

QUANTUM CHROMODYNAMIC EVOLUTION OF THE BARYON SYSTEM*

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ABSTRACT

We present a general method for solving the QCD evolution equations which govern relativistic multi-quark wavefunctions. In the case of three-quark systems, we generate a light-cone basis of completely antisymmetric wavefunctions. This provides a general covariant classification of baryonic states. We then calculate the spin-orbit mixing generated by the QCD evolution kernel in the basis of completely antisymmetric representations. We are consistent with previous results, but additionally, we obtain a distinctive classification of N and Δ wavefunctions. The corresponding Q^2 dependence of the baryon distribution amplitudes distinguishes the N and Δ form factors.

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1. Introduction

The form of the short-distance behavior of a baryon wavefunction can be computed systematically in perturbative quantum chromodynamic (QCD).¹ The leading behavior of the baryon three-quark wavefunction at large momentum transfer or short distances is controlled through an evolution equation with an irreducible hard scattering kernel which, in lowest order, is identical to the gluon exchange potential. Since the running coupling constant $\alpha_s(Q^2) = 4\pi/[\beta_0 \ln(Q^2/\Lambda^2)]$ ($\beta_0 = 11 - (2/3)n_f$, where n_f is the number of flavors) is small for large momentum transfer Q , a perturbative calculation of the short distance part of the wavefunction can be justified. The anomalous dimensions of the three-quark amplitude can also be predicted by the operator-product expansion and the renormalization group.²

A particularly convenient and physical formalism for studying processes with large momentum transfer is light-cone quantization, as discussed in Ref. 1. A systematic analysis of exclusive processes and hadron distribution amplitudes has been given, including solutions of the evolution equation of the three-quark system. Thus far the analysis of evolution equations has not been sufficiently detailed to give a complete classification of the nucleon (N) and delta (Δ) states. In particular, the distinction between QCD predictions for the N and Δ wave functions and form factors needs to be clarified.

In this paper, we develop a systematic basis for the baryon system by constructing completely antisymmetric three-quark representations. The calculation of the QCD evolution kernel matrix in the basis of completely antisymmetric representations is given by a straightforward method. The solutions obtained by the present method are consistent with the preceding results,^{1,2} but additionally we

obtain a distinctive classification of the N and Δ wavefunctions and the corresponding Q^2 dependence which discriminates the N and Δ form factors.

The methods used in this paper have general applicability to the problem of analyzing the short-distance dynamics of multi-quark systems. The analysis for four-quark systems in $SU(2)_c$ and six-quark systems in $SU(3)_c$ will be presented in subsequent publications.³

In Section 2 we classify the baryon state by constructing completely antisymmetric representations. In Section 3 we describe several properties of the three-quark evolution equation associated with the various quantum numbers and match the antisymmetric representations with the evolution equation.

To construct a basis of completely antisymmetric representations, we define products of spin and orbit representations in analogy to non-relativistic wavefunctions. Mixings of spin and orbital representations are described in Section 4. The results for the anomalous dimensions and the eigensolutions are presented and discussed in Section 5. Conclusions follow in the last section.

2. Antisymmetric Representations

Two identical fermions cannot occupy the same physical state; thus one describes interacting fermions in terms of antisymmetric wavefunctions. The systematic classification of bound states of fermion (including the baryons) is generated on the basis of antisymmetrized constituent representations. In order to describe relativistic systems we always refer to the valence Fock component of the bound state wavefunction defined at equal light-cone time in light-cone gauge. Further details may be found in Ref. 1.

A fermionic system in QCD is classified by the assignment of four quantum numbers: color (C), isospin (T), spin (S) and orbital (O). Each quantum sector of the wavefunction can be classified using irreducible representations with permutation symmetry denoted by Young diagrams.⁴ The explicit construction of totally antisymmetric representations in terms of an orbital index-power basis will be described in the next subsection.

2.1 COLOR (C), ISOSPIN (T) AND SPIN (S) STATES

We can classify the quantum numbers of C , T and S by the group of $G = \text{SU}(3)_C \times \text{SU}(2)_T \times \text{SU}(2)_S$ without loss of generality. Each quantum state assigned by C , T and S is the irreducible representation of G , and each irreducible representation is denoted by the corresponding Young diagram. Once a Young diagram is given, the explicit representation can be constructed from its permutation symmetry⁴ and Schmidt orthogonalization.

All physical baryons $\begin{array}{|c|} \hline r \\ \hline y \\ \hline b \\ \hline \end{array}$ are color singlet states. The corresponding Young diagram is given by $\begin{array}{|c|} \hline r \\ \hline y \\ \hline b \\ \hline \end{array}$ in $\text{SU}(3)_C$. Thus the explicit color representation of the baryon is fixed






$$\begin{array}{|c|} \hline r \\ \hline y \\ \hline b \\ \hline \end{array} = \frac{1}{\sqrt{6}} (ryb + ybr + bry - byr - rby - yrb) \quad (2.1)$$

$$\equiv \frac{1}{\sqrt{6}} \epsilon_{ijk} \quad ,$$

where the completely antisymmetric Cartesian tensor ϵ_{ijk} (i, j and k correspond to one of r, y and b) defines the color singlet representation. The quantum state (color in this case) of the first, second and third quark is represented by the first, second, and third location of every term in Eq. (2.1). Hereafter, we will

use this convention for each quantum number unless we specifically denote the particle number.

