

**PION-SKYRMION SCATTERING:  
COLLECTIVE COORDINATES AT WORK\***

Michael E. Peskin

Stanford Linear Accelerator Center  
Stanford University, Stanford, California 94305

## Introduction

The past two years have seen a new assault on one of the basic questions of high-energy physics, the question of the structure of the nucleon. The new developments involve an old model of the nucleon, one introduced by Skyrme in 1961<sup>1</sup> and seemingly made irrelevant a few years later by the invention of the quark model. It is more accurate, though, to say that the virtues of Skyrme's model went unappreciated by most field theorists until it was revived by the recent work of Balachandran, Nair, Rajeev, and Stern<sup>2</sup> and Witten.<sup>3</sup> The explosion of interest in this model which followed these papers has established the Skyrme model as a picture of the nucleon from which one can derive quantitative and quite nontrivial predictions.

In this lecture, I have chosen to discuss in detail one particular set of results which have been derived from the Skyrme model, results on the  $S$ -matrix for pion-nucleon scattering. I have chosen this topic, first, because I have been involved in it directly, but also, more importantly, because these results provide a direct and pleasing link between very abstract notions of field theory and concrete results of experiment. The analysis which I will discuss is due to Hayashi, Eckart, Holzwarth, and Waliser,<sup>4</sup> at Siegen, and to Mattis, Karliner, and myself,<sup>5,6</sup> at SLAC. A detailed survey of the applications of the Skyrme model has recently been presented by Nappi;<sup>7</sup> the reader should look there for an overview of the new results which have been obtained from Skyrme's picture.

The Skyrme model is a particularly appropriate topic for a symposium in honor of Niels Bohr because it is a striking example of complementarity, both in itself and in its relation to other aspects of pion-nucleon physics. It is, first of all, a picture of the nucleon which is complementary to the more conventional quark picture, in that it pulls new information from the same underlying theory by utilizing a different set of approximations. To explain this, I should first explain the other sense in which the model is complementary: it is the

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simultaneous realization of two quite different viewpoints on what makes the nucleon large and stable.

The first of these viewpoints originated with Skyrme<sup>1</sup> and Finkelstein and Rubinstein.<sup>8</sup> These authors put forward the suggestion that conserved quantum numbers arise in physics for topological reasons, that particles which carry conserved quantum numbers are built up from classical fields of nontrivial topology. This idea has a particularly straightforward realization the strong interactions with two light quark flavors. The global symmetry of the model is chiral  $SU(2) \times SU(2)$ ; this symmetry is spontaneously broken to  $SU(2)$  isospin. The symmetry-breaking can be described phenomenologically by parametrizing the local orientation of the chiral symmetry breaking condensate by an  $SU(2)$  matrix  $U(x)$ . The fluctuations of  $U(x)$  about a constant value are Goldstone boson excitations; these are the pions:

$$U(x) = \exp\left(\frac{2}{f_\pi} \vec{\pi}(x) \cdot \vec{\tau}\right). \quad (1)$$

The space of values of  $U(x)$ , the group  $SU(2)$ , is isomorphic to the 3-dimensional sphere; by mapping the 3-sphere onto 3-dimensional space (*e.g.*, sending the north pole to infinity and the south pole to the origin), one can define topologically nontrivial maps from  $SU(2)$  onto 3-space. The map in this topological class which gives the field configuration of lowest energy defines a stable, finite-energy excitation of the vacuum. Skyrme proposed to identify this object with the nucleon and the conserved topological quantum number with baryon number. Witten<sup>3</sup> has bolstered this interpretation by showing, after a remarkable analysis, that, in the phenomenological theory derived from QCD with three light flavors, an adiabatic rotation of Skyrme's object through  $2\pi$  produces a phases  $(-1)^N$ , where  $N$  is the number of colors in QCD. Thus, Skyrme's object is a fermion or a boson in accord with the properties of nucleon in  $N$ -color QCD.

The second viewpoint is due to Witten<sup>9</sup> and represents his attempt to find the systematics of baryons in QCD as the number of colors is taken to infinity. The theory of mesons was shown by 't Hooft<sup>10</sup> to be very simple in this limit; as I will discuss a bit later, mesons become well-defined, narrow states with interactions proportional to  $1/N$ . Witten proposed that baryons be viewed as the solitons of this weakly-interacting theory. He noted that the basic relations which are true of solitons in a theory with expansion parameter  $\alpha$ , that the mass of the soliton is proportional to  $\alpha^{-1}$ , and that the radius of the soliton and the soliton-meson cross-sections are independent of  $\alpha$ , are true in QCD at large  $N$  if  $\alpha$  is replaced by the expansion parameter  $1/N$ .

If the two pictures connect, the baryon must arise as a soliton in a field with the transformation law of the matrix  $U(x)$  which may be described semiclassically in the limit  $N \rightarrow \infty$ . We can see how this field arises in two complementary ways. First, consider a quark-model description. For  $N$  large, the nucleon is built of  $N$  quarks occupying the same wavefunction. This wavefunction has some fixed radial dependence and also carries spin and isospin indices:  $\Psi_a^i(\mathbf{r})$ . As  $N \rightarrow \infty$ ,  $\Psi_a^i(\mathbf{r})$  becomes macroscopically occupied and takes on classical significance. Alternatively, we might take a pion viewpoint. Consider constraining the quark bilinears  $\bar{q}q$  and  $\bar{q}\gamma^5\tau^a q$  to have fixed values:

$$\langle \bar{q}q + \bar{q}\gamma^5\tau^a q \cdot \vec{\sigma} \rangle = \Delta \cdot U(x), \quad (2)$$

and then integrating out the quark and gluon fields. As  $N \rightarrow \infty$ , the leading contributions to this integration come from planar diagrams with external insertions of  $U(x)$ ; a typical such diagram is shown in Fig. 1. Each of these diagrams is of order  $N$ ; thus, the sum yields an effective action of the form:

$$\mathcal{L}_{\text{eff}} = N \left\{ \hat{f}^2 \text{tr} [\partial_\mu U^\dagger \partial^\mu U] + \hat{e}^2 \text{tr} [\partial_\mu U^\dagger \partial_\nu U \partial^\mu U^\dagger \partial^\nu U] + \dots \right\}, \quad (3)$$

where the parameters  $\hat{f}$ ,  $\hat{e}$ ,  $\dots$ , are independent of  $N$ . In this expression,  $1/N$  stands in the place of Planck's constant, so that the  $U$  field becomes classical as  $N \rightarrow \infty$ .

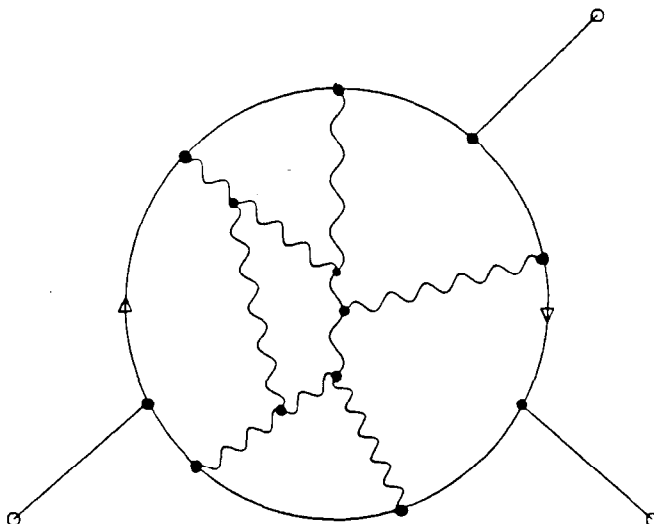


Fig. 1. A typical diagram contributing to the effective action for  $U(x)$  to the leading order in  $1/N$ .

The classical field configurations which emerge from these two arguments are not obviously equivalent, but they do clearly share a common feature: The macroscopic wavefunction of the quark picture can be written as

$$\Psi_a^i = [\psi_0(r) + \vec{\psi}(r) \cdot \sigma]_a^i, \quad (4)$$

and the topologically nontrivial  $U$  field configuration can be written as

$$U_0(x) = \exp\{iF(r)\hat{r} \cdot \sigma\}. \quad (5)$$

In each expression, the classical wavefunction defines a coordinate system for isospin space in terms of axes in physical space. Each is transformed by the separate action of  $\vec{I}$  and  $\vec{J}$  but is invariant to the combined action  $\vec{I} + \vec{J}$ .

