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Solving Field Theory in One-Space-One-Time Dimension*

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ABSTRACT

By quantizing a realistic field theory in one space and one time dimension at equal light cone time $\tau = t + x/c$ rather than at equal time t, one can find exact solutions to the bound state problem. The method is non-perturbative and amounts to the diagonalization of *finite* matrices in Fock space. It applies also for non-Abelian gauge theory in $1+1$ dimensions, but is demonstrated here for the simple case of fermions interacting with scalar fields. The success of the light cone quantization method rests on the existence of a new dynamical quantum number, the harmonic resolution K, which can be understood as the ratio of a characteristic length, the box size L, and the compton wave length of a massive particle. We emphasize the appearance of self-induced instantaneous inertias.

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1. Introduction

The underlying theory of hadrons and nuclei is presumed to be *Quantum Chromodynamics* (QCD). Its phenomenology is quite successful in the account of many experimental facets; a fact, however, which sometimes hides the fundamental difficulty: One has not solved the relevant field equations. Thus far, they can be attacked efficiently by perturbative methods, but the region of validity of perturbation theory is not clear. These difficulties are not inherent to QCD alone. Even in Abelian theories, perturbative methods are useless for large coupling constants, as for example in toy models, which let the interaction between the fermions be mediated by either vector (QED) or scalar bosons. The problems become obvious even for simple questions such as how the invariant mass of a positronium-like structure changes as function of the coupling constant α. For sufficiently small values, it behaves like $2m_0|1 - \frac{\alpha^2}{8}|$. But what happens for large values? Does the invariant mass ever go to zero, or will it increase again for sufficiently large values of α?

The goal in this paper is to investigate and to develop methods, which can be used for strongly interacting fields. As a beginning and an illustrative step, we substitute gluons by scalar bosons. For the same reason, simplicity, we restrict ourselves to one space dimension. One can not imagine treating three dimensions without being able to solve first the one dimensional problem. Thus, we consider here one of the simplest field theoretic problems: ***Fermions interacting by scalar bosons in 1+1 dimension.***

This allows also comparison with recent work. Brooks and Frautschi[2] have treated numerically the same problem in usual space-time quantization, by diagonalizing the amputated Hamiltonian matrix in the charge 0 and 1 sector. Serot, Koonin and Negele[3] have calculated the spatial density and the binding energy of one-dimensional nuclei by stochastic means. They generate the interaction between nucleons by the exchange of scalar and vector mesons, treating the problem on the tree level in non-relativistic approximation. Because of the extensive numerical work required for obtaining solutions, there has been little investigation of on the sensitivity of the solutions to the physical or non-physical parameters, like physical masses and coupling constants, or length scales and state cut-offs. The most recent approach[4] to the same problem uses the very efficient tool of matrix diagonalization, and gives a detailed numerical analysis. Exact eigenvalues and eigenfunctions for many nuclei are produced, both for the ground state and for exited states.

As we discuss here, the problem of finding solutions becomes enormously easier, if the fields are quantized at equal light cone time $t + x/c$ rather than at equal usual time t.[11] Light cone quantization was proposed originally by Dirac,[5] and rediscovered by Weinberg[6] in the context of covariant formulation of time-ordered perturbation theory. Sometimes called the infinite momentum frame approach,[7,8,9,10,11] it continues to be an important tool for many applications.[11] The formalism was thoroughly investigated and reviewed by Chang et al.[12] The rules for quantizing QCD on the light cone are given in Refs. 14 and 15.

It appears not to have been noticed that this method has a special feature, which allows a virtually exact solution of the bound state problem. In Fock space representation, the light cone Hamiltonian[5] becomes block diagonal, characterized by a new dynamic quantum number, the *harmonic resolution* K. K is closely related to the light cone momentum, when the theory is defined with periodic boundary conditions in the light cone spatial coordinates. For each fixed value of K, the Fock space dimension in the block is finite, and *finite matrices can be diagonalized numerically with unlimited precision.* Eventually, the resulting field theoretical many body problem in one space and one time dimension becomes much simpler than its non-relativistic and non-covariant approximation.[4]

In 1+1 dimensions, the light cone formalism is particularly transparent. In section 2, we give a rather explicit presentation of both concepts and notation. Here we expand the fields into a complete set of functions with periodic boundary conditions and define our Fock space representation. The primary objects of interest, the operators for charge, momentum, and energy are calculated in section 3. Conclusions on the utility of the method are given in the final section. The explicit analytical and numerical solutions are given in an accompanying paper.

The possibility of finding exact solutions is not restricted to the simple case of a scalar theory; the conclusions hold as well for Abelian and non-Abelian field theories in 1+1 dimension. Work in this direction is under way. We restrict ourselves here to the bound state problem, i.e. to the calculation of invariant masses and their Fock-space eigenstates. Eventually one has to formulate a scattering theory with the now explicitly known eigenstates, and without using perturbative methods.

