right]^{\frac{1}{2}}$$

$$\times\ \delta(W - W')\ \delta^3(\mathbf{u} - \mathbf{u}')\ \delta(v^0 - v^{0\prime})$$

$$\times\ \mathcal{N}_{J'}\ U^{(J')}_{\Lambda' M'}(l^{-1}; P)\ \underline{\Xi}_{12}(p_1, p_2, p_3; j'm') \ ,$$

where

$$\underline{\Xi}_{12}(p_1, p_2, p_3; j\, m) = \mathcal{N}_j\ e^{-i\pi s_2}\ d^j_{m\lambda}(\theta) \sum_{\nu_1\, \nu_2} d^{s_1}_{\nu_1\lambda_1}(\beta_1)\ d^{s_2}_{\nu_2\lambda_2}(\beta_2)$$

$$\times\ U^{(1)}_{\mu_1\nu_1}(p_1; l)\ U^{(2)}_{\mu_2\nu_2}(p_2; l)\ U^{(3)}_{\mu_3\lambda_3}(p_3; l) \ .$$

$\Xi_{12}(p_1, p_2, p_3; j\, m)$ satisfies

$$\delta_{JJ'}\ \delta_{MM'}\ \delta_{jj'}\ \delta_{mm'}\ \delta_{\lambda_1\lambda_1'}\ \delta_{\lambda_2\lambda_2'}\ \delta_{\lambda_3\lambda_3'} \tag{2.4}$$

$$= \sum_{\mu} \int d\Omega\ d\hat{p}\ \left[\mathcal{N}_J\ U^{(J)}_{\Lambda M}(l^{-1}; P)\ \Xi_{12}(p_1, p_2, p_3; j\, m)\right]^*$$

$$\times \left[\mathcal{N}_{J'}\ U^{(J')}_{\Lambda' M'}(l^{-1}; P)\ \Xi_{12}(p_1, p_2, p_3; j' m')\right] ,$$

where the summation over μ represents a summation over all intermediate helicities, and Ω is defined through

$$d^3 v = |\mathbf{v}|^2\ d|\mathbf{v}|\ d\Omega = \left[{v^0}^2 - 1\right]^{\frac{1}{2}}\ v^0\ dv^0\ d\Omega\ .$$

The operator scattering equations detailed in Ref. 1 will be evaluated in terms of matrix elements taken in the three-body angular momentum state basis. However, the connection between the two-body input and the three-body problem, developed for the scalar case in LM(Chap. 4), is easier to discuss in terms of another basis. We therefore define a new three-particle basis through the direct product of two-particle angular momentum states $\left|P_{12}\, j\, m\, \sigma_1\, \sigma_2\right\rangle$ and single particle plane waves $\left|p_3\, \sigma_3\right\rangle$. Since the helicities in the two-particle angular momentum states are internal variables, $\left|Q_{12}\, j\, m\, \lambda_1\, \lambda_2\right\rangle$ and $\left|q_3\, \lambda_3\right\rangle$ in W(20) can be identified with $\left|p_1^0\, \lambda_1\right\rangle$ and $\left|p_2^0\, \lambda_2\right\rangle$ in W(5). Performing the steps leading from W(5) to W(17) in an analogous manner on W(20), and noting that

$$\widetilde{d}P_{12} = \delta(P_{12}^2 - w^2)\ d^4 P_{12}\ ,$$

gives

$$\left\langle P_{12}\, j\, m\, \sigma_1\, \sigma_2 ; p_3\, \sigma_3 \middle| P' J' M' ; w' j' m' \lambda_1\, \lambda_2 ; \lambda_3 \right\rangle \tag{2.5}$$

$$= \mathcal{N}_J \left[4W/q\right]^{\frac{1}{2}} \delta^4(P - P')\ \delta({P_{12}}^2 - {w'}^2)\ \delta_{jj'}\ \delta_{\sigma_1\lambda_1}\ \delta_{\sigma_2\lambda_2}$$

$$\times\ U^{(J')}_{\Lambda' M'}(l^{-1}; P)\ U^{(j)}_{mm'}(P_{12}; l)\ U^{(3)}_{\sigma_3\lambda_3}(p_3; l)\ ,$$

where

$$P = P_{12} + p_3\ ,$$

and l is specified by the condition that l^{-1} transform P to rest ($\mathbf{P} = 0$) and $\mathbf{P}_{12}$ to a vector in the direction of the positive z-axis.