The classification of the baryon into N and Δ is given by the isospin label:

i.e., $T = 1/2$ and $3/2$, and the corresponding Young diagrams are  and  for N and Δ , respectively. The mixed symmetry  has two orthogonal permutation symmetries, represented by two different Yamanouchi labels  and . As an example, we present the explicit representation of $T_z = 1/2$ for $T = 1/2$ and $3/2$,

$$\begin{array}{|c|c|} \hline u & u \\ \hline d & \\ \hline \end{array} = \begin{cases} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \frac{1}{\sqrt{2}} (duu - udu), \\ \\ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = \frac{1}{\sqrt{6}} (duu + udu - 2uud); \end{cases} \quad (T, T_Z) = \left(\frac{1}{2}, \frac{1}{2}\right) \quad (2.2a)$$

$$\begin{array}{|c|c|c|} \hline u & u & d \\ \hline \end{array} = \frac{1}{\sqrt{3}} (uud + udu + duu); \quad (T, T_Z) = \left(\frac{3}{2}, \frac{1}{2}\right), \quad (2.2b)$$

where $(T, T_Z) = (1/2, 1/2)$ and $(3/2, 1/2)$ correspond to p and Δ^+ respectively.

The spin states of the three-quark system are classified by the Young diagrams for $S = 1/2$ and $3/2$. The explicit representations are obtained from the isospin representations with the replacement of u and d by \uparrow and \downarrow .

2.2 ORBITAL (O) STATES

The orbital states are normally defined by the quantum numbers of angular momentum L and L_Z . On the light-cone, the quark distribution amplitude $\phi(x_i, Q)$ is defined by

$$\phi(x_i, Q) = \left(\ell n \frac{Q^2}{\Lambda^2} \right)^{\frac{-3C_F}{2\beta}} \int^Q \left(\prod_{i=1}^3 \frac{d^2 \vec{k}_{\perp i}}{16\pi^3} \right) 16\pi^3 \delta^2 \left(\sum_{i=1}^3 \vec{k}_{\perp i} \right) \psi^{(Q)}(x_i, \vec{k}_{\perp i}), \quad (2.3)$$

where $\psi^{(Q)}(x_i, \vec{k}_{\perp i})$ is the wavefunction of three quarks which have longitudinal momentum fractions $x_i = k_i^+ / (\sum_{i=1}^3 k_i^+) = (k_i^0 + k_i^3) / [\sum_{i=1}^3 (k_i^0 + k_i^3)]$ and transverse momenta $\vec{k}_{\perp i}$. In this definition, the $L_Z = 0$ projection defines the amplitude for finding the constituents collinear up to the scale Q . We will use as a basis for the orbital dependence of $\phi(x_i, Q)$ the index-power space representations $x_1^{n_1} x_2^{n_2} x_3^{n_3}$ with $n = n_1 + n_2 + n_3$. The total power is analogous, as far as permutation symmetry is concerned, to the angular momentum L for non-relativistic system. In the QCD evolution equation the minimal anomalous dimension γ_n which determine hadronic amplitudes at very short distances are associated with small values of n ; only the smallest powers of x_i are important for probing the short distance behavior of $\phi(x_i, Q)$. Thus, we consider the ‘‘orbital’’ symmetry on the index-power space ($n = n_1 + n_2 + n_3$) which determines the power of x_1, x_2 and x_3 such as $x_1^{n_1} x_2^{n_2} x_3^{n_3}$.

In this power space, the orbital states are determined by filling up the possible Young diagrams with the powers of x_i . For example, if we consider $n = n_1 + n_2 + n_3 = 0$ case, then the only possible Young diagram $\begin{array}{|c|c|c|} \hline & & \\ \hline \end{array}$ gives the representation

$$\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array} = 1 \quad . \quad (2.4)$$

For $n = 1$ case, the possible diagrams and representations are

$$\boxed{0|0|1} = \frac{1}{\sqrt{3}} (x_1 + x_2 + x_3) , \quad (2.5)$$

$$\boxed{0|0|1} = \begin{cases} \boxed{1|3|2} = \frac{1}{\sqrt{2}} (x_1 - x_2) , & (2.6a) \\ \boxed{1|2|3} = \frac{1}{\sqrt{6}} (x_1 + x_2 - 2x_3) . & (2.6b) \end{cases}$$

However, the representations given by Eq. (2.5) are not independent of the representation given by Eq. (2.4) because of the conservation of momentum $\sum_{i=1}^3 x_i = 1$. Generally, the orbital representations can overlap each other between the same diagrams. Thus we use the Gram Schmidt orthogonalization procedure and normalize the states by the following rule between the orbital representations $\phi_n(x_i, Q_0)$ and $\phi_m(x_i, Q_0)$ with the same Young diagram:

$$\begin{aligned} \langle \phi_m(x_i, Q_0) | \phi_n(x_i, Q_0) \rangle &= \int [dx] \omega(x_i) \phi_m^*(x_i, Q_0) \phi_n(x_i, Q_0) \\ &= \delta_{mn} , \end{aligned} \quad (2.7)$$

where $[dx] = dx_1 dx_2 dx_3 \delta \left[1 - (\sum_{i=1}^3 x_i) \right]$ and $\omega(x_i) = x_1 x_2 x_3$.

After orthonormalization, we obtain the basis set of orbital states. The explicit representations and Young diagrams up to $n = 2$ are presented in Table I. We note that the orbital representations in power space are independent of any dynamics, and any model-dependent representation can be projected onto our representation. A state which has arbitrary angular momentum L can be projected on the corresponding index-power space.

2.3 ANTISYMMETRIZATION

In subsections 2.1 and 2.2, we showed that the quantum states for each C , T , S and O quantum number are explicitly represented by the permutation symmetry of the Young diagrams. In particular, the completely antisymmetric representation of a quark system is obtained by the inner product of C , T , S and O quantum states represented by the corresponding Young diagrams. As an example, let us construct the antisymmetric representation of the excited state of the proton with $(S, S_z) = (3/2, 1/2)$. For this state, C and T representations are given by Eqs. (2.1) and (2.2a), respectively, and the S representation is given by Eq. (2.2b) with the replacement of u and d by \uparrow and \downarrow . To construct the completely antisymmetric representations, we combine the possible orbital symmetries as given by the Clebsch Gordan series of the permutation group S_3 .