2. Quantization on the Light Cone

The Lagrangian density for interacting fermion and scalar boson fields, ψ and φ, respectively, is given by[1,3]

$$\mathcal{L} = \frac{1}{2}\,\partial^\mu\varphi\partial_\mu\varphi + \frac{1}{2}\,m_B^2\varphi^2 + \frac{i}{2}\,\overline{\psi}\gamma^\mu\partial_\mu\psi - \frac{i}{2}\,\partial_\mu\overline{\psi}\gamma^\mu\psi - (m_F + \lambda\varphi)\overline{\psi}\psi\ . \qquad (2.1)$$

m_F and m_B are the bare masses for the fermions and the bosons, respectively, to be determined below, and the bare coupling constant λ is considered a free parameter. The Lagrangian density is manifestly hermitean, although only the total Lagrangian $\int d\tau\,\mathcal{L}$ has to be so by physical reasons. The volume element $d\tau$ denotes integration over all covariant coordinates x^μ.

The metric tensors $g^{\mu\nu}$ and $g_{\mu\nu}$, are defined as the raising and lowering operators, $x^\mu = g^{\mu\nu}x_\nu$ and $x_\mu = g_{\mu\nu}x^\nu$, respectively, such that the scalar product $x^\mu x_\mu \equiv g^{\mu\nu}x_\nu x_\mu$ remains an invariant under Lorentz transformations. This implies, that they are inverse to each other, i.e. $g^{\mu\nu}g_{\nu\kappa} = \delta^\mu_\kappa$. As long as one does not write out the sums explicitly, the Lagrangian in four is the same as in two dimensions. Henceforth we shall restrict ourselves to the latter case. In the usual parameterization with $x^0 = ct$ being the time and $x^1 = x$ being the space coordinate, $g^{\mu\nu}$ has the nonvanishing elements $g^{11} = -g^{00} = 1$.

There is no compelling reason, why the fields must be treated always as functions of the usual time and space coordinates. Any invertible parameterization of space and time is admissible as well. For example, one can consider them as functions like $\varphi(x^-, x^+) = \varphi(x - ct, x + ct)$. If one transforms the coordinates to a rotated frame, the metric tensor becomes ($g^{ij} = \partial x^i / \partial x_j$)

$$g^{\mu\nu} = \begin{pmatrix} g^{++} & g^{+-} \\ g^{-+} & g^{--} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{2.2}$$

Scalar products become, for example

$$k_\mu x^\mu \equiv k_+ x^+ + k_- x^- = \frac{1}{2}(k^- x^+ + k^+ x^-) = 2(k_- x_+ + k_+ x_-), \tag{2.3}$$

or even simpler, $k_\mu k^\mu = k^+ k^- = 4k_- k_+$. As shown in Fig. 1, the above transformation corresponds to a *real* rotation in phase space by -45^0, combined with an irrelevant stretching of scale. Upon rotation, time and space lose their meaning. Nevertheless, in line with familiar phrasing[11,12] , one refers to $x^+ = x^0 + x^1$ as the *light cone time*, and correspondingly, to $x^- = x^0 - x^1$ as the *light cone position.* It is somewhat unfortunate, that this way of parameterization has also been described as the *infinite momentum frame approach.* In fact, the light cone formulation is frame-independent, the momentum is always finite, and it is not really correct to think of the above rotation as a Lorentz transformation which boosts the system to high momenta. Dirac's original formulation seems to be more adequate[5] .

In a quantized theory, the Lagrangian does not completely specify the problem, one has to know the commutation properties of the operators φ and ψ. But before one can formulate these, one must be clear about which of the field components may be considered as independent variables[12] . This can be investigated through the equations of motion, as obtained by the canonical variation[1] of the

Lagrangian, i.e.

$$\partial^\mu \partial_\mu \varphi + m_B^2 \varphi + \lambda \overline{\psi} \psi = 0 \ , \tag{2.4}$$

and

$$\begin{aligned} i\partial_\mu \overline{\psi} \gamma^\mu + m\overline{\psi} = 0, \quad i\gamma^\mu \partial_\mu \psi - m\psi = 0 \quad , \\ \text{with} \quad m(x) \equiv \quad m_F + \lambda \varphi(x), \end{aligned} \tag{2.5}$$

When transforming the frame, the differential equations change their structure, as we shall demonstrat now for the Dirac equation (2.5). The Dirac matrices γ^μ obey the relation $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$. In 1+1 dimensions they are 2 by 2 matrices[2] ,i.e.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.6}$$

and transform like coordinates $\gamma^+ = \gamma^0 + \gamma^1$ and $\gamma^- = \gamma^0 - \gamma^1$. Written out, the commutation relation become $\gamma^+\gamma^+ = 0$, $\gamma^-\gamma^- = 0$, and $\gamma^+\gamma^- + \gamma^-\gamma^+ = 4$. Thus, the two operators

$$\Lambda^{(+)} = \frac{1}{4}\gamma^-\gamma^+ \quad \text{and} \quad \Lambda^{(-)} = \frac{1}{4}\gamma^+\gamma^- \tag{2.7}$$

have the property of projectors with $\Lambda^{(+)} + \Lambda^{(-)} = 1$; i.e.