The orthonormality and completeness of this new basis follows from the properties of the direct product

$$\left\langle P_{12}\, j\, m\, \sigma_1\, \sigma_2 ; p_3\, \sigma_3 \middle| P'_{12}\, j' m' \sigma'_1\, \sigma'_2 ; p'_3\, \sigma'_3 \right\rangle \tag{2.6}$$

$$= \delta^4(P_{12} - P'_{12})\ \widetilde{\delta}(p_3, p'_3)\ \delta_{jj'}\ \delta_{mm'}\ \delta_{\sigma_1\sigma'_1}\ \delta_{\sigma_2\sigma'_2}\ \delta_{\sigma_3\sigma'_3}\ ,$$

where

$$\widetilde{\delta}(p_3, p'_3) = 2p_3^0\ \delta^3(p_3 - p'_3)\ .$$

Completeness involves an integration which is convenient to write as

$$d^4 P_{12}\ \widetilde{dp}_3 = \left[u^0\ \varpi(w,v^0)\right]^{-1}\ d^3u\ d^3v\ dw \tag{2.7}$$

$$= \left[W^3/u^0\right]\ \left[(4W/q)\ \xi(W,v^0)\right]^{-1}\ dW\ d^3u\ dv^0\ d\Omega\ .$$

The projection onto the plane wave basis is

$$\left\langle p_1\,\mu_1, p_2\,\mu_2, p_3\,\mu_3 \middle| P_{12}'\, j'm'\sigma_1\,\sigma_2; p_3'\,\sigma_3\right\rangle \tag{2.8}$$

$$= \mathcal{N}_j\ \left[4w/p\right]^{\frac{1}{2}}\ u^0\ \varpi(w,v^0)\ \delta^3(\mathbf{u}-\mathbf{u}')\ \delta^3(\mathbf{v}-\mathbf{v}')\ \delta(w-w')$$

$$\times\ \delta_{\mu_3\sigma_3}\ U^{(12)}_{\sigma m'}(l^{-1};P_{12})\ U^{(1)}_{\mu_1\sigma_1}(p_1;l)\ U^{(2)}_{\mu_2\sigma_2}(p_2;l)\ ,$$

where

$$\sigma = \sigma_1 - \sigma_2\ ,$$

and l is specified by the condition that l^{-1} transform P_{12} to rest and $\mathbf{p}_1$ to a vector in the direction of the positive z-axis. Thus

$$\left\langle p_1\,\mu_1, p_2\,\mu_2, p_3\,\mu_3 \middle| P_{12}'\, j'm'\sigma_1\,\sigma_2; p_3'\,\sigma_3\right\rangle \tag{2.9}$$

$$= u^0\ \left[\varsigma(w,v^0)\ \varpi(w,v^0)\right]^{\frac{1}{2}}\ \delta^3(\mathbf{u}-\mathbf{u}')\ \delta^3(\mathbf{v}-\mathbf{v}')$$

$$\times\ \delta(w-w')\ \underline{\Gamma}_{12}(p_1,p_2;j'm')\ ,$$

where

$$\underline{\Gamma}_{12}(p_1, p_2; j\, m) \equiv \mathcal{N}_j \; \delta_{\mu_3 \sigma_3} \; U^{(j)}_{\sigma m}(l^{-1}; P_{12}) \; U^{(1)}_{\mu_1 \sigma_1}(p_1; l) \; U^{(2)}_{\mu_2 \sigma_2}(p_2; l) \; .$$

$\underline{\Gamma}_{12}(p_1, p_2; j\, m)$ satisfies

$$\delta_{jj'} \; \delta_{mm'} \; \delta_{\sigma_1 \sigma_1'} \; \delta_{\sigma_2 \sigma_2'} \; \delta_{\sigma_3 \sigma_3'} = \sum_{\mu} \int d\hat{p} \; \underline{\Gamma}^*_{12}(p_1, p_2; j\, m) \; \underline{\Gamma}_{12}(p_1, p_2; j' m') \; . \quad (2.10)$$

3. Two-Body Input

To the requirement of Lorentz invariance and unitarity imposed on the two-body input in LM(4.5) we add here angular momentum conservation. The two-body transition matrix elements taken between angular momentum states must conserve both the total angular momentum and its projection along the z-axis, and must be independent of the particular value of the projection. Therefore

$$\left\langle P_{12}\, j\, m\, \lambda_1\, \lambda_2 \right| t(z) \left| P'_{12}\, j' m' \lambda'_1\, \lambda'_2 \right\rangle \tag{3.1}$$

$$= \left[u^0_{12}\right]^2 \left[w w'\right]^{-\frac{3}{2}} \delta^3(\mathbf{u}_{12} - \mathbf{u}'_{12})\, \delta_{jj'}\, \delta_{mm'}\, \tau^j_{\lambda_1\lambda_2,\lambda'_1\lambda'_2}(w|w'; Z)\ .$$

As in LM(4.5), Z is given by

$$Z = z/u^0_{12}\ .$$

τ^j is a matrix in which each element is associated with a particular combination of incoming and outgoing helicities. The λ subscripts will not be shown explicitly in subsequent equations when their values are obvious from the context.