Gervais and Sakita<sup>11</sup> and Manohar<sup>12</sup> have emphasized that this group-theoretic property leads directly to many of the pleasing phenomenological results of the Skyrme model. The symmetry  $\vec{I} + \vec{J}$ , which we call  $\vec{K}$ , will in fact play a central role in the analysis I will present here. More model-specific properties of the Skyrme model also tend to work quite well, however, as was first emphasized by the detailed results on the static properties of nucleons presented by Adkins, Nappi, and Witten.<sup>13</sup> I will try to strike a balance, then, between purely group-theoretic and more scheme-dependent results concerning pion-nucleon scattering.

### Baryon resonances in large- $N$ QCD

Since I will use  $1/N$  as an expansion parameter, we should begin this analysis by discussing qualitatively the properties of the nucleon and its excited states as  $N \rightarrow \infty$ . These properties will guide us in setting up a proper analysis and in simplifying it to the greatest extent possible.

We should first recall the basis of 't Hooft's result that mesons become weakly coupled in the large- $N$  limit.<sup>10</sup> Let us take as our starting point the effective Lagrangian (3), which was derived by summing the planar diagrams. (A more general argument would involve an effective Lagrangian involving fields which create higher-mass mesons.) Expand  $U = \exp(i\vec{\Pi} \cdot \vec{\sigma})$  in terms of pion fields, and renormalize the pion fields by the replacement

$$\Pi(x) = \pi(x) \cdot \frac{C}{\sqrt{N}} \quad (6)$$

in order to normalize the kinetic energy term to  $\int \frac{1}{2}(\partial_\mu \pi)^2$ . (Our formula (1) contains this prescription, since, for large  $N$ ,  $f_\pi \sim \sqrt{N}$ .) Then the terms cubic and quartic in pion

fields have coefficients proportional to  $1/\sqrt{N}$  and  $1/N$ , respectively, and so the width and scattering cross-sections of mesons fall off as

$$\Gamma \sim \frac{1}{N}, \quad \sigma \sim \frac{1}{N^2}. \quad (7)$$

For baryons, analogous results apply. The expansion of  $\mathcal{L}_{\text{eff}}$  in small fluctuations  $\delta\Pi(x)$  about a classical solution  $U_0(x)$  takes the form

$$\mathcal{L}_{\text{eff}} = N \left\{ S_0[U_0] + \int \delta\Pi \cdot \mathcal{L}_2[U_0] \cdot \delta\Pi + \int \mathcal{L}_3 \cdot \delta\Pi \cdot \delta\Pi \cdot \delta\Pi + \dots \right\} \quad (8)$$

Again, one should rescale  $\delta\Pi$  according to (6); after this rescaling, one can see that the cubic and higher terms are small in the large- $N$  limit. However, the quadratic term, which is of order 1 in powers of  $1/N$ , already generates nontrivial scattering of pions from the soliton. Excited states of the nucleon appear as resonances in this scattering process. These states are not particularly narrow or long-lived; their features follow the motion of the pion-soliton phase shifts. In this respect, the results of the leading order in  $1/N$  contrast markedly with those of the nonrelativistic quark model, in which baryon resonances are zero-width states to leading order.

I have already remarked that, in the field of a soliton of fixed orientation,  $\vec{I}$  and  $\vec{J}$  are not necessarily conserved, since only their combination  $\vec{K} = \vec{I} + \vec{J}$  leaves the soliton invariant. This implies that the eigenfunctions of  $\mathcal{L}_2[U_0]$ , and, thus, the nucleon resonances, are classified only by  $\vec{K}$ . The idea of testing the Skyrme model by assigning resonances to  $\vec{K}$  channels has been explored by several groups;<sup>14-16</sup> Hayashi and Holzwarth<sup>16</sup> have given a particularly complete catalogue. However, since resonances are not necessarily narrow in this model, it is useful to develop a formalism which treats both resonant and nonresonant  $\vec{K}$  channels contributing to the same physical pion-nucleon amplitude.

Such an analysis is possible because the limit  $N \rightarrow \infty$  allows some additional simplification in the way one connects the stability matrix  $\mathcal{L}_2[U]$  to the amplitudes for pion-nucleon scattering. To see this, let us first recall that nucleons arise in the Skyrme model as rotational eigenstates of the soliton:<sup>13</sup> Since the soliton is not invariant to spatial rotations, one can obtain a family of soliton solutions, all degenerate with a given solution  $U_0$ , by rotating  $U_0$  through an arbitrary set of angles. We may write these solutions as

$$U(A) = AU_0A^{-1}, \quad (9)$$

where  $A$  is an  $SU(2)$  matrix acting on  $\vec{\sigma}$  in (5). To break the degeneracy, one treats  $A$  as a set of canonical coordinates (collective coordinates). Since the energy of the soliton is

independent of  $A$ , one should diagonalize the corresponding canonical momenta. For the problem at hand,  $A$  gives the coordinates of a symmetric top; diagonalizing the angular momenta conjugate to  $A$  yields the eigenfunctions of the symmetric top, characterized by  $I = J$  and independently chosen  $I_z, J_z$ . For the case of two quark flavors, it is permissible to quantize  $I$  and  $J$  as integers or as half-integers.<sup>8</sup> To describe baryons, we make the latter choice and identify the  $I = J = \frac{1}{2}$  and  $I = J = \frac{3}{2}$  states with the  $N$  and  $\Delta$ , respectively.

As  $N \rightarrow \infty$ , both the mass and the moment of inertia of the soliton increase as  $N^1$ . Therefore, the translational and rotational energies of the soliton

$$\frac{p^2}{2M}, \quad \frac{J(J+1)}{2I} \quad (10)$$

are of order  $1/N$ , and so the corresponding velocities are of order  $1/N$ . Thus, if the energy of the incident pion is of order 1 (that is, of order  $m_\rho$ , or another typical meson mass), we may neglect the motion of the soliton during the scattering process to leading order in  $1/N$ . To compute the amplitude for pion-nucleon scattering, then, we need only decompose a nucleon state into soliton states of fixed orientation, compute the pion-soliton  $S$ -matrix in these fixed orientation states by diagonalizing  $\mathcal{L}_2[U(A)]$ , and then combine the results coherently to form a final-state baryon of definite spin and isospin. Let me now explain how to carry out this program explicitly.

### Reduction of the pion-nucleon $S$ -matrix

Let us first work out the general properties of the pion-nucleon  $S$ -matrix which follow only from the symmetries of the problem. Following the program sketched in the previous section, we can perform this analysis in two steps, first, the computation of  $S$  for a fixed orientation of the soliton, and, afterward, the reconstruction of initial and final states of fixed spin and isospin. This treatment is quite similar to the analysis of excitations of deformed nuclei in the adiabatic approximation of Bohr and Mottelson.<sup>17</sup>

Let us begin by considering the soliton to be fixed in a particular orientation, that of the reference configuration  $U_0$  given by (5). In this background, the wavefunctions of pions may be classified by the conserved quantum numbers  $K, K_z, I, L$ , where  $I$  and  $L$  refer to the pion alone ( $I = 1$ ), and  $\vec{K} = \vec{I} + \vec{L}$ . The dependence on isospin and angle of such a state is that of a vector spherical harmonic

$$(\Pi_L^{KK_z}(\Omega))^{i_z} = \langle L1 K_z - i_z i_z | L1 K K_z \rangle Y_{L K_z - i_z}(\Omega). \quad (11)$$

Three values of  $L$  couple to each  $K$ . However, since the parity of a pion wavefunction is  $(-1)^{L-1}$ , the states with  $L = K$  do not mix with those of  $L = K \pm 1$ . Now imagine

that we have computed the  $S$ -matrix in a given  $K$  channel; we can represent this by a reduced  $S$ -matrix  $s_{KL'L}$ , where  $L'$  and  $L$  are, respectively, the final and initial pion angular momenta. The  $S$ -matrix is then given, in a basis of definite  $I, L$ , by

$$S_{(L'L_z I'; LL_z I_z)} = \sum_{K, K_z} \langle L'_z I'_z | K K_z \rangle s_{KL'L} \langle K K_z | L_z I_z \rangle. \quad (12)$$

In this equation, and henceforth, primes denote the quantum numbers of the final state.