$$\Lambda^{(+)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^{(-)} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.8}$$

Acting from the left with $\Lambda^{(+)}$ and $\Lambda^{(-)}$, Eq. (2.5) separates into a set of two coupled equations

$$\partial_- \psi^{(-)} = \frac{1}{2i}\, \gamma^0 m \psi^{(+)} \qquad \text{and} \tag{2.9}$$

$$\partial_+\psi^{(+)} = \frac{1}{2i}\,\gamma^0 m\psi^{(-)}\,, \tag{2.10}$$

where $\psi^{(+)} \equiv \Lambda^{(+)}\psi$, $\psi^{(-)} \equiv \Lambda^{(-)}\psi$, $\partial_+ \equiv \partial/\partial x^+$, and $\partial_- \equiv \partial/\partial x^-$. Suppose, one has arbitrarily fixed both the fermion component $\psi^{(+)}$ and the boson field φ at some particular light cone time $x^+ = x_0^+$ on the interval $x^- \in (-L, L)$. Then, one can integrate Eq. (2.9),

$$\psi^{(-)}(x^-, x_0^+) = F(x_0^+) + \frac{i}{4}\gamma^0 \int\limits_{-L}^{+L} dy^- \epsilon(x^- - y^-) m(y^-, x_0^+)\psi^{(+)}(y^-, x_0^+), \tag{2.11}$$

with ϵ being the antisymmetric step function; i.e. $\epsilon'(x) = -2\delta(x)$. The function F depends only on x_0^+, but is otherwise arbitrary. A consistent boundary condition is[12]

$$F(x_0^+) = 0. \tag{2.12}$$

Inserting Eqs. (2.11) and (2.12) into the second couple, Eq. (2.10), one obtains the time derivative as a functional of $\psi^{(+)}$ alone, i.e.

$$\partial_+\psi^{(+)}(x^-, x^+) = \frac{1}{8}m(x^-, x^+) \int\limits_{-L}^{+L} dy^- \epsilon(x^- - y^-) m(y^-, x^+)\psi^{(+)}(y^-, x^+). \tag{2.13}$$

A similar analysis can be given for the boson field. With φ and $\psi^{(+)}$ being fixed, and therefore also $\rho = \overline{\psi}\psi$, the equation of motion (2.4), i.e. $4\partial_+\partial_-\varphi + \lambda\rho + m_B^2\varphi = 0$, can be integrated

$$\partial_+\varphi(x^-, x^+) = -\frac{1}{8} \int\limits_{-L}^{+L} dy^- \epsilon(y^- - x^-)(\lambda\rho(y^-, x^+) + m_B^2\varphi(y^-, x^+)). \tag{2.14}$$

In other words, only $\psi^{(+)}$ and φ are independent variables. Neither $\psi^{(-)}$ or its derivatives nor the light cone time derivatives $\partial_+\psi^{(+)}$ or $\partial_+\varphi$ are independent, they must satisfy Eqs. (2.13) and (2.14) everywhere. These constraints are a consequence of first order partial differential equations, in sharp contrast with the second order equation in the usual space-time parameterization.

The dependent components having been found, one can determine the canonical commutation relations. By means of Schwinger's action principle,[13] one obtains[12]

$$i[\varphi(x^-, x^+), \varphi(x^{-\prime}, x^+)] = \frac{1}{4}\epsilon(x^- - x^{-\prime}), \qquad \text{and} \tag{2.15}$$

$$\{\psi_\alpha^{(+)}(x^-, x^+), \psi_\beta^{(+)\dagger}(x^{-\prime}, x^+)\} = \Lambda_{\alpha\beta}^{(+)}\delta(x^- - x^{-\prime}) \quad . \tag{2.16}$$

All other (anti-) commutators vanish. A more thorough discussion can be found in the literature.[12] As an alternative, one can proceed canonically.[1,14,15] Taking x^+ as the time-like coordinate, one defines the momentum conjugate to the field φ as $\Pi^+ \equiv \frac{\partial \mathcal{L}}{\partial \partial_+ \varphi} = \partial^+\varphi = 2\partial_-\varphi$. The canonical procedure at equal time-like coordinate gives thus

$$[\partial_-\varphi(x^-, x^+), \varphi(x^{-\prime}, x^+)] = \frac{i}{2}\delta(x^- - x^{-\prime}),$$

which is identical to the space derivative of Eq. (2.15). The fermion fields behave in the same manner.