Two-body unitarity in operator form is given by LM(4.4). Taking matrix element between two-body angular momentum states leads to the matrix equation

$$\tau^j(w|w'; Z_1) - \tau^j(w|w'; Z_2) \tag{3.2}$$

$$= (Z_2 - Z_1) \int dw''\, \tau^j(w|w''; Z_1)\, \frac{1}{w'' - Z_1}\, \frac{1}{w'' - Z_2}\, \tau^j(w''|w'; Z_2)\ .$$

The connection to the three-body problem is governed by the same considerations as in the scalar case. Matrix elements of the non-interacting resolvent conserve linear momenta, angular momenta, and helicities. Clustering requires velocity conservation in both the two- and three-body systems. Therefore

$$\left\langle P_{12}\, j\, m\, \sigma_1\, \sigma_2; p_3\, \sigma_3 \right| T_{12}(Z) \left| P'_{12}\, j' m' \sigma'_1\, \sigma'_2; p'_3\, \sigma'_3 \right\rangle \tag{3.3}$$

$$= \left[u^0\right]^2 \left[WW'\right]^{-\frac{3}{2}} \left[(16\, WW'/qq')\; \xi(W, v^0)\; \xi(W', v^0)\right]^{\frac{1}{2}}$$

$$\times\; \delta^3(\mathbf{u} - \mathbf{u}')\; \delta(v^0 - v^{0\prime})\; \delta^2(\Omega - \Omega')\; \delta_{jj'}\, \delta_{mm'}\; U^{(3)}_{\sigma_3 \sigma'_3}(p_3; p'_3)$$

$$\times\; \theta(W - \varepsilon_3^{\rm par} - m_{12} v^0)\; \theta(W' - \varepsilon_3^{\rm par} - m_{12} v^0)\; \tau^j_{12}(\widetilde{w} | \widetilde{w}'; \mathcal{Z})\ ,$$

where

$$\widetilde{w} = (W - \varepsilon_3^{\rm par})/v^0$$

$$\widetilde{w}' = (W' - \varepsilon_3^{\rm par})/v^0$$

$$\mathcal{Z}_{12} = (Z^{\rm c} - \varepsilon_3^{\rm par})/v^0\ .$$

We have defined

$$Z^{\rm c} = Z/u^0$$

and

$$U(p; p') \equiv U(l; p') = U(p; l)$$

for a Lorentz transformation l which satisfies $p = l\, p'$. Using (3.3) to evaluate the matrix elements of LM(3.19), the unitarity condition for $T_{12}(Z)$, reproduces the two-body unitarity condition (3.2).

The matrix elements of $T_{12}(Z)$ between three-body angular momentum states follow from (2.2), (2.5), (2.7), and (3.3)

$$\left\langle P\,J\,M; w\,j\,m\,\lambda_1\lambda_2;\lambda_3\right| T_{12}(Z) \left|P'J'M'; w'j'm'\lambda_1'\,\lambda_2';\lambda_3'\right\rangle \tag{3.4}$$

$$= \left[u^0\right]^2 \left[WW'\right]^{-\frac{3}{2}} \left[\xi(W,v^0)\;\xi(W',v^0)\right]^{-\frac{1}{2}} \delta^3(\mathbf{u}-\mathbf{u}')\;\delta(v^0-v^{0\prime})\;\delta_{jj'}$$

$$\times\; \mathcal{N}_J\;\mathcal{N}_{J'}\;\theta(W-\varepsilon_3^{\mathrm{par}}-m_{12}v^0)\;\theta(W'-\varepsilon_3^{\mathrm{par}}-m_{12}v^0)\;\tau_{12}^{j}(\widetilde{w}|\widetilde{w}';Z)$$

$$\times \sum_{\sigma_3\,\sigma_3'\,m''} \int d\Omega''\;d\Omega'''\;\delta^2(\Omega''-\Omega''')\;U^{(3)}_{\sigma_3\sigma_3'}(p_3'';p_3''')$$

$$\times \left[U^{(J)}_{\Lambda M}(l''^{-1};P)\;U^{(j)}_{m''m}(P_{12}'';l'')\;U^{(3)}_{\sigma_3\lambda_3}(p_3'';l'')\right]^*$$

$$\times \left[U^{(J')}_{\Lambda'M'}(l'''^{-1};P')\;U^{(j')}_{m''m'}(P_{12}''';l''')\;U^{(3)}_{\sigma_3'\lambda_3'}(p_3''';l''')\right],$$

where

$$P_{12}'' \text{ and } p_2'' \text{ are functions of } u,\; v^0,\; W,\; \Omega'$$

$$P_{12}''' \text{ and } p_2''' \text{ are functions of } u,\; v^0,\; W',\; \Omega''\,.$$

l'' is defined by the condition that l''^{-1} transform P to rest and $\mathbf{P}_{12}$ to a vector in the direction of the positive z-axis. Therefore, it is a function of u and Ω''.