In this case, the only possible orbital Young diagram is $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}$. The lowest state is $O^2 1$ ($n = 1$) and the representation is given in Table I. If we consider the Clebsch-Gordan coefficients of the permutation group ⁴

$$\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} = \frac{1}{\sqrt{2}} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} + \frac{1}{\sqrt{2}} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}, \quad (2.8)$$

then we can obtain the completely antisymmetric representation

$$\begin{aligned}
\begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} &_{CTSO} = \begin{array}{|c|} \hline r \\ \hline y \\ \hline b \\ \hline \end{array} \times \begin{array}{|c|c|} \hline u & u \\ \hline d & \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline \uparrow & \uparrow & \downarrow \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} \\
&= \frac{\sqrt{N}}{18} \epsilon_{ijk} (\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow) \\
&\times \left\{ duu(2x_1 - x_2 - x_3) + udu(-x_1 + 2x_2 - x_3) \right. \\
&\left. + uud(-x_1 - x_2 + 2x_3) \right\} ,
\end{aligned} \tag{2.9}$$

where $N = 21 \times 5!$ and ϵ_{ijk} is defined by Eq. (2.1). In a similar way, we can classify all possible three-quark states and obtain the explicit antisymmetric representations. In Table II we present the classification and the representations of the baryon system up to the power $n = 2$.

3. The Baryon Evolution Equation

The three-quark evolution equation for the three-quark distribution amplitude $\phi(x, Q)$ with $L_Z = 0$ is given by¹

$$x_1 x_2 x_3 \left(\frac{\partial}{\partial \xi} + \frac{3C_F}{2\beta} \right) \tilde{\phi}(x, Q) = \frac{C_B}{\beta} \int_0^1 [dy] V(x, y) \tilde{\phi}(y, Q) ,$$

where the reduced amplitude $\tilde{\phi}(x, Q)$ and the variable ξ are defined by

$$\phi(x, Q) = x_1 x_2 x_3 \tilde{\phi}(x, Q) ,$$

and

$$\xi(Q^2) = \frac{\beta}{4\pi} \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \alpha_s(k^2) \sim \left(\frac{\ln \frac{Q^2}{\Lambda^2}}{\ln \frac{Q_0^2}{\Lambda^2}} \right) .$$

The color factor $C_B = (n_c + 1)/2n_c = 2/3$ is fixed . The evolution kernel $V(x, y)$ is the sum over interactions between quark pairs i, j due to exchange of a single gluon:

$$\begin{aligned}
V(x, y) &= 2 x_1 x_2 x_3 \sum_{i \neq j} \theta(y_i - x_i) \delta(x_k - y_k) \frac{y_j}{x_j} \left(\frac{\delta_{h_i \bar{h}_j}}{x_i + x_j} + \frac{\Delta}{y_i - x_i} \right) \\
&= V(y, x) \quad , \tag{3.2}
\end{aligned}$$

where $\delta_{h_i \bar{h}_j} = 1(0)$ when the helicities of quark pairs i, j are antiparallel (parallel). The infrared singularity at $x_i = y_i$ is cancelled by $\Delta \tilde{\phi}(y, Q) \equiv \tilde{\phi}(y, Q) - \tilde{\phi}(x, Q)$ reflecting the fact that the baryon is a color singlet.

We use the following general properties of the kernel:

(a) The color states are evolved by the operator $\sum_{a=1}^8 \lambda_a/2 \lambda_a/2$ [the λ_a are the Gell–Mann matrices of $SU(3)_C$ group]. Under the action of this operator, the antisymmetric color singlet representation does not change. The color factor C_B is then fixed as $2/3$ for the baryon.

(b) QCD evolution conserves isospin.

(c) At high momentum, the helicity of the quarks is conserved. However, the evolution kernel given by Eq. (3.2) has the $\delta_{h_i \bar{h}_j}$ term, which means the spin evolution operator is not diagonal for spin multiplets classified by total spin while it is diagonal for S_Z components.

(d) For the orbital evolution, the total power n of orbital representation [total power of $\tilde{\phi}_n(y)$ before gluon exchange and $\tilde{\phi}_n(x)$ after gluon exchange] is conserved upon integration $\int_0^1 [dy]$. However, many different representations can have the same power n . Thus we must allow mixing between different orbital states under evolution.

The evolution equation Eq. (3.1) has a general solution of the form

$$\phi(x, Q) = x_1 x_2 x_3 \sum_{n=0}^{\infty} A_n \tilde{\phi}_n(x) \left(\ln \frac{Q^2}{\Lambda^2} \right)^{-\gamma_n}, \quad (3.3)$$

where γ_n and ϕ_n satisfy the following equation

$$x_1 x_2 x_3 \left(\frac{3C_F}{2\beta} - \gamma_n \right) \tilde{\phi}_n = \frac{C_B}{\beta} \int_0^1 [dy] V(x, y) \tilde{\phi}_n(y) . \quad (3.4)$$

The γ_n are the anomalous dimensions corresponding to the three-quark eigensolutions $\tilde{\phi}_n$.

The calculations of the γ_n and $\tilde{\phi}_n$ were already given in Ref. 1 for the proton ($h = 1/2$). However, the complete classification of all baryon states was not given. To do this, we use the basis elements in Table II to diagonalize the three-quark evolution equation, Eq. (3.4). In general, the eigensolutions $\tilde{\phi}_n$ are linear combinations of antisymmetric representations. As a result, we find different γ_n for the nucleon and isobar states. We present the general solutions for each baryon in Table III.