3. The Free Field Solutions and the Fock Space

At some initial time $x^+ = x_0^+ = 0$, the independent fields $\psi^{(+)}$ and φ can be chosen arbitrarily, as long as they satisfy the commutation relations and Lorentz covariance. The former is easy to arrange, and the latter is enforced by letting them be solutions of the equations of motion with vanishing coupling constant, i.e.

$$\psi^{(+)}(x^-, 0) = \psi_{\text{free}}^{(+)}(x^-, 0) \quad \text{and} \quad \varphi(x^-, 0) = \varphi_{\text{free}}(x^-, 0) \tag{3.1}$$

The free fields can be constructed easily, and in turn define the Hilbert space in which $\psi^{(+)}$ and φ act as operators, the so-called Fock space.

The free fields obey

$$(\partial^\mu\partial_\mu + m_B^2)\varphi_{\text{free}} = 0 \quad \text{and} \quad (\partial^\mu\partial_\mu + m_F^2)\,\psi_{\text{free}}^{(+)} = 0 \ . \tag{3.2}$$

Although φ_{free} is a real scalar and $\psi_{\text{free}}^{(+)}$ a complex spinor, they obey the same equation. A particular solution to the latter is $\psi^{(+)} \sim e^{ik_\mu x^\mu}$, provided one satisfies

$$k_\mu k^\mu - m_{F,B}^2 = 0 \quad \text{or} \quad k^+k^- = m_{F,B}^2 \ . \tag{3.3}$$

The relation between (k^+, k^-) and (k^0, k^1) is the same as for the covariant coordinates, and displayed in the Fig. 2. Because of the rotation, the usual meaning of energy and momentum gets lost, but it is justified to speak of a *single particle light cone momentum* $k^+ \equiv k^0 + k^1$ and a *single particle light cone energy* $k^- \equiv k^0 - k^1$. But there is a distinct difference. For a fixed momentum k^1 one has both a particle-state with energy $(k^0)_p = +\sqrt{m^2 + (k^1)^2}$ and a hole-state with energy $(k^0)_h = -\sqrt{m^2 + (k^1)^2}$. But in light cone parametrization, one has only one value of the single particle energy, i.e.

$$k^- = \frac{m_F^2}{k^+} \tag{3.4}$$

for a fixed single particle momentum. Moreover, particles have only positive and holes only negative values of k^+ and k^-. In line with field theoretic conventions,[1] one counts energies and momenta relative to a reference state, the Fock space vacuum. After a renormalization, particles and antiparticles have both positive momenta and energies. The positive-definite momenta are responsible for the great simplicity of the present approach.

The single particle energy, Eq. (3.4), seems to have a singularity at $k^+ = 0$. But a free massive particle will actually never have a vanishing light cone momentum. Its (space-time) energy can become arbitrarily close, but is never identical to its (space-time) momentum, no matter how large its momentum is, simply because $(k^0)^2 - (k^1)^2 = m^2 \neq 0$ [see also Fig. 2 for an illustration of this fact]. The construction

$$k^+ \to k^+_n = \frac{2\pi}{L} n \,, \quad n = 1, 2, 3, \ldots, \Lambda, \tag{3.5}$$

accounts for this aspect. For the lowest possible value $n = 1$, L regulates the vicinity of $k^+ = 0$, while Λ determines the highest possible value of k^+ for each fixed L. A glance at Fig. 2 reveals, that the left running states ($k^1 < 0, k^+$ small) have a different cut-off in space-time momentum than the right runners, as opposed to space-time where they are treated symmetrically.[2]

In their most general form the free field solutions can hence be written as

$$\varphi_{\text{free}}(x^-, x^+) = \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\Lambda} \frac{1}{\sqrt{n}} \left(a_n e^{-ik^{(n)}_\mu x^\mu} + a^\dagger_n e^{+ik_\mu x^\mu} \right) \,, \quad \text{and} \tag{3.6}$$

$$\psi^{(+)}_{\text{free}}(x^-, x^+) = \frac{1}{\sqrt{2L}}\, u \sum_{n=1}^{\Lambda} \left(b_n e^{-ik^{(n)}_\mu x^\mu} + d^\dagger_n e^{+ik^{(n)}_\mu x^\mu} \right) \,. \tag{3.7}$$

The spinor u is normalized to unity, $u = \frac{1}{\sqrt{2}} \binom{1}{1}$, and is independent of the momenta. Fermions and antifermions are created by the operators $b^\dagger_n$ and $d^\dagger_n$, respectively, subject to the anti-commutators

$$\{b_n, b^\dagger_m\} = \delta_{n,m} \quad \text{and} \quad \{d_n, d^\dagger_m\} = \delta_{n,m} \,. \tag{3.8}$$

The boson creation and destruction operators obey the commutator

$$[a_n, a^\dagger_m] = \delta_{n,m} \,. \tag{3.9}$$

Boson and fermion operators commute. The quantization rules , Eq. (2.16) and

(2.15), are satisfied by means of the completeness relation for Fourier series, i.e.