Similarly l''' is a function of u and Ω'''. Integration over the final delta function sets

$$l'' = l'''$$

$$\mathbf{u}_{12}'' = \mathbf{u}_{12}''' \ .$$

Then

$$\sum_{m''} U^{(j)}_{m''m}{}^{*}(P_{12}''; l'')\ U^{(j')}_{m''m'}(P_{12}'''; l'') = \delta_{mm'} \ .$$

Since $q_3'' = l''^{-1} p_3''$ and $q_3''' = l''^{-1} p_3'''$ differ only in magnitude of velocity,

$$\sum_{\sigma_3 \sigma_3'} U^{(3)}_{\sigma_3\lambda_3}{}^{*}(p_3'', l'')\ U^{(3)}_{\sigma_3\sigma_3'}(p_3'', p_3''')\ U^{(3)}_{\sigma_3'\lambda_3'}(p_3''', l'') = \delta_{\lambda_3\lambda_3'} \ .$$

Finally,

$$\int d\Omega''\ U^{(J)}_{\Lambda M}{}^{*}(l''^{-1}; P)\ U^{(J')}_{\Lambda' M'}(l''^{-1}; P') = \int d\Omega''\ D^{J}_{\Lambda M}{}^{*}(\Omega''^{-1})\ D^{J'}_{\Lambda' M'}(\Omega''^{-1})$$

$$= \frac{4\pi}{2J+1}\ \delta_{JJ'}\ \delta_{MM'}\ \delta_{\Lambda\Lambda'} \ .$$

Therefore

$$\left\langle P\,J\,M; w\,j\,m\,\lambda_1\,\lambda_2; \lambda_3 \right| T_{12}(Z) \left| P'J'M'; w'j'm'\lambda_1'\,\lambda_2'; \lambda_3' \right\rangle \tag{3.5}$$

$$= \left[u^0\right]^2 \left[W W'\right]^{-\frac{3}{2}} \left[\xi(W, v^0)\; \xi(W', v^0)\right]^{\frac{1}{2}}$$

$$\times\; \delta^3(\mathbf{u} - \mathbf{u}')\; \delta(v^0 - v^{0\,\prime})\; \delta_{JJ'}\; \delta_{MM'}\; \delta_{jj'}\; \delta_{mm'}$$

$$\times\; \theta(W - \varepsilon_3^{\mathrm{par}} - m_{12} v^0)\; \theta(W' - \varepsilon_3^{\mathrm{par}} - m_{12} v^0)\; \underline{\tau}^{j}_{12}(\widetilde{w}|\widetilde{w}'; Z)\ ,$$

where

$$\underline{\tau}^{j}_{12}(\widetilde{w}|\widetilde{w}'; Z) = \delta_{\lambda_3 \lambda_3'}\; \tau^{j}_{12}(\widetilde{w}|\widetilde{w}'; Z)\ .$$

Using (2.7), (2.9), and (3.3) to re-express the matrix elements of $T_{12}(Z)$ in terms of the plane wave basis gives

$$\left\langle p_1\,\mu_1, p_2\,\mu_2, p_3\,\mu_3 \right| T_{12}(Z) \left| p_1'\,\mu_1', p_2'\,\mu_2', p_3'\,\mu_3' \right\rangle \tag{3.6}$$

$$= \left[u^0\right]^2 \left[W W'\right]^{-\frac{3}{2}} \left[\rho(W, v^0)\; \rho(W', v^0)\right]^{\frac{1}{2}}$$

$$\times\; \delta^3(\mathbf{u} - \mathbf{u}')\; \delta^3(\mathbf{v} - \mathbf{v}')\; \theta(W - \varepsilon_3^{\mathrm{par}} - m_{12} v^0)$$

$$\times\; \theta(W' - \varepsilon_3^{\mathrm{par}} - m_{12} v^0)\; \underline{\tau}_{12}(p_1, p_2, p_3 | p_1', p_2', p_3'; Z)\ ,$$

where

$$\underline{\mathcal{T}}_{12}(p_1, p_2, p_3 | p_1', p_2', p_3' ; Z)$$

$$= \sum_{j\,m} \sum_{\sigma\,\sigma'} \underline{\Gamma}_{12}(p_1, p_2; j\,m)\ U^{(3)}_{\sigma_3\sigma_3'}(p_3; p_3')$$

$$\times\ \tau^{j}_{12}(\widetilde{w}|\widetilde{w}' ; Z)\ \underline{\Gamma}^*_{12}(p_1', p_2'; j\,m)\ .$$