For specific calculations, one does not always have to use the full antisymmetric representation since we can use an effective representation in which the helicity configuration is fixed. As an example, we can choose an $\uparrow\downarrow\uparrow$ term for $S_Z = 1/2$ state as an effective representation. We can also narrow the effective representation by fixing the isospin configuration: e.g. , choose the uud term for $(T, T_Z) = (1/2, 1/2)$ and $S_Z = 1/2$. The orbital wavefunction coefficient then provides a practical representation. However, we must choose a proper term in order to calculate the mixing coefficients. If we use an improper term (such as

the udu term in the above example), then we cannot represent the eigensolution because the mixed states ($S = 3/2$ and $1/2$ states) have the same orbital representation.

4. Mixing and Diagonalization

The lowest power orbital state (O^3 state) of each baryon is unique. These special states are eigensolutions by themselves, giving the eigenvalues γ_0 by Eq. (3.4). However as the orbital power becomes larger, the number of different representations is increased and the mixing between the different representations occurs as described in the last section. Here we describe the mixing of different spin representations for the example given at the end of the last section. We define

$$\phi^{3/2} = \begin{array}{|c|} \hline r \\ \hline y \\ \hline b \\ \hline \end{array} \times \begin{array}{|c|c|} \hline u & u \\ \hline d & \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline \uparrow & \uparrow & \downarrow \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} \quad (4.1)$$

which is the state $(T, T_Z) = (1/2, 1/2)$, $(S, S_Z) = (3/2, 1/2)$ and $n = 1$, and

$$\phi^{1/2} = \begin{array}{|c|} \hline r \\ \hline y \\ \hline b \\ \hline \end{array} \times \begin{array}{|c|c|} \hline u & u \\ \hline d & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} \quad (4.2)$$

which is the state $(T, T_Z) = (1/2, 1/2)$, $(S, S_Z) = (1/2, 1/2)$ and $n = 1$. For convenience, we rewrite Eq. (3.4) as

$$-b \tilde{\phi}(x) = \frac{2C_B}{\beta} \tilde{V}(x, y) \tilde{\phi}(y) \quad , \quad (4.3)$$

where b is defined in

$$\gamma_n \equiv \left(2b_n C_B + \frac{3}{2} C_F \right) / \beta \quad (4.4)$$

and $\tilde{V}(x, y) = \tilde{V}_\delta(x, y) + \tilde{V}_\Delta(x, y)$ is defined by

$$\begin{aligned}\tilde{V}_\delta(x, y) &\equiv \int_0^1 [dy] \sum_{i \neq j} \mathcal{O}(y_i - x_i) \delta(x_k - y_k) \frac{y_j}{x_j} \frac{\delta_{h_i \bar{h}_j}}{x_i + x_j} \\ \tilde{V}_\Delta(x, y) &\equiv \int_0^1 [dy] \sum_{i \neq j} \mathcal{O}(y_i - x_i) \delta(x_k - y_k) \frac{y_j}{x_j} \frac{\Delta}{y_i + x_i}.\end{aligned}\tag{4.5}$$

Putting the y representations of $\phi^{3/2}$ and $\phi^{1/2}$ [see Eqs. (4.1) and (4.2), and Table II] into the three-quark evolution equation [Eqs. (4.3) and (4.5)], we find

$$\tilde{V}_\delta(x, y) \begin{bmatrix} \tilde{\phi}^{3/2}(y) \\ \tilde{\phi}^{1/2}(y) \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \tilde{\phi}^{3/2}(x) \\ \tilde{\phi}^{1/2}(x) \end{bmatrix}$$

and (4.6)

$$\tilde{V}_\Delta(x, y) \begin{bmatrix} \tilde{\phi}^{3/2}(y) \\ \tilde{\phi}^{1/2}(y) \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 0 \\ 0 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} \tilde{\phi}^{3/2}(x) \\ \tilde{\phi}^{1/2}(x) \end{bmatrix},$$

which gives the eigenvalues and eigensolutions after diagonalization as

$$\begin{aligned}b = \frac{2}{3} &\quad \text{for} \quad \tilde{\phi} = \frac{1}{\sqrt{2}} \tilde{\phi}^{3/2} + \frac{1}{\sqrt{2}} \tilde{\phi}^{1/2} \\ b = 1 &\quad \text{for} \quad \tilde{\phi} = -\frac{1}{\sqrt{2}} \tilde{\phi}^{3/2} + \frac{1}{\sqrt{2}} \tilde{\phi}^{1/2}\end{aligned}\tag{4.7}$$

This gives the expected mixing between $\phi^{3/2}$ and $\phi^{1/2}$ through the QCD evolution.

For the antisymmetric representations given by Table II, we find the mixing between the following states: for the nucleon case, $n = 1$ mixes $\phi^{3/2}$ and $\phi^{1/2}$ (the previous example), and $n = 2$ mixes

$$\begin{array}{|c|c|c|} \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array}$$

For the Δ case, $n = 2$ has mixing between

$$\begin{array}{|c|c|c|} \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array}.$$

As in the previous example ($\phi^{3/2}$ and $\phi^{1/2}$), we can diagonalize the mixing matrix $\tilde{V} = \tilde{V}_\delta + \tilde{V}_\Delta$ and find the eigenvalues b and the eigensolutions $\tilde{\phi}$ for all antisymmetric representations as given in Table II. The results for b and $\tilde{\phi}$ are given in Table III.

5. Results and Discussion

We have presented the antisymmetric representations for the three-quark system and shown the action of the three-quark evolution equation. We have used the following properties of the evolution kernel:

1. The unique color singlet state of the baryon is preserved.
2. The isospin state is preserved.
3. S_Z component of the spin state is conserved but different spin multiplets ($S = 3/2$ and $S = 1/2$ states) can mix with each other.
4. $L_Z = 0$ is fixed on the light-cone and n value is conserved, but different orbital multiplets (corresponding to different Young diagrams) can mix with each other.

After the diagonalization of the mixing matrix $\tilde{V} = \tilde{V}_\delta + \tilde{V}_\Delta$, we find the eigenvalues and the eigensolutions as summarized in Table III.