$$\lim_{\Lambda,L\to\infty}\left[\frac{1}{2L}+\frac{1}{2L}\sum_{n=1}^{\Lambda}\left(e^{i\frac{\pi n}{L}(x-x')}+e^{i\frac{\pi n}{L}(x'-x)}\right)\right]=\delta(x-x')\ . \tag{3.10}$$

Choosing the fields according to Eq. (3.1), i.e.

$$\begin{aligned}\psi^{(+)}(x^-,0)&\equiv\frac{n}{\sqrt{2L}}\ \Psi\left(\frac{\pi x^-}{L}\right)\ ,\quad\text{and}\\ \varphi(x^-,0)&\equiv\frac{1}{\sqrt{4\pi}}\ \Phi\left(\frac{\pi x^-}{L}\right)\ ,\end{aligned} \tag{3.11}$$

one can express the fields in terms of the scalar and dimensionless operator functions

$$\begin{aligned}\Psi(\xi)&=\sum_{n=1}^{\Lambda}b_n\,e^{-in\xi}+d_n^\dagger e^{+in\xi}\ ,\quad\text{and}\\ \Phi(\xi)&=\sum_{n=1}^{\Lambda}c_n\,e^{-in\xi}+c_n^\dagger\,e^{+in\xi},\quad\text{with}\quad c_n=\frac{1}{\sqrt{n}}\,a_n.\end{aligned} \tag{3.12}$$

Because of the discretized momenta, Eq. (3.5), the operators Ψ and Φ , and therefore also the fields ψ and φ are periodic functions with period $2L$ in the light cone position x^-. We define them on the interval $x^-\in(-L,+L)$. On this interval, the plane wave states are orthonormal and complete, and the series, Eq. (3.11) and (3.12) can be understood as the special case of an expansion into a denumerable and complete set $< x|n >$.

The operator part of this expansion, the creation and destruction operators act in Fock space, i.e. in the representation which diagonalizes simultaneously the number operators $a_n^\dagger a_n$, $b_n^\dagger b_n$ and $d_n^\dagger d_n$. Since one has to specify exactly which momentum states are occupied and which are empty (c.f. also Refs. 2 and 16), denumerability seems compulsory, rather than only a formal trick.

All these advantages have the price of introducing into the formalism two at first non-physical, mathematical parameters, the *cut-off* Λ and the *length* L. Since they are redundant, one must be able to show at the end that the physical results do not depend on either of them.

4. The Constants of Motion

The Lagrangian, Eq.(2.1) , has two kinds of conserved currents, $\partial_\mu j^\mu = 0$ and $\partial_\mu J^{\mu\nu} = 0$. The first arises since $\mathcal{L}$ does not depend explictly on the phase of ψ and is $j^\mu = \overline{\psi}\gamma^\mu\psi$. The second, the energy momentum stress tensor, is a consequence of coordinate invariance and has the form

$$J^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi + \frac{i}{2}\,(\overline{\psi}\gamma^\mu\partial^\nu\psi - \partial^\nu\overline{\psi}\gamma^\mu\psi) + \frac{1}{2}\,g^{\mu\nu}(m_B^2\varphi^2 - \partial_\kappa\varphi\partial^\kappa\varphi)\ . \tag{4.1}$$

Integrating the currents over a closed hypersurface $\int d\sigma_a$ conjugate to the time-like coordinate x^a, i.e. $d\tau = dx^a d\sigma_a$, one generates conserved charges

$$Q = \int d\sigma_a j^a \quad \text{and} \quad P^\mu = \int d\sigma_a J^{\mu a}\ . \tag{4.2}$$

They are independent of x^a. In a quantized theory, the total charge Q and the components of the energy-momentum vector P^μ are operators, as well as the contraction of the latter, the Lorentz scalar $M^2 = P_\mu P^\mu$. In space-time quantization, P^1 is the operator for the total momentum, P^0 for the total energy , and M^2 is the operator for the square of the invariant mass, i.e. $M^2 = (P^0)^2 - (P^1)^2$. They mutually commute.[1] In light cone quantization, P^+ is the operator for the *total light cone momentum*, P^- for the *total light cone energy*, and M^2 again the operator for the *invariant mass squared*,

$$P^+ = P^0 + P^1, \quad P^- = P^0 - P^1 \quad \text{and} \quad M^2 = P^+P^-. \tag{4.3}$$

The notation implies that M^2 is a positive operator, i.e. one which has only positive eigenvalues. We shall come back to this question, below. Chang et al.[12] have shown that Q, P^+ and P^- mutually commute if the fields satisfy the commutation relations, Eqs. (2.15) and (2.16) . Thus, they can be diagonalized simultaneously, for example in Fock space representation which is equivalent to solving the equations of motion.[1]