This expression can be simplified by substituting the explicit form for $\underline{\Gamma}_{12}$ and noting that

$$\sum_m U^{(j)}_{\sigma m}(l^{-1}; P_{12})\ U^{(j)\,*}_{\sigma' m}(l'^{-1}; P_{12}') = D^j_{\sigma\sigma'}(r^{-1}\,r')\ ,$$

where r and r' are defined by

$$U^{(j)}_{\sigma m}(l^{-1}; P_{12}) = D^j_{\sigma m}(r^{-1})$$

$$U^{(j)}_{\sigma' m}(l'^{-1}; P_{12}') = D^j_{\sigma' m}(r'^{-1})\ .$$

Then

$$\underline{\mathcal{T}}_{12}(p_1, p_2, p_3 | p_1', p_2', p_3' ; Z)$$

$$= \sum_j \mathcal{N}_j^2\ U^{(3)}_{\mu_3\mu_3'}(p_3; p_3')$$

$$\times \sum_{\sigma_1\,\sigma_2} \sum_{\sigma_1'\sigma_2'} U^{(1)}_{\mu_1\sigma_1}(p_1; l)\ U^{(2)}_{\mu_2\sigma_2}(p_2; l)\ \tau^j_{12}(\widetilde{w}; \widetilde{w}' ; Z)$$

$$\times\ D^j_{\sigma\sigma'}(r^{-1}\,r')\ U^{(1)\,*}_{\mu_1'\sigma_1'}(p_1'; l')\ U^{(2)\,*}_{\mu_2'\sigma_2'}(p_2'; l')\ .$$

4. Integral Equations

A particular coupling for the three-body angular momentum states is defined by the spectator a and the pair $(a+, a-)$, labeled by A. The recoupling coefficient is

$$\left\langle P\,J\,M; w_{A}\,j\,m\,\lambda_{a+}\,\lambda_{a-};\lambda_{a}\middle| P'J'M'; w'_{B}\,j'm'\lambda'_{b+}\,\lambda'_{b-};\lambda'_{b}\right\rangle \tag{4.1}$$

$$= \left[u^0/W^3\right]\,\left[\xi(W, v^0_{A})\;\xi(W, v^{0\,\prime}_{B})\right]^{\frac{1}{2}}\,\delta(W-W')\;\delta^3(\mathbf{u}-\mathbf{u}')$$

$$\times\;\delta_{JJ'}\;\delta_{MM'}\;\underline{\Delta}^{J}_{AB}(w\,j\,m|w'j'm';W)\ ,$$

where

$$\underline{\Delta}^{J}_{AB}(w\,j\,m|w'j'm';W) = \left[\xi(W, v^0_{A})\;\xi(W, v^{0\,\prime}_{B})\right]^{-\frac{1}{2}}\,\theta(1-|\cos\theta|)\;\left\langle (A)a\middle|(B)b\right\rangle\ .$$

The angle θ and the abbreviated coefficient $\left\langle (A)a\middle|(B)b\right\rangle$ are given by W(31) and W(35).

The integral equations generated by LM(3.22) for particles with spin are written in terms of the matrix $\underline{\mathcal{W}}^{J}_{AB}$ defined by

$$\Big\langle P\,J\,M\,;w_{A}\,j\,m\,\lambda_{a+}\,\lambda_{a-};\lambda_{a}\Big|\;W_{AB}(Z)\;\Big|P'J'M';w_{B}'\,j'm'\lambda_{b+}'\,\lambda_{b-}';\lambda_{b}'\Big\rangle \tag{4.2}$$

$$= \left[u^{o}\right]^{2}\,\left[WW'\right]^{-\frac{3}{2}}\,\delta^{3}(\mathbf{u}-\mathbf{u}')\,\delta_{JJ'}\,\delta_{MM'}\,\left[\xi(W,v_{A}^{0})\;\xi(W',v_{B}^{0}{}')\right]^{\frac{1}{2}}$$

$$\times\;\theta(W-\varepsilon_{a}^{\mathrm{par}}-m_{A}v_{A}^{0})\;\theta(W'-\varepsilon_{b}^{\mathrm{par}\,\prime}-m_{B}v_{B}^{0}{}')$$

$$\times\;\underline{\mathcal{W}}_{AB}^{J}(W\,w\,j\,m|W'w'j'm';Z^{c})\;.$$

Then

$$\underline{\mathcal{W}}^J_{AB}(W\,w\,j\,m|W'w'j'm';Z^c) \tag{4.3}$$

$$= -\,\bar{\delta}_{AB}\sum_{\lambda''\lambda'''}\int dW''\,\Big[\theta(W''-\varepsilon_a^{\rm par}-m_A v_A^0)\;\theta(W''-\varepsilon_b^{\rm par\,\prime}-m_B v_B^{0\,\prime})$$

$$\times\;\frac{1}{W''-Z^c}\;\underline{\tau}^j_A(\widetilde{w}|\widetilde{w}'';Z)$$