To find the  $S$ -matrix for a different fixed orientation of the soliton, we need only rotate the soliton through  $A$ , or, alternatively, rotate the pions backwards through  $A$ . If we denote the matrix representation of  $A$  on a state of spin  $s$  by  $A_{(s)}$ , the rotated  $S$ -matrix is given by

$$S_{(L'L_z I'; LL_z I_z)}^{(A)} = \sum_{K, K_z} A_{(1)}^{I'_z I'_z} \langle L'_z \bar{I}'_z | K K_z \rangle s_{KL'L} \langle K K_z | L_z \bar{I}_z \rangle A_{(1)}^{-1 I_z I_z}. \quad (13)$$

Let us now project this amplitude onto external nucleon states, or, more generally, onto external baryons chosen from the tower of rotational excitations. Denote any one of these baryons by  $I = J = R$ . The corresponding wavefunction is given by the symmetric-top eigenfunction:<sup>13</sup>

$$\langle A | R i_z s_z \rangle = \frac{i}{\pi} \sqrt{\frac{2R+1}{2}} (\epsilon_{(R)} A_{(R)}^{-1})_{s, i_z}, \quad (14)$$

where  $\epsilon_{ab} = (-1)^a \delta(a+b)$  serves to lower the first index. As a shorthand, let me denote the action of  $\epsilon_{(R)}$  by writing this index as  $\tilde{s}_z$ . The physical  $S$ -matrix

$$S(\pi(LL_z I_z)B(R i_z s_z)) \rightarrow \pi(L'L'_z I'_z)B(R' i'_z s'_z) \quad (15)$$

is given by the matrix element of (13) between states (14).

I will give only the most important steps in the evaluation of this matrix element. Write the matrix element as an integral over  $A$ , and project the initial and final states onto definite total isospin. This gives an expression of the form

$$\int dA \langle \mathbf{I}' I'_z | i'_z I'_z \rangle (A_{(R')})_{i'_z \tilde{s}'_z} (A_{(1)})_{I'_z I'_z} \cdots \cdots (A_{(1)}^{-1})_{\tilde{I}_z I_z} (A_{(R')}^{-1})_{\tilde{s}_z i_z} \langle i_z I_z | \mathbf{I} I_z \rangle \quad (16)$$

The matrices  $A$  act in reducible representations of isospin which are then projected onto irreducible representations  $\mathbf{I}', \mathbf{I}$ . We could, alternatively, reduce first and then rotate. This

converts (16) to

$$\int dA (A_{(I)})_{I_1 \bar{I}_1} \langle I_1 \bar{I}_1 | \tilde{s}'_z I'_z \rangle \dots \langle \tilde{s}_z \bar{I}_z | I \bar{I}_z \rangle (A_{(R)}^{-1})_{I_1 \bar{I}_1} \quad (17)$$

Now only two  $A$ 's appear, and so we can use orthogonality to perform the integral. This procedure yields the structure

$$\frac{2\pi^2}{2I+1} \delta_{I_1 I} \delta_{I_1 \bar{I}_1} \cdot \delta_{I_1 \bar{I}_1} \dots \quad (18)$$

The first two delta functions impose isospin conservation in the reaction; this conservation law was not manifest at any earlier stage of the calculation.

Now project the initial and final states onto definite total angular momentum. The resulting expression has three Clebsch-Gordan coefficients for each side of the reaction. For the initial state, these are:

$$\langle K K_z | L_z \bar{I}_z \rangle \langle \tilde{s}_z \bar{I}_z | I \bar{I}_z \rangle \langle s_z L_z | J J_z \rangle; \quad (19)$$

the first factor comes from (13), the second from (17). (19) is of exactly the right form to be reassembled into a 6- $j$  symbol and one remaining Clebsch-Gordan coefficient:

$$\begin{pmatrix} I & J & K \\ \bar{I}_z & J_z & -K_z \end{pmatrix} \cdot \left\{ \begin{matrix} I & J & K \\ L & I=1 & R \end{matrix} \right\} \quad (20)$$

Since  $K, K_z$  are the same in the initial and final states, the sum over  $\bar{I}_z, \bar{I}'_z$ , constrained by the third Kronecker delta of (18), becomes the orthogonality sum for the remaining two 3- $j$  symbols. This produces

$$\delta_{J' J} \delta_{J'_z J_z}, \quad (21)$$

the reflection of angular momentum conservation, which was also not obvious at the early stages of the calculation. The final result is

$$S_{L'R'LRIJ} = \sum_K (-1)^{R-R'} \sqrt{(2R'+1)(2R+1)(2K+1)} \cdot \left\{ \begin{matrix} I & J & K \\ L' & 1 & R' \end{matrix} \right\} s_{KL'L} \left\{ \begin{matrix} I & J & K \\ L & 1 & R \end{matrix} \right\} \quad (22)$$

This expression has the form one might have expected from a nonrelativistic  $SU(6)$  (actually  $SU(4)$ ) analysis, except that the  $SU(4)$  Clebsch-Gordan coefficients have been replaced by  $SU(2)$  6- $j$  symbols. The pattern of combination of the 6 angular momenta is illustrated in Fig. 2.



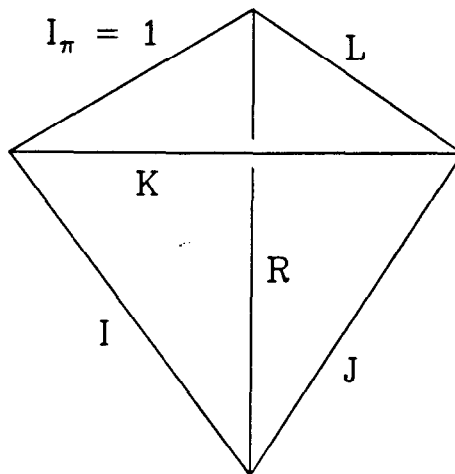


Fig. 2. Relation of the 6 angular momenta in the initial or final state of pion-Skyrmion scattering.

### Model-Independent Results

We have now derived the general structure of the pion-nucleon elastic scattering amplitude, and of the amplitudes for some related quasielastic processes, to leading order in  $1/N$ . Though I have presented this analysis in the language of the Skyrme model, it actually applies to any description of the nucleon as a soliton which is invariant to motions generated by  $\vec{K}$  but not by  $\vec{I}$  and  $\vec{J}$  separately. Had we taken the model of eq. (4) as our starting point, we would have reached the same conclusion. The result (22), then, can be viewed as a consequence of the validity of the  $1/N$  expansion for the description of baryons, independently of the details of any particular model. It is therefore interesting to ask if there is experimental evidence for the structure displayed in (22).

Let us write (22) more explicitly for the special case of pion-nucleon elastic scattering. For each value of  $L$ , there are four possible channels  $(\mathbf{I}, \mathbf{J})$ , corresponding to  $\mathbf{I} = \frac{1}{2}, \frac{3}{2}$  and  $\mathbf{J} = L \pm \frac{1}{2}$ . These channels couple, in general, to all three possible values of  $K$ :  $K = L, L \pm 1$ . Parity and angular momentum conservation insist that  $L$  is conserved in pion-nucleon elastic scattering. Thus, the full set of equations involves only three reduced  $S$ -matrix elements:  $s_{KLL}$  for  $K = L, L \pm 1$ . Let us abbreviate this quantity as  $s_K$ , with the correct value substituted for  $K$ . Then the four channels have partial-wave amplitudes given by:

$$\begin{aligned}
S_{\frac{1}{2}L-\frac{1}{2}} &= \frac{2L-1}{3L}s_{L-1} + \frac{L+1}{3L}s_L \\
S_{\frac{1}{2}L+\frac{1}{2}} &= \frac{L}{3L+1}s_L + \frac{2L+3}{3L+3}s_{L+1} \\
S_{\frac{3}{2}L-\frac{1}{2}} &= \frac{(2L-1)(L-1)}{6L(2L+1)}s_{L-1} + \frac{2L-1}{6L}s_L + \frac{2L+3}{4L+2}s_{L+1} \\
S_{\frac{3}{2}L+\frac{1}{2}} &= \frac{2L-1}{4L+2}s_{L-1} + \frac{2L+3}{6L+6}s_L + \frac{(L+2)(2L+3)}{(6L+6)(2L+1)}s_{L+1}
\end{aligned} \tag{23}$$