Written out in light cone metric, the operators are

$$Q = \frac{1}{2}\int\limits_{-L}^{+L} dx^- 2[\psi^{(+)}]^\dagger\psi^{(+)}, \tag{4.4}$$

$$P^+ = \frac{1}{2}\int\limits_{-L}^{+L} dx^- \left[4\partial_-\varphi\partial_-\varphi + 2i([\psi^{(+)}]^\dagger\partial_-\psi^{(+)} - [\partial_-\psi^{(+)}]^\dagger\psi^{(+)})\right] , \tag{4.5}$$

$$P^- = \frac{1}{2}\int\limits_{-L}^{+L} dx^- \left[m_B^2\varphi\varphi + 2i([\psi^{(+)}]^\dagger\partial_+\psi^{(+)} - [\partial_+\psi^{(+)}]^\dagger\psi^{(+)})\right] . \tag{4.6}$$

The factor $\frac{1}{2}$ arises from the Jacobian $dx^0dx^1 = \frac{1}{2}dx^+dx^-$. The momentum P^+ and the charge Q are independent of the coupling constant $\lambda \equiv g\sqrt{4\pi}$, but P^- depends on λ through $\partial_+\psi^{(+)}$. By means of the dimensionless operators as defined above, one can extract the dependence on the box size L,

$$P^+ = \frac{2\pi}{L}\,K \quad \text{and} \quad P^- = \frac{L}{2\pi}\,H\ . \tag{4.7}$$

After some algebraic manipulations, one obtains with $\xi = \frac{\pi x^-}{L}$

$$Q = \frac{1}{2\pi}\int\limits_{-\pi}^{+\pi} d\xi\ \Psi^\dagger\Psi \quad \text{and} \quad K = \frac{1}{4\pi}\int\limits_{-\pi}^{+\pi} d\xi\left[\frac{\partial\Phi}{\partial\xi}\frac{\partial\Phi}{\partial\xi} + 2i\Psi^\dagger\frac{\partial\Psi}{\partial\xi}\right] . \tag{4.8}$$

The modified momentum operator K is dimensionless, while the modified energy operator H carries the dimension of a mass squared,

$$H = +i\,\frac{m_F^2}{4\pi}\int\limits_{-\pi}^{+\pi} d\xi \int\limits_{-\pi}^{+\pi} d\xi'\epsilon(\xi-\xi')\,\Psi^\dagger(\xi)\Psi(\xi') + \frac{m_B^2}{4\pi}\int\limits_{-\pi}^{+\pi} d\xi\,\Phi(\xi)\Phi(\xi)$$

$$+i\,\frac{m_F g}{4\pi}\int\limits_{-\pi}^{+\pi} d\xi \int\limits_{-\pi}^{+\pi} d\xi'\epsilon(\xi-\xi')\,\Psi^\dagger(\xi)\Phi(\xi)\Psi(\xi') \tag{4.9}$$

$$+i\,\frac{g^2}{4\pi}\int\limits_{-\pi}^{+\pi} d\xi \int\limits_{-\pi}^{+\pi} d\xi'\epsilon(\xi-\xi')\,\Psi^\dagger(\xi)\Phi(\xi)\Psi(\xi')\Phi(\xi')\,.$$

The appearance of a term quadratic in the coupling constant reflects the instantaneous, Coulomb-like interaction, which is not propagated by the exchange of bosons.[11]

The integrals over ξ can be carried out in closed form. With the identity $\frac{1}{2\pi}\int_{-\pi}^{+\pi} d\xi\, e^{im\xi} = \delta_{m,0}$, the normal ordered operators Q and K become

$$Q = \sum_n b_n^\dagger b_n - d_n^\dagger d_n \quad \text{and} \quad K = \sum_n n(a_n^\dagger a_n + b_n^\dagger b_n + d_n^\dagger d_n)\;. \tag{4.10}$$

Some of the most important conclusions can be drawn even without knowing the Fock space structure of H, the reader not interested in these details may skip the remainder. The terms quadratic in g with an *even* number of creation operators become most directly

$$\begin{aligned} g^2 \sum_{k,l,m,n} \Big[\; & b_k^\dagger c_l^\dagger d_m^\dagger c_n^\dagger \{+k+l|+m+n\} + b_k^\dagger c_l^\dagger b_m c_n \{+k+l|-m-n\} \\ & + b_k^\dagger c_l d_m^\dagger c_n \{+k-l|+m-n\} + b_k^\dagger c_l b_m c_n^\dagger \{+k-l|-m+n\} \\ & + d_k c_l^\dagger d_m^\dagger c_n \{-k+l|+m-n\} + d_k c_l^\dagger b_m c_n^\dagger \{-k+l|-m+n\} \\ & + d_k c_l d_m^\dagger c_n^\dagger \{-k-l|+m+n\} + d_k c_l b_m c_n \{-k-l|-m-n\}\Big]\,. \end{aligned} \tag{4.11}$$