The method by which we obtain the above results is sufficiently straightforward that we can find the basis of the eigensolutions before solving the evolution equation and see the evolution of each state explicitly. Furthermore, we can check every step of the calculation explicitly. For example, the symmetry of the evolution potential given by Eq. (3.2) can be checked by obtaining the matrix representation such as Eq. (4.6) for the mixing of $\tilde{\phi}^{3/2}$ and $\tilde{\phi}^{1/2}$. (Note that the matrices \tilde{V}_δ and \tilde{V}_Δ are symmetric separately.)

From Table III, we find that the eigenvalues and the eigensolutions are consistent with the previous results, but we have the classification in terms of physical baryons. The eigensolutions of the proton with $S_Z = 1/2$ case coincide with the result of Ref.1 for the distribution amplitude for the proton ($h = 1/2$);

$$\begin{aligned}
\phi_p^{1/2} = & \left\{ \left[\frac{d_\uparrow(1) u_\uparrow(3) + u_\uparrow(1) d_\uparrow(3)}{\sqrt{6}} \quad u_\downarrow(2) \right. \right. \\
& \left. \left. - \left(\frac{2}{3} \right)^{1/2} u_\uparrow(1) d_\downarrow(2) u_\uparrow(3) \right] \phi^S(x_i, Q) \right. \\
& \left. + \left[\frac{d_\uparrow(1) u_\uparrow(3) - u_\uparrow(1) d_\uparrow(3)}{\sqrt{2}} \quad u_\downarrow(2) \right] \phi^A(x_i, Q) \right\} \\
& + (1 \leftrightarrow 2) + (2 \leftrightarrow 3) ,
\end{aligned} \tag{5.1}$$

where ϕ^S and ϕ^A are symmetric and antisymmetric under the interchange $x_1 \leftrightarrow x_3$ and the color singlet representation $\epsilon_{ijk}/\sqrt{6}$ is understood. The representation inside the curly bracket turns out to be the same as our effective antisymmetric representation given by Table III. Furthermore, we can give the general

distribution amplitudes for the other baryons such as the excited proton $\phi_p^{3/2}$, and isobars $\phi_\Delta^{1/2}$, $\phi_\Delta^{3/2}$:

$$\begin{aligned} \phi_p^{3/2} &= \frac{1}{2} (d_\uparrow u_\uparrow u_\uparrow - u_\uparrow d_\uparrow u_\uparrow) \phi^\alpha(x_i, Q) \\ &+ \frac{1}{2\sqrt{3}} (d_\uparrow u_\uparrow u_\uparrow + u_\uparrow d_\uparrow u_\uparrow - 2u_\uparrow u_\uparrow d_\uparrow) \phi^\beta(x_i, Q) \end{aligned} \quad , \quad (5.2)$$

where ϕ^α and ϕ^β have the symmetry represented by the Young diagrams

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad ,$$

respectively,

$$\begin{aligned} \phi_\Delta^{1/2} \left(T_Z = \frac{3}{2} \right) &= u_\uparrow u_\downarrow u_\uparrow \phi^S(x_i, Q) + (1 \leftrightarrow 2) + (2 \leftrightarrow 3) \\ \phi_\Delta^{1/2} \left(T_Z = \frac{1}{2} \right) &= \frac{1}{\sqrt{3}} (u_\uparrow u_\downarrow d_\uparrow + u_\uparrow d_\downarrow u_\uparrow + d_\uparrow u_\downarrow u_\uparrow) \phi^S(x_i, Q) \\ &+ (1 \leftrightarrow 2) + (2 \leftrightarrow 3) \end{aligned} \quad (5.3)$$

where ϕ^S is symmetric under the interchange $x_1 \leftrightarrow x_3$, and

$$\begin{aligned} \phi_\Delta^{3/2} \left(T_Z = \frac{3}{2} \right) &= u_\uparrow u_\uparrow u_\uparrow \phi^\gamma(x_i, Q) \\ \phi_\Delta^{3/2} \left(T_Z = \frac{1}{2} \right) &= \frac{1}{\sqrt{3}} (u_\uparrow u_\uparrow d_\uparrow + u_\uparrow d_\uparrow u_\uparrow + d_\uparrow u_\uparrow u_\uparrow) \phi^\gamma(x_i, Q) \end{aligned} \quad , \quad (5.4)$$

where the ϕ^γ are totally symmetric under any interchange between x_1 , x_2 , and x_3 .

As stated in the introduction, we can apply the above method to multi-quark systems which have several color singlet representations.³ In this case a much richer phenomenology of QCD states exists including hidden color configurations. We can also combine this approach with the fractional parentage technique⁵ to predict the effective interaction between baryonic clusters within a multi-quark system.

In conclusion, we have presented a general technique which combined with evolution equations predicts the short-distance behavior and classifies the spectrum of relativistic many-fermion systems. This approach thus provides a fundamental method for studying short-distance dynamics even in the domain of the multi-quark systems of nuclear physics.

References

1. G. P. Lepage and S. J. Brodsky, Phys. Rev. D22, 2157 (1980). For a recent review of light cone quantization and earlier references, see H. C. Pauli and S. J. Brodsky, SLAC-PUB-3701 (1985).
2. M. Peskin, Phys. Lett. 88B, 128 (1979); S. J. Brodsky, Y. Frishman, G. P. Lepage and C. Sachrajda, Phys. Lett. 91B, 239 (1980); A. V. Efremov and A. V. Radyushkin, Phys. Lett. 94B, 245 (1980).
3. S. J. Brodsky and Chueng-Ryong Ji, SLAC-PUB-3707; Chueng-Ryong Ji and S. J. Brodsky, SLAC-PUB-3148
4. For a review, see M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Massachusetts, 1962).
5. M. Harvey, Nucl. Phys. A352, 301 (1981); A352, 326 (1981).