Since there are four equations in three unknown  $s_K$ 's, there should be a model-independent linear relation among the  $S_{IJ}$ 's. Actually, there are two, and they allow us to express both isospin- $\frac{3}{2}$  amplitudes in terms of energy-independent linear combinations of the isospin- $\frac{1}{2}$  amplitudes:

$$\begin{aligned}
S_{\frac{3}{2}L-\frac{1}{2}} &= \frac{L-1}{4L+2}S_{\frac{1}{2}L-\frac{1}{2}} + \frac{3L+3}{4L+2}S_{\frac{1}{2}L+half} \\
S_{\frac{3}{2}L+\frac{1}{2}} &= \frac{3L}{4L+2}S_{\frac{1}{2}L-\frac{1}{2}} + \frac{L+2}{4L+2}S_{\frac{1}{2}L+\frac{1}{2}}
\end{aligned} \tag{24}$$

The relations (24) can be checked directly against experimental data, or, rather, against sets of partial wave amplitudes extracted from experiments on pion-nucleon elastic scattering. The most complete compilation and analysis of the data for this reaction has been done by Höhler, Kaiser, Koch, and Pietarinen.<sup>18</sup> Using their partial wave amplitudes to give the values of  $S_{\frac{1}{2}J}$  and  $S_{\frac{3}{2}J}$ , we can directly compare the left and right-hand sides of these two relations. Some representative results are shown in Figs. 3-5. (The complete set of comparisons, up to K-waves, is presented in ref. 5.) In these figures, the solid lines represent the isospin- $\frac{3}{2}$  amplitudes and the dotted lines represent the particular linear combinations of isospin- $\frac{1}{2}$  amplitudes dictated by eq. (24). In these figures, and henceforth, I refer to the partial wave amplitude  $S_{\frac{1}{2}k}(L)$  as  $Ljk$  (e.g.,  $S_{\frac{3}{2}\frac{1}{2}}(L=1) = P31$ ).

Clearly, the linear relations do not work especially well for the P waves. They actually are comparably bad for the S and D waves. A sign of what is wrong can be seen from the P33 comparison: The true P33 amplitude includes a complete tour of the Argand circle; this is the  $\Delta$ . In the amplitude as reconstructed from  $I = \frac{1}{2}$  amplitudes, there is a small bump at relatively low energy, which is the Roper resonance  $N(1410)$  trying feebly to simulate the  $\Delta$ . In the F and H waves, however, the reconstructed curve traces out every nuance of the true isospin- $\frac{3}{2}$  amplitude. The only major difference between the two sets of curves is a small ( $\sim 200$  MeV) displacement of the energy scale. But this is actually to be expected: This displacement presumably comes from the rotational energy term which produces the  $N$ - $\Delta$  mass splitting; like that splitting, such a contribution would be of order

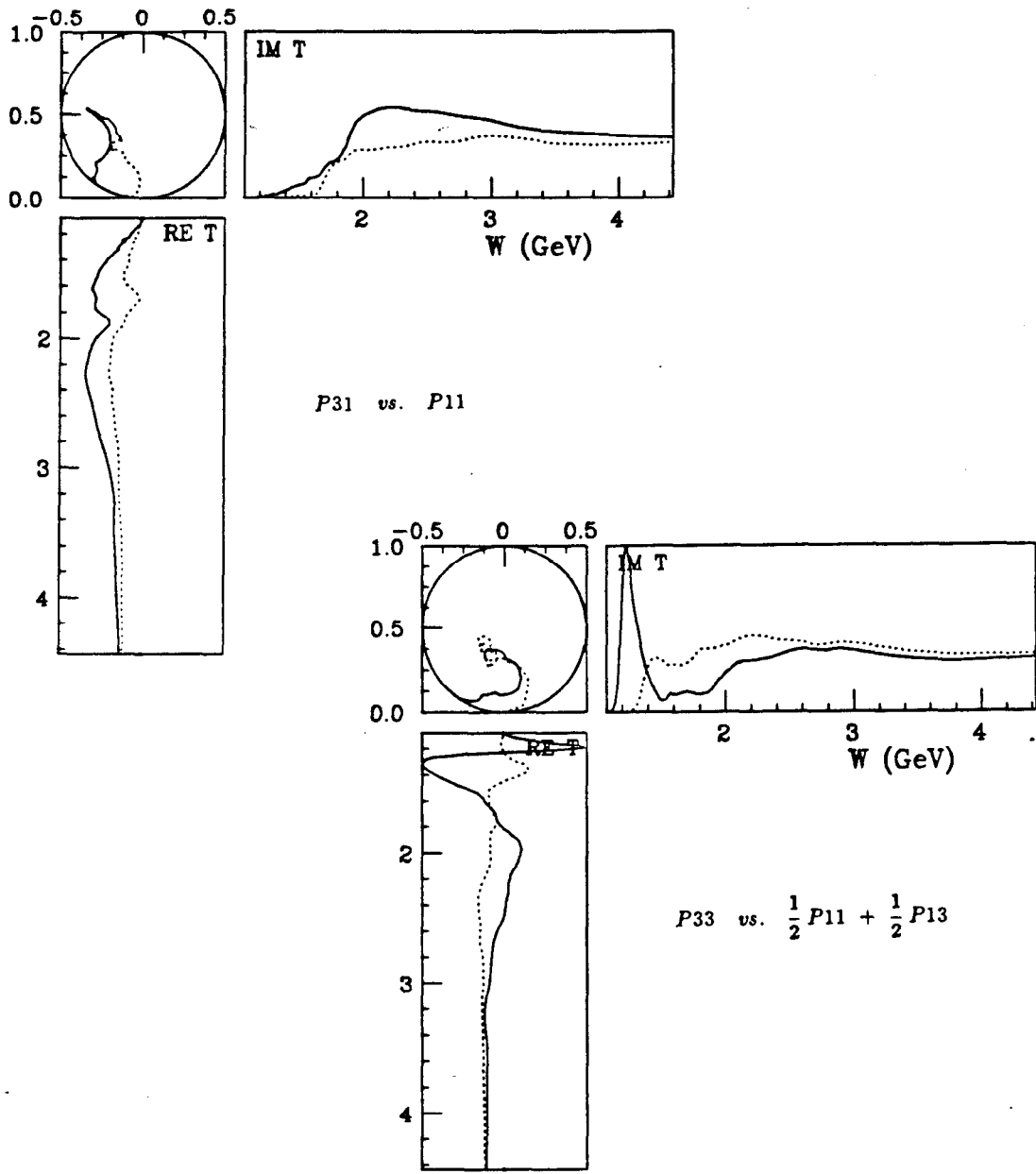


Fig. 3. Test of eq. (24) for  $\pi N$  elastic scattering in the P waves. The solid and dotted lines represent the evaluation of the left- and right-hand sides of (24), respectively, using the partial wave amplitudes of H6hler, *et. al.*, ref. 18. Each result is expressed as a  $T$ -matrix ( $T = (S - 1)/2i$ ).