$$\times\;\underline{\Delta}^J_{AB}(w''j\,m|w''j'm';W'')\;\underline{\tau}^{j'}_B(\widetilde{w}''|\widetilde{w}';Z')\Big]$$

$$-\sum_D\bar{\delta}_{AD}\sum_{j''m''}\;\sum_{\lambda''\lambda'''}\int dW''\,dv_D^{0\,\prime\prime}\,\Big[\theta(W''-\varepsilon_a^{\rm par}-m_A v_A^0)$$

$$\times\;\theta(W''-\varepsilon_d^{\rm par}-m_D v_D^{0\,\prime\prime})\;\frac{1}{W''-Z^c}$$

$$\times\;\underline{\tau}^j_A(\widetilde{w}|\widetilde{w}'';Z)\;\underline{\Delta}^J_{AD}(w''j\,m|w''j''m'';W'')$$

$$\times\;\underline{\mathcal{W}}^J_{DB}(W''w''j''m''|W'w'j'm';Z^c)\Big]\;,$$

where

$$w_A'' = \omega(W'',v_A^0,m_a^2)$$

$$w_B'' = \omega(W'',v_B^{0\,\prime},m_b^2)$$

$$w_D'' = \omega(W'',v_D^{0\,\prime\prime},m_d^2)\;.$$

The summations over λ'' and λ''' represent summations over all intermediate helicities.

5. Probability Amplitude

In order to connect the solutions of the integral equations (4.3) to the physical probability amplitude, the matrix elements of $W_{AB}(Z)$ must be re-expressed in terms of the plane wave basis. Using (2.3) and (4.2) gives

$$\left\langle p_1\,\mu_1, p_2\,\mu_2, p_3\,\mu_3 \right| W_{AB}(Z) \left| p_1'\,\mu_1', p_2'\,\mu_2', p_3'\,\mu_3' \right\rangle \tag{5.1}$$

$$= \left[u^0\right]^2 \left[W W'\right]^{-\frac{3}{2}} \left[\rho(W, v_A^0)\;\rho(W', v_B^{0\,\prime})\right]^{\frac{1}{2}} \delta^3(\mathbf{u}-\mathbf{u}')$$

$$\times\ \theta(W - \varepsilon_a^{\rm par} - m_A v_A^0)\ \ \theta(W' - \varepsilon_b^{\rm par\,\prime} - m_B v_B^{0\,\prime})$$

$$\times\ \underline{\mathcal{W}}_{AB}(p_1, p_2, p_3 | p_1', p_2', p_3'; Z^{\rm c})\ ,$$

where

$$\underline{\mathcal{W}}_{AB}(p_1,p_2,p_3|p_1',p_2',p_3';Z^c)$$

$$= \left[v^{0^2}\ (v^{0^2}-1)\ v^{0'^2}\ (v^{0'^2}-1)\right]^{-\frac{1}{4}}$$

$$\times \sum_{JM}\sum_{jm}\sum_{j'm'}\sum_{\lambda\lambda'} \mathcal{N}_J^2\ U_{\Lambda M}^{(J)}(l^{-1};P)\ \Xi_A(p_1,p_2,p_3;j\,m)$$

$$\times\ \underline{\mathcal{W}}_{AB}^J(W\,w\,j\,m|W'w'j'm';Z^c)$$

$$\times\ U_{\Lambda' M}^{(J)\ *}(l'^{-1};P')\ \Xi_B^*(p_1',p_2',p_3';j'm')\ .$$

This can be simplified by noting that

$$\sum_M U_{\Lambda M}^{(J)}(l^{-1};P)\ U_{\Lambda' M}^{(J)\ *}(l'^{-1};P') = D_{\Lambda\Lambda'}^J(s^{-1}\,s')\ ,$$

where s and s' are defined by

$$U_{\Lambda M}^{(J)}(l^{-1};P) = D_{\Lambda M}^J(s^{-1})$$

$$U_{\Lambda' M}^{(J)}(l'^{-1};P') = D_{\Lambda' M}^J(s'^{-1})\ .$$

An interacting two-particle state is characterized by an invariant mass w_A, angular momentum quantum numbers j and m, and other internal quantum numbers summerized by the single parameter γ_A. The clustered channel states, formed by the direct product of two-particle interacting states with plane waves for the third particle, satisfy

$$\Big\langle u, u_A, \psi_A(w, j\, m, \gamma), \sigma_a \Big| u', u'_A, \psi_A(w', j'm', \gamma'), \sigma'_a \Big\rangle \tag{5.2}$$

$$= u^0\, \varpi(w_A, v^0_A)\, \delta^3(\mathbf{u} - \mathbf{u}')\, \delta^3(\mathbf{v}_A - \mathbf{v}'_A)\, \delta(w_A, w'_A)\, \delta_{jj'}\, \delta_{mm'}\, \delta_{\gamma_A \gamma'_A}\, \delta_{\sigma_a \sigma'_a} .$$