Table I

The orbital representations in the index-power space for the baryon system; the normalization constant is multiplied by the representation for the correct normalization.

| Index-Power | Young Diagram and Representation | Normalization |
|-------------|---|-------------------------------|
| $n = 0$ | $\boxed{000} = 1$ | $\sqrt{5!}$ |
| $n = 1$ | $\begin{array}{ c c } \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} = \begin{cases} \begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \frac{1}{\sqrt{2}} (x_1 - x_2) \\ \\ \begin{array}{ c c } \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = \frac{1}{\sqrt{6}} (x_1 + x_2 - 2x_3) \end{cases}$ | $\sqrt{21} \times \sqrt{5!}$ |
| $n = 2$ | $\boxed{011} = \frac{1}{\sqrt{3}} \left(x_2 x_3 + x_3 x_1 + x_1 x_2 - \frac{2}{7} \right)$ | $42\sqrt{5!}$ |
| $n = 2$ | $\begin{array}{ c c } \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} = \begin{cases} \begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \frac{1}{\sqrt{2}} \left\{ (x_2 x_3 - x_1 x_3) + \frac{1}{4} (x_1 - x_2) \right\} \\ \\ \begin{array}{ c c } \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = \frac{1}{\sqrt{6}} \left\{ (x_2 x_3 + x_1 x_3 - 2x_1 x_2) + \frac{1}{4} (x_1 + x_2 - 2x_3) \right\} \end{cases}$ | $6\sqrt{28} \times \sqrt{5!}$ |

Table II

Completely antisymmetric three-quark representations for A) the N system and B) the Δ system. For all representations, the color singlet representation factor $\epsilon_{ijk}/\sqrt{6}$ given by Eq. (2.1) is abbreviated and the orbital normalization constant given by Table I must be multiplied for the correct normalization. Even though we present the representations for positive T_Z and S_Z values, those for negative T_Z and S_Z values can be simply obtained by the replacement of $u \leftrightarrow d$ and $\uparrow \leftrightarrow \downarrow$, respectively, for the corresponding representations of positive T_Z and S_Z values. For example, $T_Z = -1/2$ representation is obtained by the replacement of $u \leftrightarrow d$ in the $T_Z = 1/2$ representation.

A. N system ($T = 1/2$); $T_Z = 1/2$ is fixed for convenience

Index-Power Symmetry Representation of the Three-Quark System

1. $(S, S_Z) = (3/2, 3/2)$

$$\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} \quad \frac{1}{3\sqrt{2}} (\uparrow\uparrow\uparrow) \quad [duu(2y_1 - y_2 - y_3) \\ + udu(-y_1 + 2y_2 - y_3) \\ + uud(-y_1 - y_2 + 2y_3)]$$

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \quad \frac{1}{3\sqrt{2}} (\uparrow\uparrow\uparrow) \quad [duu\{(2y_2y_3 - y_1y_3 - y_1y_2) + \frac{1}{4}(2y_1 - y_2 - y_3)\} \\ + udu\{(-y_2y_3 + 2y_1y_3 - y_1y_2) + \frac{1}{4}(-y_1 + 2y_2 - y_3)\} \\ + uud\{(-y_2y_3 - y_1y_3 + 2y_1y_2) + \frac{1}{4}(-y_1 - y_2 + 2y_3)\}]$$

2. $(S, S_Z) = (3/2, 1/2)$

$$\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} \quad \frac{1}{3\sqrt{6}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \quad [duu(2y_1 - y_2 - y_3) \\ + udu(-y_1 + 2y_2 - y_3) \\ + uud(-y_1 - y_2 + 2y_3)]$$

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \quad \frac{1}{3\sqrt{6}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \quad [duu\{(2y_2y_3 - y_1y_3 - y_1y_2) + \frac{1}{4}(2y_1 - y_2 - y_3)\} \\ + udu\{(-y_2y_3 + 2y_1y_3 - y_1y_2) + \frac{1}{4}(-y_1 + 2y_2 - y_3)\} \\ + uud\{(-y_2y_3 - y_1y_3 + 2y_1y_2) + \frac{1}{4}(-y_1 - y_2 + 2y_3)\}]$$

3. $(S, S_z) = (1/2, 1/2)$

$$\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline \end{array} \frac{1}{3\sqrt{2}} [(\downarrow\uparrow\uparrow)(2duu - udu - uud) \\ + (\uparrow\downarrow\uparrow)(-duu + 2udu - uud) \\ + (\uparrow\uparrow\downarrow)(-duu - udu + 2uud)] \times 1$$

$$\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} \frac{1}{3\sqrt{6}} [(\downarrow\uparrow\uparrow) \{ duu(-2y_1 + y_2 + y_3) + udu(y_1 + y_2 - 2y_3) + uud(y_1 - 2y_2 + y_3) \} \\ + (\uparrow\downarrow\uparrow) \{ duu(y_1 + y_2 - 2y_3) + udu(y_1 - 2y_2 + y_3) + uud(-2y_1 + y_2 + y_3) \} \\ + (\uparrow\uparrow\downarrow) \{ duu(y_1 - 2y_2 + y_3) + udu(-2y_1 + y_2 + y_3) + uud(y_1 + y_2 - 2y_3) \}]$$

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 1 \\ \hline \end{array} \frac{1}{3\sqrt{6}} [(\downarrow\uparrow\uparrow)(2duu - udu - uud) \\ + (\uparrow\downarrow\uparrow)(-duu + 2udu - uud) \\ + (\uparrow\uparrow\downarrow)(-duu - udu + 2uud)] \times \left(y_2 y_3 + y_3 y_1 + y_1 y_2 - \frac{2}{7} \right)$$