The matrix elements

$$\{n|m\} = \frac{i}{4\pi} \int_{-\pi}^{+\pi} d\xi \int_{-\pi}^{+\pi} d\xi' \epsilon(\xi - \xi')\, e^{i(n\xi + m\xi')} \tag{4.12}$$

take upon calculation the values

$$\{n|m\} = \begin{cases} 0 & \text{if } n = 0 \textit{ and } m = 0, \\ \frac{1}{n}\delta_{m,-n} & \text{if } n \neq 0 \textit{ and } m \neq 0. \end{cases} \tag{4.13}$$

With the symmetry properties $\{n|m\} = -\{m|n\} = \{-m|-n\} = -\{-n|-m\}$, the normal ordered product can be cast into the *seagull part H_S of the Hamiltonian*

$$\begin{aligned} H_S = g^2 \sum_{k,l,m,n} & b_k^\dagger b_m c_l^\dagger c_n \, [\{k-n|l-m\} + \{k+l|-m-n\}] \\ & + d_k^\dagger d_m c_l^\dagger c_n \, [\{k-n|l-m\} + \{k+l|-m-n\}] \\ & + (d_k b_m c_l^\dagger c_n^\dagger + b_m^\dagger d_k^\dagger c_n c_l) \, \{l-k|n-m\} \quad . \end{aligned} \tag{4.14}$$

The nomenclature[11] has its origin in the structure of the graphs of Fig. 3. The terms corresponding to a simultaneous creation of bosons and fermion-antifermion pairs do not contribute. They are kinematically suppressed in light cone quantization,[11] because $\{+k+l|+m+n\}$ vanishes for positive values of the momenta. However, the time and the normal ordered product, Eqs. (4.11) and (4.14) , respectively, are not the same! Consider for example the fourth term in Eq. (4.11) , i.e. $\sum_{k,l,m,n} b_k^\dagger c_l b_m c_n^\dagger \{+k-l|-m+n\}$. Using the commutation relations to generate the normal ordered product $b_k^\dagger b_m c_n^\dagger c_l$, leaves one with $\sum_{m,n} \frac{1}{n} b_m^\dagger b_m \{+m-n|-m+n\}$. Contrary to a c-number,this operator can not be omitted. It represents *instantaneous, self-induced inertias*, which so far have apparently not been mentioned in the literature. These inertias are naturally combined with the mass terms of Eq. (4.9) to yield the *massive part H_M of the Hamiltionian*, i.e.

$$H_M = \sum_n \frac{1}{n}\left[a_n^\dagger a_n \left(m_B^2 + g^2 \alpha_n\right) + b_n^\dagger b_n \left(m_B^2 + g^2 \beta_n\right) + d_n^\dagger d_n \left(m_B^2 + g^2 \gamma_n\right)\right], \quad (4.15)$$

with the coefficients α, β and γ given by

$$\begin{aligned}\alpha_n &= \sum_{m=1}^{\Lambda} \{n-m|m-n\} - \{n+m|-m-n\}\,, \\ \beta_n &= \sum_{m=1}^{\Lambda} \frac{n}{m}\{n-m|m-n\} \quad \text{and} \quad \gamma_n = \sum_{m=1}^{\Lambda} \frac{n}{m}\{n+m|-m-n\}\,.\end{aligned} \quad (4.16)$$

The terms quadratic in g with an *odd* number of creation operators are computed in the same manner. The normal ordering does *not* induce new terms and one can cast them into the *fork part H_F of the Hamiltonian*, i.e.

$$\begin{aligned}H_F = g^2 \sum_{k,l,m,n} & (b_k^\dagger b_m c_l^\dagger c_n^\dagger + b_m^\dagger b_k c_n c_l)\ \{k+l|n-m\} \\ & +(d_k^\dagger d_m c_l^\dagger c_n^\dagger + d_m^\dagger d_k c_n c_l)\ \{k+l|n-m\} \\ & +b_k^\dagger d_m^\dagger c_l^\dagger c_n\ [\{k-n|m+l\} + \{k+l|m-n\}] \\ & +d_m b_k c_n^\dagger c_l\ [\{k-n|m+l\} + \{k+l|m-n\}]\,,\end{aligned} \quad (4.17)$$

Graphical representations of H_F and H_V are given in Fig. 3 . The *vertex part H_V of the Hamiltonian* includes all terms linear in the coupling constant, i.e.

$$\begin{aligned}H_V = g m_F \sum_{k,l,m} & (b_k^\dagger b_m c_l^\dagger + b_m^\dagger b_k c_l)\ [\{k+l|-m\} + \{k|+l-m\}] \\ & (d_k^\dagger d_m c_l^\dagger + d_m^\dagger d_k c_l)\ [\{k+l|-m\} + \{k|+l-m\}] \\ & (b_k d_m c_l^\dagger + d_m^\dagger b_k^\dagger c_l)\ [\{k-l|+m\} + \{k|-l+m\}]\,.\end{aligned} \quad (4.18)$$

For the same reason as above, the terms with *only* creation or *only* destruction operators vanish by the selection rules of the matrix elements. Collecting all terms, the Hamiltonian $H = H_M + H_V + H_S + H_F$ is the sum of four parts defined above.