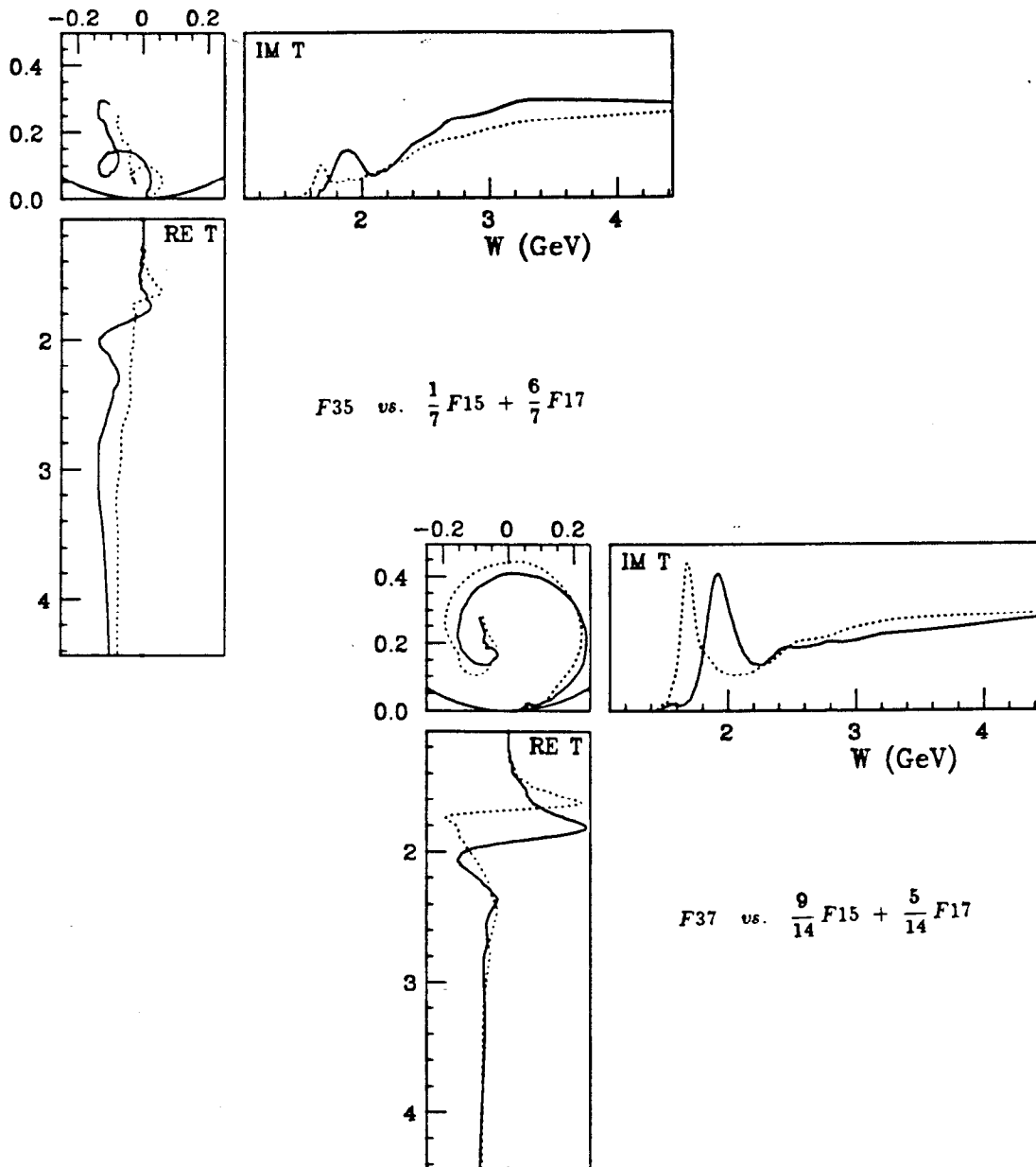


Fig. 4. Test of eq. (24) for F waves.

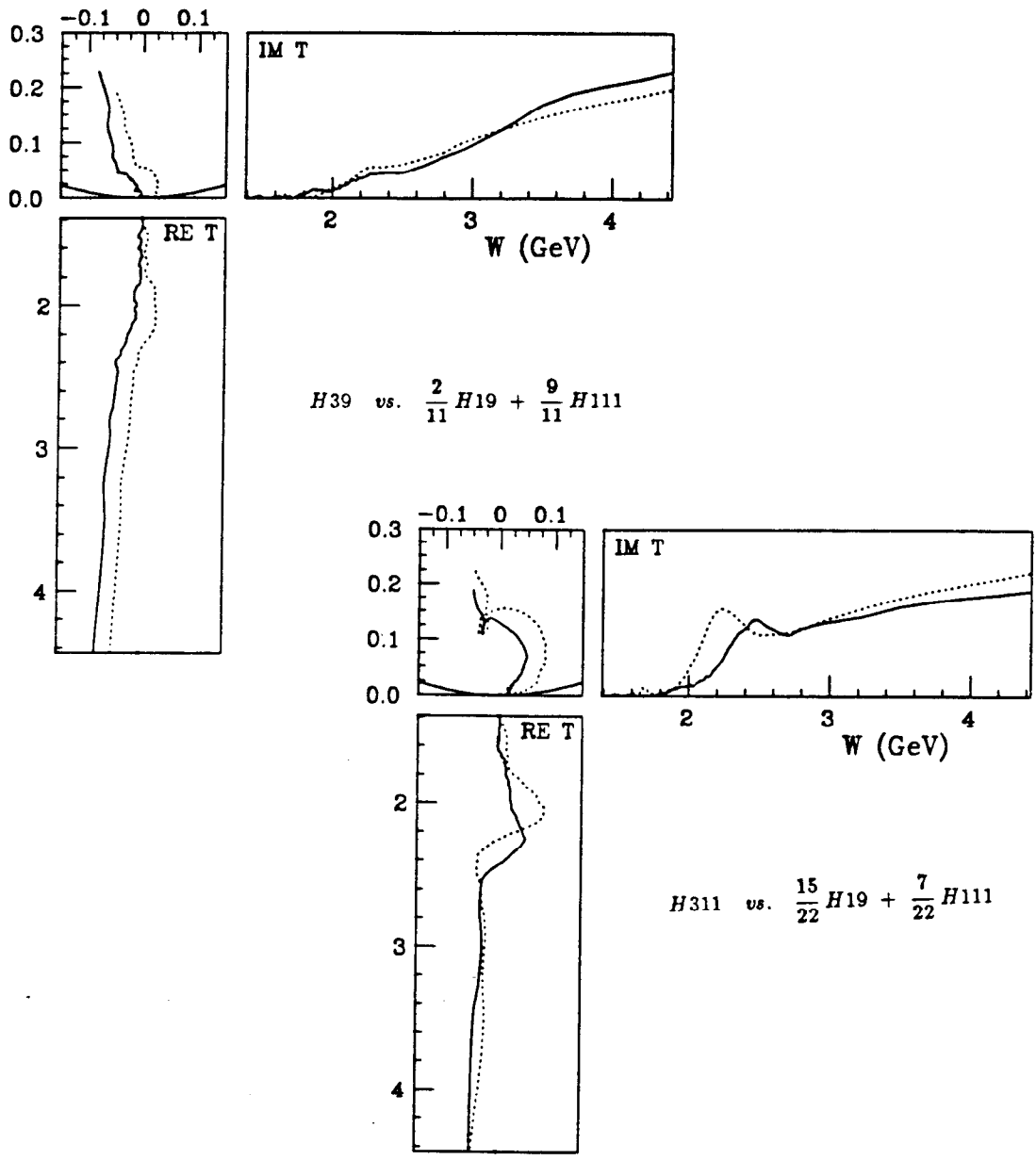


Fig. 5. Test of eq. (24) for H waves.

$N^{-1}$  and thus beyond the level of the leading-order analysis I have presented. Comparably good results hold for all the higher partial waves. A similar set of comparisons can also be made for the process  $\pi N \rightarrow \pi \Delta$ ; this is also presented in ref. 5.

Can we trust the good agreement found in the F and higher waves if the lowest partial waves are so problematic? This could make sense if the Skyrme model itself predicts difficulties in the S, P, and D waves which do not occur in the higher partial waves. An example of such a difficulty would be a bound state near threshold. A pole of the  $S$ -matrix in the complex energy plane is accompanied by a zero at the complex conjugate point. If the pole occurs at or just below threshold, but on the real axis, the pole and zero coalesce. A problem can occur if the pole lies on the real axis only in the leading order of the  $1/N$  expansion; then higher-order corrections can push the pole either to the right, into the physical region, where it produces a dramatic resonance, or to the left, where it might produce a strong repulsion near threshold. This precise difficulty actually arises directly from the chiral soliton picture.<sup>6</sup> It is well-known that symmetries of the underlying theory which do not leave the soliton invariant lead to zero-energy eigenstates of the stability matrix  $\mathcal{L}_2[U_0]$  defined in eq. (8). These states correspond to pion modes which have exactly zero energy in the leading order in  $1/N$ . The Skyrmion has 6 such zero modes, corresponding to 3 translational and 3 rotational (in space or isospace separately) motions. The translational zero modes form a  $K = 1$  multiplet mixing  $L = 0, 2$ ; these couple to the S and D waves. The rotational zero modes form a  $K = 1$  multiplet with  $L = 1$  and couple to the P waves. There are no such zero modes which couple to the F or higher waves.