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array} \frac{1}{3\sqrt{6}} \left((\downarrow\uparrow\uparrow) \left[duu \left\{ (-2y_2 y_3 + y_1 y_3 + y_1 y_2) + \frac{1}{4} (-2y_1 + y_2 + y_3) \right\} \right. \right. \\ \left. \left. + udu \left\{ (y_2 y_3 + y_1 y_3 - 2y_1 y_2) + \frac{1}{4} (y_1 + y_2 - 2y_3) \right\} \right. \right. \\ \left. \left. + uud \left\{ (y_2 y_3 - 2y_1 y_3 + y_1 y_2) + \frac{1}{4} (y_1 - 2y_2 + y_3) \right\} \right] \right. \\ \left. + (\uparrow\downarrow\uparrow) \left[duu \left\{ (y_2 y_3 + y_1 y_3 - 2y_1 y_2) + \frac{1}{4} (y_1 + y_2 - 2y_3) \right\} \right. \right. \\ \left. \left. + udu \left\{ (y_2 y_3 - 2y_1 y_3 + y_1 y_2) + \frac{1}{4} (y_1 - 2y_2 + y_3) \right\} \right. \right. \\ \left. \left. + uud \left\{ (-2y_2 y_3 + y_1 y_3 + y_1 y_2) + \frac{1}{4} (-2y_1 + y_2 + y_3) \right\} \right] \right. \\ \left. + (\uparrow\uparrow\downarrow) \left[duu \left\{ (y_2 y_3 - 2y_1 y_3 + y_1 y_2) + \frac{1}{4} (y_1 - 2y_2 + y_3) \right\} \right. \right. \\ \left. \left. + udu \left\{ (-2y_2 y_3 + y_1 y_3 + y_1 y_2) + \frac{1}{4} (-2y_1 + y_2 + y_3) \right\} \right. \right. \\ \left. \left. + uud \left\{ (y_2 y_3 + y_1 y_3 - 2y_1 y_2) + \frac{1}{4} (y_1 + y_2 - 2y_3) \right\} \right] \right)$$

B. Δ system ($T = 3/2$); for $T_Z = 3/2$, $\begin{bmatrix} u & u & u \end{bmatrix} = uuu$, and for $T_Z = 1/2$, $\begin{bmatrix} u & u & d \end{bmatrix} = [(1/\sqrt{3})(uud + udu + duu)]$ are factors for all representations.

Index-Power Symmetry Representation of the Three-Quark System

1. $(S, S_Z) = (3/2, 3/2)$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} (\uparrow\uparrow\uparrow) \times 1$$

$$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \frac{1}{\sqrt{3}} (\uparrow\uparrow\uparrow) \left(y_2 y_3 + y_3 y_1 + y_1 y_2 - \frac{2}{7} \right)$$

2. $(S, S_Z) = (3/2, 1/2)$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \frac{1}{\sqrt{3}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \times 1$$

$$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \frac{1}{3} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \left(y_2 y_3 + y_3 y_1 + y_1 y_2 - \frac{2}{7} \right)$$

3. $(S, S_Z) = (1/2, 1/2)$

$$\begin{bmatrix} 0 & 0 \\ 1 & \end{bmatrix} \frac{1}{3\sqrt{2}} [(\downarrow\uparrow\uparrow)(2y_1 - y_2 - y_3) + (\uparrow\downarrow\uparrow)(-y_1 + 2y_2 - y_3) + (\uparrow\uparrow\downarrow)(-y_1 - y_2 + 2y_3)]$$

$$\begin{bmatrix} 0 & 1 \\ 1 & \end{bmatrix} \frac{1}{3\sqrt{2}} \left[(\downarrow\uparrow\uparrow) \left\{ 2y_2 y_3 - y_1 y_3 - y_1 y_2 + \frac{1}{4}(2y_1 - y_2 - y_3) \right\} + (\uparrow\downarrow\uparrow) \left\{ -y_2 y_3 + 2y_1 y_3 - y_1 y_2 + \frac{1}{4}(-y_1 + 2y_2 - y_3) \right\} + (\uparrow\uparrow\downarrow) \left\{ -y_2 y_3 - y_1 y_3 + 2y_1 y_2 + \frac{1}{4}(-y_1 - y_2 + 2y_3) \right\} \right]$$

Table III

Eigenvalues and eigensolutions for A) N system and B) Δ system: the anomalous dimensions are related to b by Eq. (4.4), i.e., $\gamma = (2bC_B + 3C_F/2)/\beta$. The normalization factor \sqrt{N} for the effective representation is also given.

A. N system ($T = 1/2$); $T_Z = 1/2$ is fixed.

| Spin Configuration | b | Spin \times Orbital | \sqrt{N} | $\tilde{\phi}(y)$ (Effective Representation) |
|------------------------------|----------------|--|-------------------------------|--|
| $\uparrow\uparrow\uparrow$ | $\frac{3}{2}$ | $\begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array}$ | $\sqrt{21} \times \sqrt{5!}$ | $\frac{1}{3\sqrt{2}} [duu(2y_1 - y_2 - y_3) + udu(-y_1 + 2y_2 - y_3) + uud(-y_1 - y_2 + 2y_3)]$ |
| $\uparrow\uparrow\uparrow$ | $\frac{17}{6}$ | $\begin{array}{ c c c } \hline \uparrow & \uparrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 & \\ \hline \end{array}$ | $6\sqrt{28} \times \sqrt{5!}$ | $\frac{1}{3\sqrt{2}} [duu\{(2y_2y_3 - y_1y_3 - y_1y_2) + \frac{1}{4}(2y_1 - y_2 - y_3)\} + udu\{1 \leftrightarrow 2\} + uud\{1 \leftrightarrow 3\}]$ |
| $\uparrow\downarrow\uparrow$ | -1 | $\begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 0 & 0 \\ \hline \end{array}$ | $\sqrt{5!}$ | $\frac{1}{\sqrt{6}} (-duu + 2udu - uud) \times 1$ |
| $\uparrow\downarrow\uparrow$ | $\frac{2}{3}$ | $\frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} + \frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array}$ | $\sqrt{21} \times \sqrt{5!}$ | $\frac{1}{2} (duu - uud) (y_1 - y_3)$ |
| $\uparrow\downarrow\uparrow$ | 1 | $-\frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array} + \frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow & \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 0 \\ \hline 1 & \\ \hline \end{array}$ | $\sqrt{21} \times \sqrt{5!}$ | $\frac{1}{6} (duu - 2udu + uud) (-y_1 + 2y_2 - y_3)$ |