The overlap of these states with non-interacting states defines wavefunctions

$$\Big\langle P_A\, j\, m\, \sigma_{a+}\, \sigma_{a-}; p_a\, \sigma_a \Big| u', u'_A, \psi_A(w', j'm', \gamma'), \sigma'_a \Big\rangle \tag{5.3}$$

$$= u^0\, \left[\varpi(w_A, v^0_A)\, \varpi(w'_A, v^0_A)\right]^{\frac{1}{2}}\, \delta^3(\mathbf{u} - \mathbf{u}')\, \delta^3(\mathbf{v}_A - \mathbf{v}'_A)$$

$$\times\, \delta_{jj'}\, \delta_{mm'}\, \delta_{\sigma_a \sigma'_a}\, \psi^j_A(w, \sigma_{a+}\, \sigma_{a-} | w', \gamma') .$$

Then

$$\Big\langle p_1\, \mu_1, p_2\, \mu_2, p_3\, \mu_3 \Big| u', u'_A, \psi_A(w', j'm', \gamma'), \sigma'_a \Big\rangle \tag{5.4}$$

$$= u^0\, \left[\varsigma(w_A, v^0_A)\, \varpi(w'_A, v^0_A)\right]^{\frac{1}{2}}\, \delta^3(\mathbf{u} - \mathbf{u}')\, \delta^3(\mathbf{v}_A - \mathbf{v}'_A)$$

$$\times\, \delta_{\mu_a \sigma'_a}\, \psi(p_{a+}\, \mu_{a+}, p_{a-}\, \mu_{a-} | w'_A, j'm', \gamma'_A) ,$$

where

$$\delta_{\mu_a\sigma_a'}\ \psi(p_{a+}\,\mu_{a+},p_{a-}\,\mu_{a-}|w_A',j'm',\gamma_A')$$

$$=\sum_{\sigma''}\underline{\Gamma}_A(p_1,p_2;j',m')\ \psi_A^{j'}(w,\sigma_{a+}''\,\sigma_{a-}''|w',\gamma')\ \delta_{\sigma_a''\sigma_a'}\ .$$

(5.1), (5.2), and (5.4)are the generalizations of LM(5.1, 2.15, and 2.16), respectively, for particles with spin. The techniques of LM(Chap. 6) are directly applicable. We assume, again, that there are no degeneracies in the two-body bound state spectrum. LM(6.11) becomes

$$\frac{1}{E-E^{\mathrm{p}}}\ \lim_{\epsilon\to 0}\ (-i\epsilon)^2\left\langle p_1\,\mu_1,p_2\,\mu_2,p_3\,\mu_3\middle|W_{AB}(E^{\mathrm{p}}+i\epsilon)\middle|p_1'\,\mu_1',p_2'\,\mu_2',p_3'\,\mu_3'\right\rangle\ \frac{1}{E'-E^{\mathrm{p}}}$$

$$=\ \left[\varsigma(w_A,v_A^0)\ \varsigma(w_B',v_B^{0\,\prime})\right]^{\frac{1}{2}}\ \left[\varpi(\mu_A^{\mathrm{p}},v_A^0)\ \varpi(\mu_B^{\mathrm{p}\,\prime},v_B^{0\,\prime})\right]^{-\frac{1}{2}} \tag{5.5}$$

$$\times\ \psi(p_{a+}\,\mu_{a+},p_{a-}\,\mu_{a-}|\mu_A^{\mathrm{p}},j^{\mathrm{p}}\,m^{\mathrm{p}},\gamma_A^{\mathrm{p}})\ \psi^*(p_{b+}'\,\mu_{b+}',p_{b-}'\,\mu_{b-}'|\mu_B^{\mathrm{p}\,\prime},j^{\mathrm{p}\,\prime}m^{\mathrm{p}\,\prime},\gamma_B^{\mathrm{p}\,\prime})$$

$$\times\ \left\langle u,u_A,\psi_A(\mu^{\mathrm{p}},j^{\mathrm{p}}\,m^{\mathrm{p}},\gamma^{\mathrm{p}}),\mu_a\middle|Q_{AB}^{(+)}(E^{\mathrm{p}})\middle|u,u_B',\psi_B(\mu^{\mathrm{p}\,\prime},j^{\mathrm{p}\,\prime}\,m^{\mathrm{p}\,\prime},\gamma^{\mathrm{p}\,\prime}),\mu_b'\right\rangle\ .$$

The $\varepsilon^{\mathrm{par}}$ factors are fixed by LM(6.14)

$$\varepsilon_i^{\mathrm{par}}=W^{\mathrm{p}}-\omega(W^{\mathrm{p}},v_I^0,m_i^2)\ v_I^0$$

$$\varepsilon_i^{\mathrm{par}\,\prime}=W^{\mathrm{p}}-\omega(W^{\mathrm{p}},v_I^{0\,\prime},m_i^2)\ v_I^{0\,\prime}\ .$$