We know already that the  $\Delta$  appears as a resonance in  $\pi N$  scattering in precisely this way. At the level at which we have been working, the  $N$  and  $\Delta$  are degenerate, and so the  $\Delta$  resonance should decouple from the  $P33$  scattering amplitude. Small corrections, though, can push the  $\Delta$  above the  $N$  and make it a strong resonance. Mattis and Karliner<sup>6</sup> have suggested that the other relatively low-lying  $\pi N$  resonances—the Roper resonance  $P_{11}(1410)$ , the  $S_{11}(1526)$ , and the  $D_{13}(1519)$ —also arise from this mechanism. It is interesting that the other channels in low partial waves—the  $S_{13}$ ,  $P_{13}$ ,  $P_{31}$ , and  $D_{35}$ —are the only channels in  $\pi N$  scattering which show repulsive behavior near threshold. In this picture, that would correspond to a small displacement of these poles in the opposite direction. One can, in principle, check this picture by making more detailed calculations in the Skyrme model. I will discuss the technique, at least, a bit later.

By adding a very simple bit of model-dependent information to the relations (23), one can extract an additional, rather striking, result. Let us assume that, for given  $L$ , the reduced  $T$ -matrix is larger for the lower values of  $K$ . This would coincide with the insight

one obtains from the study of, for example, the scattering of fermions from magnetic monopoles, all the interesting action takes place in the modes of lowest combined angular momentum  $\vec{J}$ . In particular, I would like to assume that the reduced  $T$ -matrix associated with  $s_{K=L+1}$  is negligible compared to the reduced  $T$ -matrices associated with  $s_{K=L-1}$  and  $s_{K=L}$ . To use this information, notice that the coefficients in each row of (23) sum to 1. Thus, if the coefficient of  $s_{L+1}$  is large, the important terms in the  $T$ -matrix will be small. Checking the values of these coefficients, we predict the pattern:<sup>5</sup>

$$(S-1)_{\frac{1}{2}L-\frac{1}{2}} > (S-1)_{\frac{3}{2}L+\frac{1}{2}} \gg (S-1)_{\frac{1}{2}L+\frac{1}{2}} \sim (S-1)_{\frac{3}{2}L-\frac{1}{2}}, \quad (25)$$

which one might call the big-small-small-big rule. This rule is actually quite visible in the Höhler, *et. al.*, partial wave amplitudes, as is shown in Fig. 6. This behavior was noted in the classical period of the study of  $\pi N$  scattering, and Donnachie, Hamilton, and Lea<sup>19</sup> showed how it followed from a particle exchange model. It is striking, however, that, in chiral soliton models, this rule follows very simply from the structure of the nucleon itself.

### Model-Dependent Results

Thus far, I have discussed results which follow from  $K$  symmetry alone, without making reference to any particular Lagrangian. If one assumes a particular form for the Lagrangian, however, one may compute the reduced  $S$ -matrix elements directly by a numerical calculation of the eigenstates of  $\mathcal{L}_2[U_0]$ . Most of the calculations of this sort done to date make use of Skyrme's Lagrangian

$$\mathcal{L} = \frac{f_\pi^2}{16} \text{tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{tr}([\partial_\mu U U^\dagger, \partial_\nu U U^\dagger])^2. \quad (26)$$

Breit and Nappi<sup>14</sup> and Zahed, *et. al.*,<sup>15</sup> studied the specific case of the breathing mode  $K=0$ ,  $L=1$ . Hayashi, *et. al.*,<sup>4</sup> and Mattis and Karliner<sup>6</sup> have presented quite general analyses of the structure of  $s_{KL'L}$ . I will not discuss these calculations in detail, especially since the actual equations which must be solved numerically are rather lengthy. Suffice it to say that these two latter groups have obtained all the information necessary to completely construct predictions for the  $\pi N$  elastic partial wave amplitudes from the formulae (23).

Hayashi, *et. al.*, have presented their results for the F wave amplitudes; Mattis and Karliner have presented predictions for all the partial waves considered in the data compilation of Höhler and collaborators.<sup>18</sup> These calculation fail badly for the S, P, and D waves (presumably for the reasons discussed above), but do reproduce the general form of

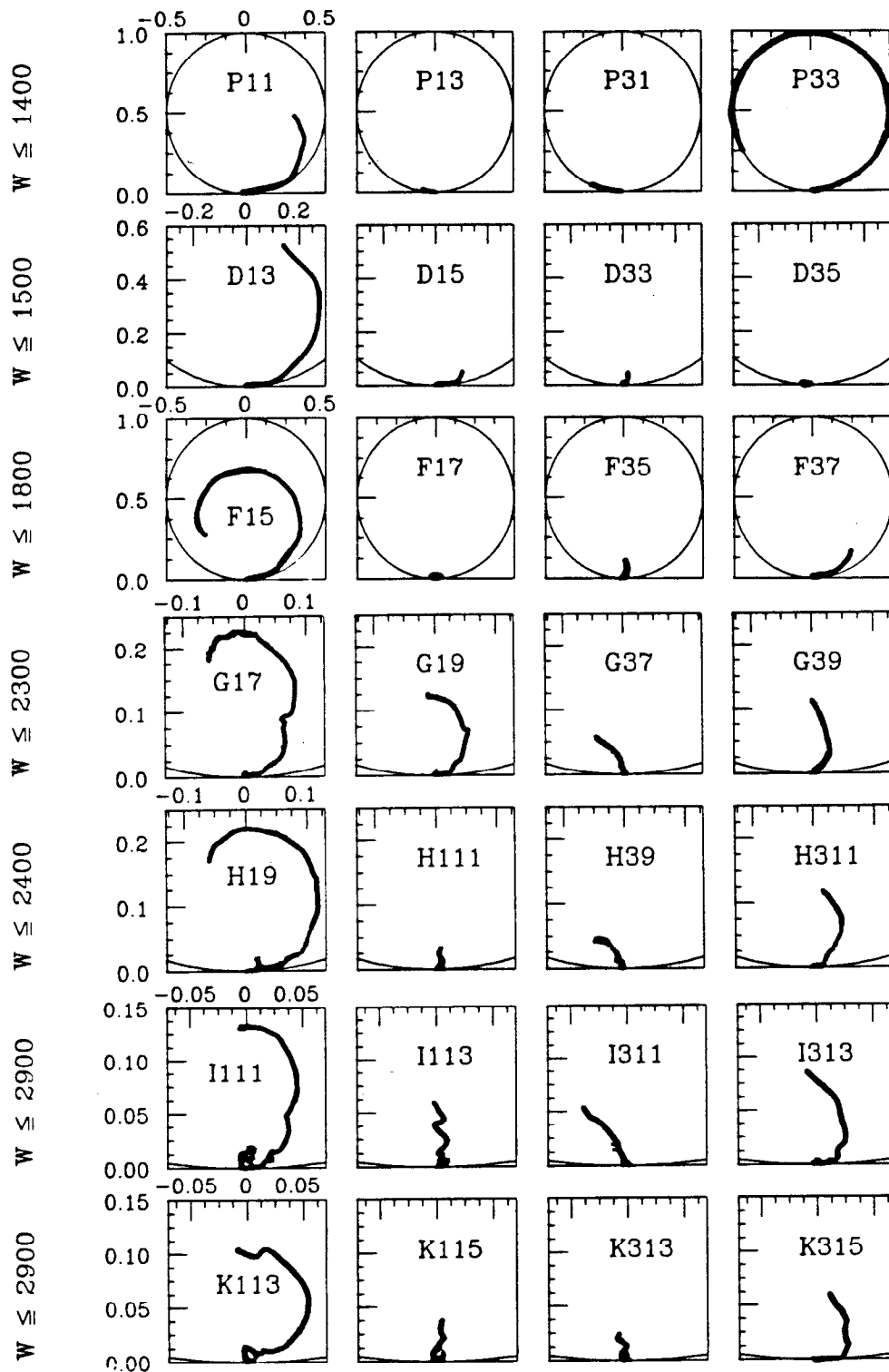


Fig. 6. Motion of the various  $\pi N$  elastic scattering amplitudes in the unitarity circle, over a range of energy  $W$  from threshold through the first major resonance in that channel.