| Spin Configuration | b | Spin × Orbital | \sqrt{N} | $\tilde{\phi}(y)$ (Effective Representation) |
|------------------------------|---------------|---|----------------------------------|--|
| $\uparrow\downarrow\uparrow$ | $\frac{5}{3}$ | $+\frac{1}{\sqrt{30}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ $-\sqrt{\frac{28}{30}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 1 & 1 \\ \hline \end{array}$ $-\sqrt{\frac{1}{30}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ | $6\sqrt{28} \times \sqrt{5!}$ | $\frac{1}{6\sqrt{15}} (duu - 2udu + uud)$ $\times \{-6y_2y_3 - 9y_1y_3 - 6y_1y_2$ $+ \frac{1}{4} (9y_1 + 6y_2 + 9y_3)\}$ |
| $\uparrow\uparrow\uparrow$ | $\frac{7}{3}$ | $\frac{1}{\sqrt{2}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ $+\frac{1}{\sqrt{2}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ | $6\sqrt{28} \times \sqrt{5!}$ | $\frac{1}{2} (duu - uud)$ $\times \left\{ y_2y_3 - y_1y_2 + \frac{1}{4} (y_1 - y_3) \right\}$ |
| $\uparrow\downarrow\uparrow$ | $\frac{5}{2}$ | $-\sqrt{\frac{7}{15}} \begin{array}{ c c c } \hline \uparrow & \downarrow & \uparrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ $-\sqrt{\frac{1}{15}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c c } \hline 0 & 1 & 1 \\ \hline \end{array}$ $+\sqrt{\frac{7}{15}} \begin{array}{ c c } \hline \uparrow & \uparrow \\ \hline \downarrow \\ \hline \end{array} \times \begin{array}{ c c } \hline 0 & 1 \\ \hline 1 \\ \hline \end{array}$ | $\frac{42}{\sqrt{15}} \sqrt{5!}$ | $\frac{1}{3\sqrt{2}} (duu - 2udu + uud)$ $\times (-3y_2y_3 + 3y_1y_3 - 3y_1y_2$ $+ \frac{2}{7} - \frac{1}{2} y_1 + y_2 - \frac{1}{2} y_3)$ |

B. Δ system ($T = 3/2$); for $T_Z = 3/2$, $\boxed{u} \boxed{u} \boxed{u} = uuu$, and for $T_Z = 1/2$, $\boxed{u} \boxed{u} \boxed{d} = [(1/\sqrt{3})(uud + udu + duu)]$ are factors for the effective representations.

| Spin Configuration | b | Spin \times Orbital | \sqrt{N} | $\tilde{\phi}(y)$ (Effective Representation) |
|------------------------------|---------------|---|---|---|
| $\uparrow\uparrow\uparrow$ | 0 | $\boxed{\uparrow\uparrow\uparrow} \times \boxed{000}$ | $\sqrt{5!}$ | 1 |
| $\uparrow\uparrow\uparrow$ | $\frac{7}{3}$ | $\boxed{\uparrow\uparrow\uparrow} \times \boxed{011}$ | $42\sqrt{5!}$ | $\frac{1}{3} \left(y_2 y_3 + y_3 y_1 + y_1 y_2 - \frac{2}{7} \right)$ |
| $\uparrow\downarrow\uparrow$ | -1 | $\boxed{\uparrow\downarrow\uparrow} \times \boxed{000}$ | $\sqrt{5!}$ | 1 |
| $\uparrow\downarrow\uparrow$ | 1 | $\boxed{\uparrow\uparrow} \times \boxed{00}$ $\boxed{\downarrow} \times \boxed{1}$ | $\sqrt{21} \times \sqrt{5!}$ | $\frac{1}{6} (-y_1 + 2y_2 - y_3)$ |
| $\uparrow\downarrow\uparrow$ | $\frac{5}{3}$ | $\sqrt{\frac{14}{15}} \boxed{\uparrow\downarrow\uparrow} \times \boxed{011}$ $+ \sqrt{\frac{1}{15}} \boxed{\uparrow\uparrow} \times \boxed{01}$ $\boxed{\downarrow} \times \boxed{1}$ | $\frac{6\sqrt{14} \times \sqrt{5!}}{\sqrt{15}}$ | $\sqrt{3} \left(2y_2 y_3 + 3y_1 y_3 + 2y_1 y_2 - \frac{3}{4} y_1 - \frac{1}{2} y_2 - \frac{3}{4} y_3 \right)$ |
| $\uparrow\downarrow\uparrow$ | $\frac{5}{2}$ | $-\sqrt{\frac{1}{15}} \boxed{\uparrow\downarrow\uparrow} \times \boxed{011}$ $+ \sqrt{\frac{1}{15}} \boxed{\uparrow\uparrow} \times \boxed{01}$ $\boxed{\downarrow} \times \boxed{1}$ | $\frac{42 \times \sqrt{5!}}{\sqrt{15}}$ | $\frac{1}{\sqrt{3}} \left(-3y_2 y_3 + 3y_1 y_3 - 3y_1 y_2 + \frac{2}{7} - \frac{1}{2} y_1 + y_2 - \frac{1}{2} y_3 \right)$ |