By taking the matrix element of LM(3.18) between a bra and a ket formed from the same helicity plane wave state we obtain an equation similar to LM(6.16)

$$v_A^0 \, W^{-3} \, \rho(w, v_A^0) \quad \lim_{\epsilon_A \to 0} \, (-i\epsilon_A) \; \underline{T}_A(p_1, p_2, p_3 | p_1, p_2, p_3 \,; \mu_A^{\mathrm{P}} + i\epsilon_A) \tag{5.6}$$

$$= - \, \varsigma(w_A, v_A^0) \; (W - W^{\mathrm{P}})^2 \; \left| \psi(p_{a+} \, \mu_{a+}, p_{a-} \, \mu_{a-} | \mu_A^{\mathrm{P}}, j^{\mathrm{P}} \, m^{\mathrm{P}}, \gamma_A^{\mathrm{P}}) \right|^2 .$$

Define

$$\chi(p_1, p_2, p_3, \mu_I^{\mathrm{P}}) \tag{5.7}$$

$$= \left[v_I^0 \, W^3 \, \varpi(\mu_I^{\mathrm{P}}, v_I^0) \right]^{\frac{1}{2}} \left[- \lim_{\epsilon_I \to 0} \, (-i\epsilon_I) \; \underline{T}_I(p_1, p_2, p_3 | p_1, p_2, p_3 \,; \mu_I^{\mathrm{P}} + i\epsilon_I) \right]^{-\frac{1}{2}} .$$

Then, in analogy to LM(6.18), we find the probability amplitude for elastic and rearrangement scattering

$$\mathcal{A}^{(+)} \big(\Phi_A \big(u, u_A, \psi_A(\mu^{\mathrm{P}}, j^{\mathrm{P}} \, m^{\mathrm{P}}, \gamma^{\mathrm{P}}), \mu_a \big) \big| \Phi_B \big(u, u_B', \psi_B(\mu^{\mathrm{P}\prime}, j^{\mathrm{P}\prime} \, m^{\mathrm{P}\prime}, \gamma^{\mathrm{P}\prime}), \mu_b' \big) \big) \,; W^{\mathrm{P}} \big)$$

$$= - \, \chi(p_1, p_2, p_3, \mu_A^{\mathrm{P}}) \; \chi(p_1', p_2', p_3', \mu_B^{\mathrm{P}\prime}) \tag{5.8}$$

$$\times \lim_{\epsilon_A \to 0} \; \lim_{\epsilon_B' \to 0} \; (-\epsilon_A \epsilon_B') \; \underline{\mathcal{W}}_{AB}(p_1, p_2, p_3 | p_1', p_2', p_3' \,; W^{\mathrm{P}} + i\epsilon^{\mathrm{c}}) .$$

The amplitude is not explicitly invariant because the helicities of single particle plane waves are defined with respect to the frame of the observer.

The probability amplitude for free particle scattering is similar to LM(6.20)

$$\mathcal{A}^{(+)}\big(\Phi_0(p_1\,\mu_1, p_2\,\mu_2, p_3\,\mu_3)\,|\,\Phi_0(p_1'\,\mu_1', p_2'\,\mu_2', p_3'\,\mu_3')\,;W\big) \tag{5.9}$$

$$= -\sum_{A,B} \left[\rho(W, v_A^0)\ \rho(W, v_B^{0\,\prime})\right]^{\frac{1}{2}} \left[\delta_{AB}\ \delta^3(\mathbf{v}_A - \mathbf{v}_A')\ \underline{\mathcal{T}}_A^{(+)}(p_1, p_2, p_3 | p_1', p_2', p_3'\,; w)\right.$$

$$\left. + \ \underline{\mathcal{W}}_{AB}^{(+)}(p_1, p_2, p_3 | p_1', p_2', p_3'\,; W)\right] .$$

The probabilty amplitudes for breakup and coalescence are, just as in the scalar case, obvious extensions of (5.8) and (5.9).

6. Conclusion

The techniques of Ref. 1 have been extended to include the effects of angular momentum conservation and individual particle spin. The resulting angular momentum decomposed equations exhibit the same properties as the scalar equations: exact unitary and physical clustering.

REFERENCES

1. J. V. Lindesay, A. J. Markevich, H. P. Noyes, and G. Pastrana, "A Self-Consistent, Poincaré Invarinat and Unitary Three-Particle Scattering Theory",

   June 1985 (submitted to Phys. Rev. D), referred to as LM.

2. G. C. Wick, Ann. Phys. 18, 65 (1962), referred to as W. See also M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).

3. W(24) contains a typographical error. The state being represented is $|P^0 J\, M \ldots\rangle$, not $|P\, J\, M \ldots\rangle$.