the true amplitudes for F waves and above. As an illustration, I have reproduced in Figs. 7 and 8 the predictions of Mattis and Karliner for the isospin- $\frac{3}{2}$  F and H wave amplitudes. These can be compared directly to the solid curves of Figs. 4 and 5. Since the quantity being computed is of order  $N^0$ , the results depend only on the combination  $ef_\pi$ , and this one parameter serves only to set the overall energy scale. The shapes of the curves are completely parameter-independent, though they do, of course, depend on the particular choice for the Lagrangian, eq. (26). Note that the big-small-small-big rule is properly reflected: One sees pronounced resonances in the higher- $J$ , but not in the lower- $J$ , partial waves. Note also that this direct calculation does very poorly in predicting the elasticity of the higher partial waves. This result is not unexpected, since very few inelastic channels are included in the calculation. The scheme does allow  $\pi N$  to turn into  $\pi\Delta$  or even a heavier baryon; however, direct multiple pion emission is ignored as being of higher order in  $1/N$ , an approximation valid only at low energies. Ignoring the effects of elasticity, however, one would say that the phenomenologically determined amplitudes are reproduced fairly well by this simple scheme.

Using the language of these model-dependent calculations, it is possible to indicate what is missing from the analysis of the partial waves of low angular momentum and what might be added to describe the threshold region more accurately. In the whole previous discussion, I assumed that the pion moved quickly compared to the rate of rotational or translational motion of the soliton. This allowed one to solve separately the differential equations for each  $K$  mode, extract a reduced  $S$ -matrix from the asymptotic behavior of the eigenfunctions  $\pi_K$ , and then reconstruct the physical  $S$ -matrix elements by the prescription (23). Let us refer to these differential equations schematically as:

$$D_K \pi_K = 0, \quad (27)$$

and denote the reconstruction (23) by:

$$S^{IJ} = \sum_K P_K^{IJ} s_K. \quad (28)$$

The assumption that the pion moves quickly was justified earlier in this paper on the basis of the  $1/N$  expansion; however, for that argument, the energy of the pion was taken to be of order 1. The argument breaks down for pion energies close to threshold. When the pion energy is of order  $1/N$ , which might be relevant in practice when the pion energy is a few hundred MeV, then one must take into account the rotation of the soliton during the scattering process.

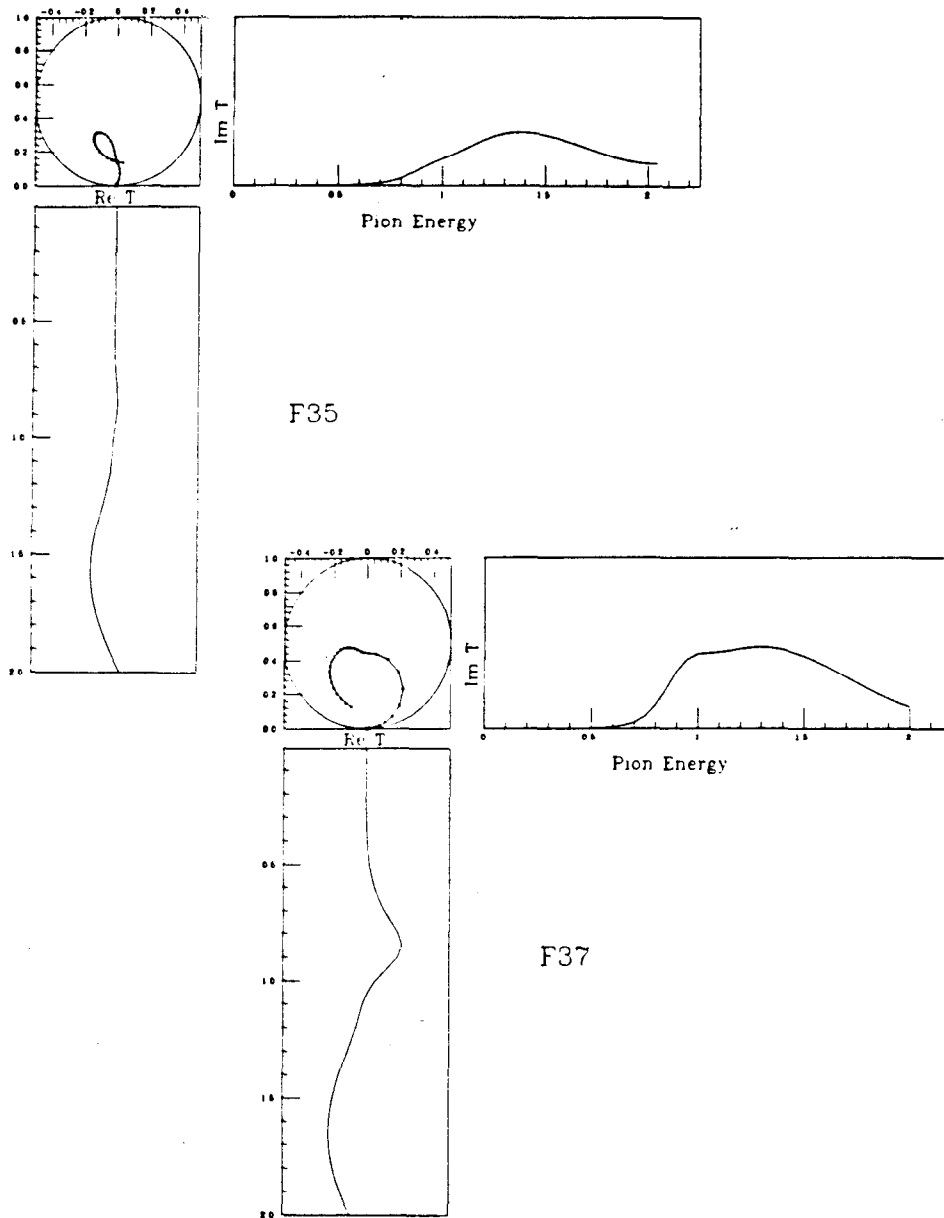


Fig. 7. Skyrme model predictions for the  $F_{35}$  and  $F_{37}$  pion-nucleon elastic scattering amplitudes, from ref. 6.

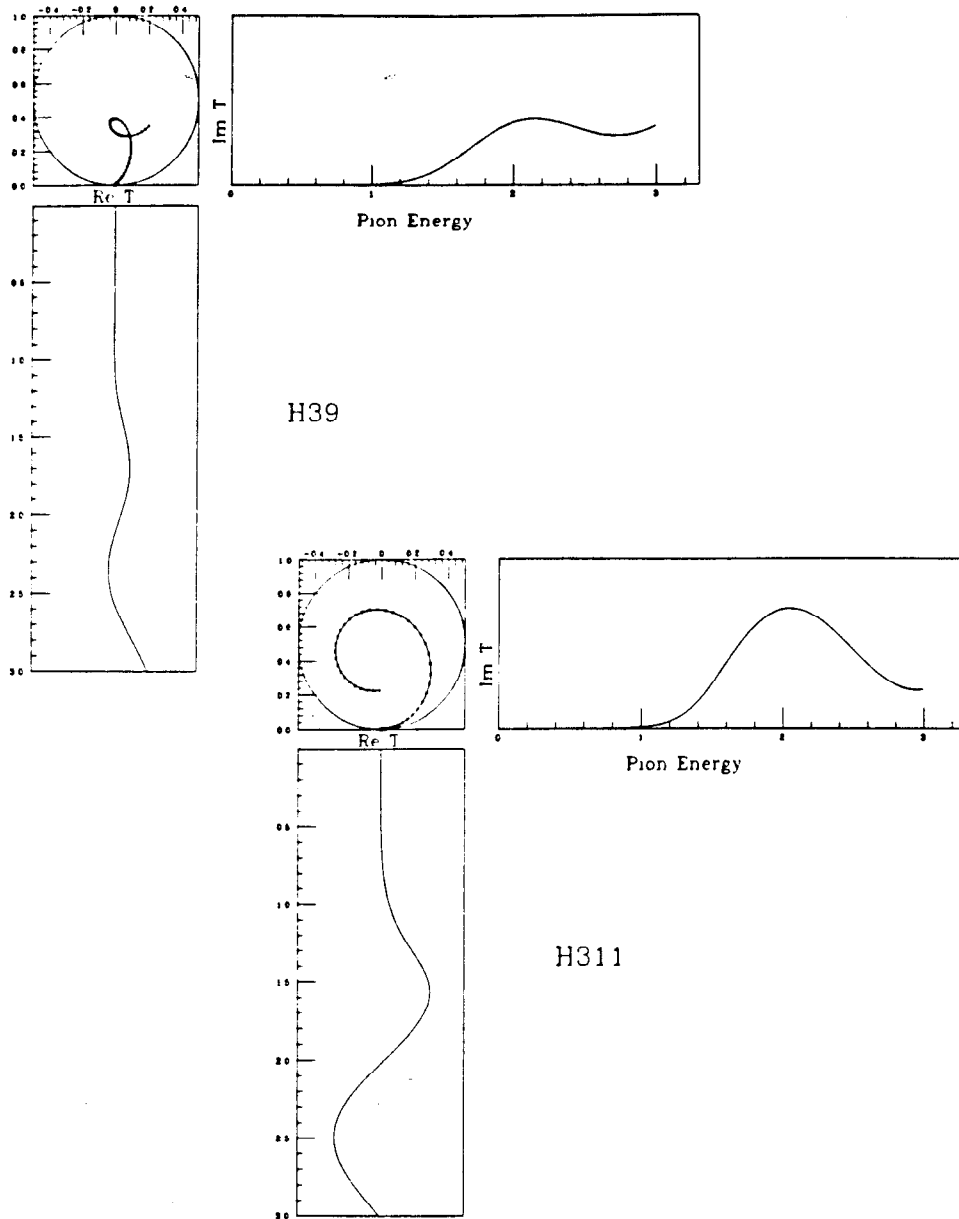


Fig. 8. Skyrme model predictions for the *H39* and *H311* pion-nucleon elastic scattering amplitudes, from ref. 6.

Let us examine how this can be done in the limit of low pion energies, by assuming that the soliton is rotating rapidly compared to the time the pion takes to interact. One's first guess would be to average the effective Hamiltonian, or equivalently, the differential equations (27), before computing the  $S$ -matrix. That is, one might try to solve

$$\left[ \sum_K P_K^{IJ} D_K \right] \delta\pi^{IJ} = 0. \quad (29)$$

However, this is not quite correct. In putting the nucleon into rotational eigenstates, we have assigned some deformations of the pion field to be parametrized by collective coordinates. If one implements this reassignment by a canonical transformation,<sup>20-22</sup> one must consider the new canonical pion field to be orthogonal to these motions. Since the infinitesimal rotations are zero modes of the  $D_K$ , our original procedure gave finite-energy pion eigenstates which were automatically orthogonalized to these zero modes. But our new equation (29) does not necessarily possess zero modes. We must, then, impose this orthogonality by hand, by adding to the Hamiltonian a term

$$\frac{1}{2}V \cdot \left( \int d^3x \delta\pi^i(x) Z^i(x) \right)^2, \quad (30)$$

where  $Z^i(x)$  is the form of the zero mode and  $V$  is a parameter to be taken to infinity. Let us consider for definiteness the  $L = 1$  partial waves; then the zero modes have the form:

$$Z^i(x) = \epsilon^{ijk} n^j \hat{r}^k F(r), \quad (31)$$

where  $n^j$  is a fixed unit vector and  $F(r)$  is the Skyrmion profile defined in eq. (5). This adds a new term to the equation for  $\delta\pi^{IJ}$ ; eq. (29) should then properly be written:

$$\left[ \sum_K P_K^{IJ} D_K \right] \delta\pi^{IJ} + W \cdot F(r) = 0, \quad (32)$$

where

$$W = P_K^{IJ} \cdot V \cdot \int dr r^2 \pi(r) F(r). \quad (33)$$

$V$  must be taken to infinity. This prescription can make sense only if the integral in eq. (33) vanishes in the same limit. Thus,  $W$  in (32) is an unknown constant, to be determined self-consistently so that the radial pion wavefunction is always orthogonal to  $F(r)$ . The correct treatment of collective coordinates, then, does introduce extra attraction or repulsion in the region just near threshold. It remains to be seen, however, whether these terms can account quantitatively for the structure of the low-lying resonances in low partial waves.

## Conclusions

I have argued that the Skyrme model, and, more generally, the picture of the nucleon as a chiral soliton, can give a qualitatively correct picture of pion-nucleon scattering. Clearly, many aspects of this description need to be understood better. These include the treatment of nonleading corrections near threshold and the inclusion of inelastic channels. One would also like to extend this analysis to the case of three flavors. But the model already succeeds, at the first level of calculation, in producing many of the features of this process which are revealed by experiment. It will be instructive to see how precise a picture of this reaction one can eventually build from the basic ingredients of solitons in quantum field theory.

## REFERENCES

1. T. H. R. Skyrme, *Proc. Roy. Soc. A* **260**, 127 (1961); *Nucl. Phys.* **31**, 556 (1962).
2. A. P. Balachandran, V. P. Nair, S. G. Rajeev, and A. Stern, *Phys. Rev. Lett.* **49**, 1124 (1982); *Phys. Rev.* **D27**, 1153 (1983).
3. E. Witten, *Nucl. Phys.* **B223**, 422, 433 (1983).
4. A. Hayashi, G. Eckart, G. Holzwarth, and H. Walliser, *Phys. Lett.* **147B**, 5 (1984).
5. M. P. Mattis and M. E. Peskin, *Phys. Rev. D*, (to appear)
6. Mattis and Karliner, *Phys. Rev.* **D31**, 2833 (1985).
7. C. Nappi, in *Proceedings of the Symposium on Anomalies, Geometry, and Topology*, Argonne National Lab., March 28-30, 1985.
8. D. Finkelstein, *J. Math. Phys.* **7**, 1218 (1966); D. Finkelstein and J. Rubinstein, *J. Math. Phys.* **9**, 1762 (1968).
9. E. Witten, *Nucl. Phys.* **B160**, 57 (1979).
10. G. 't Hooft, *Nucl. Phys.* **B72**, 461 (1974).
11. J.-L. Gervais and B. Sakita, *Phys. Rev. Lett.* **52** 87 (1984); *Phys. Rev.* **D30**, 1795 (1984).
12. A. Manohar, *Nucl. Phys.* **B248**, 19 (1984).
13. G. Adkins, C. Nappi, and E. Witten, *Nucl. Phys.* **B228**, 552 (1983)
14. J. D. Breit and C. R. Nappi, *Phys. Rev. Lett.* **53**, 889 (1984).
15. I. Zahed, U.-G. Meissner, and U. B. Kaulfuss, *Nucl. Phys.* **A426**, 525 (1984).

16. A. Hayashi and G. Holtzwarth, *Phys. Lett.* **140B**, 175 (1984).
17. A. Bohr and B. R. Mottelson, *Nuclear Structure*, vol. II. (W. A. Benjamin, Reading, Mass., 1975).
18. G. Höhler, F. Kaiser, R. Koch, and E. Pietarinen, *Handbook of Pion-Nucleon Scattering*. Physik Daten No. 12-7 (Fachinformationszentrum, Karlsruhe, 1979).
19. A. Donnachie, J. Hamilton, and A. T. Lea, *Phys. Rev.* **135**, B515 (1964).
20. J.-L. Gervais and B. Sakita, *Phys. Rev.* **D11**, 2943 (1975).
21. N. H. Christ and T. D. Lee, *Phys. Rev.* **D12**, 1606 (1975).
22. E. Tomboulis, *Phys. Rev.* **D12**, 1678 (1975).