The self-induced inertias are the only parts of the Hamiltonian, which depend on the cut-off Λ [see Eq. (4.16)]. Approximating sums by integrals, this dependence can be worked out explicitly.[16] For vanishing $1/\Lambda$, the fermion and the antifermion inertias become independent of the cut-off, while the boson inertias diverge logarithmically; however, such that the divergence cancels in the differences $\alpha_n - \alpha_m$. In this limit, the eigenvalues and eigenfunctions of the Hamiltonian become strictly independent of the cut-off in the limit of vanishing $1/\Lambda$, as can be shown numerically, and for some of the cases even analytically.

5. Conclusions: Finite Dimensional Representations, Labelled by the Harmonic Resolution.

The discretization of the momentum eigenvalues k^+ allows one to denumerate the momentum eigenstates. The price one has to pay is the appearance of two additional, formal parameters in the theory, i.e. the length L and the cut-off Λ. One must be able to show, that the physical results do not depend on either of the these, at least not in the limit $L \to \infty$ and $\Lambda \to \infty$.

In light cone quantization, discretization has rather unexpected consequences, which seem not to have been noticed so far.

First, and perhaps most remarkably, the length cancels in the only Lorentz scalar of the theory, the invariant mass squared, i.e. $M^2 = P^+P^- = KH$. The eigenvalues of I are *independent* of L for *any* value of L.

The eigenvalues and eigenfunctions of the Hamiltonian H, or of the invariant mass squared M^2, are also independent of the cut-off Λ and positive definite. This is shown in an accompanying paper in the context of mass renormalization. For sufficiently simple cases, i.e. for small g^2 or for $K = 1$ and $K = 2$, it can be done analytically.

Second, the number operators of Fock space representation are diagonal and have positive or zero eigenvalues. Therefore, both the operators for charge and momentum are diagonal, with eigenvalues Q and K. The single particle momenta are positive by definition, and consequently K has only positive or zero eigenvalues. But, by the same reason, only a *finite* number of Fock states can have the same eigenvalue K. Since Q, K and H commute, the latter can be arranged in block diagonal form. Each block is labelled by the eigenvalues K and Q and has a *finite dimension: Since diagonalization is a closed operation, the eigenvalue problem on the light cone can be solved exactly in 1+1 dimension.* This aspect of light cone quantization is profoundly different from space-time quantization. There, too, charge and total momentum are diagonal, but the momentum operator has infinite degeneracy. The energy matrix must be truncated by brute force,[2] in order to become numerically tractable.

Third, K is a *dynamical quantum number.* Its value characterizes a wave function as much as the charge Q. What is its physical meaning? Suppose one has diagonalized H for some charge, for a given value of K, and for some value of the coupling constant λ, the bare fermion mass m_F and the bare boson mass m_B. Suppose, the lowest eigenvalue KH is identical with M^2, the invariant mass squared of a physical particle. Can one go back to space-time representation and calculate the momentum P and the energy E of this particle? In a way one can, since $E = \frac{1}{2}(P^+ + P^-)$ and $P = \frac{1}{2}(P^+ - P^-)$. But actually one has to know the length L, since

$$P^+ = \frac{2\pi}{L} K \quad \text{and} \quad P^- = \frac{L}{2\pi} \frac{M^2}{K} .$$

However, one can fix L by the requirement of vanishing center of mass momentum, $P = 0$, which implies $P^+ = P^-$. This in turn requires

$$2\pi K/L = M^2\, L/(2\pi K) \quad ,$$

or upon restoring the correct units

$$K = \frac{L}{\lambda_C} \qquad \text{with} \quad \lambda_C = \frac{2\pi\,\hbar}{Mc}\,.$$

Thus in the rest frame the *dynamical quantum number, the 'harmonic resolution' K becomes the ratio of the length L to the Compton wavelength of the particle.* The larger one chooses the period of the wavefunction in phase space, the larger K becomes and therefore the dimension of the Hamiltonian matrix. Thus, K plays the role of a resolving power. Increasing K allows the observation of a more detailed and more complex structure of the eigenfunction in terms of Fock states. One must conclude, that the wavefunction of a particle in one space and one time dimension depends on the resolution, on the accuracy one imposes by the choice of L or, more precisely, by the value of the harmonic resolution K.

The length L thus has apparently two aspects. One the one hand, for a particle a rest, it has to be a multiple of the Compton wave length. On the other hand, for a particle in motion, it can take any value required for the continuum limit $K \to \infty$ and $L \to \infty$.

Last but not least, these conclusions do not depend on the detailed structure of the Hamiltonian. They hold as well for other field theories.

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FIGURE CAPTIONS

1. The light cone in space-time (left) and in light-cone (right) representation.
2. Particle and hole energies in space-time (left) and in light-cone (right) representation.
3. Diagrams : (a) Vertices, (b) Seagulls and, (c) Forks.

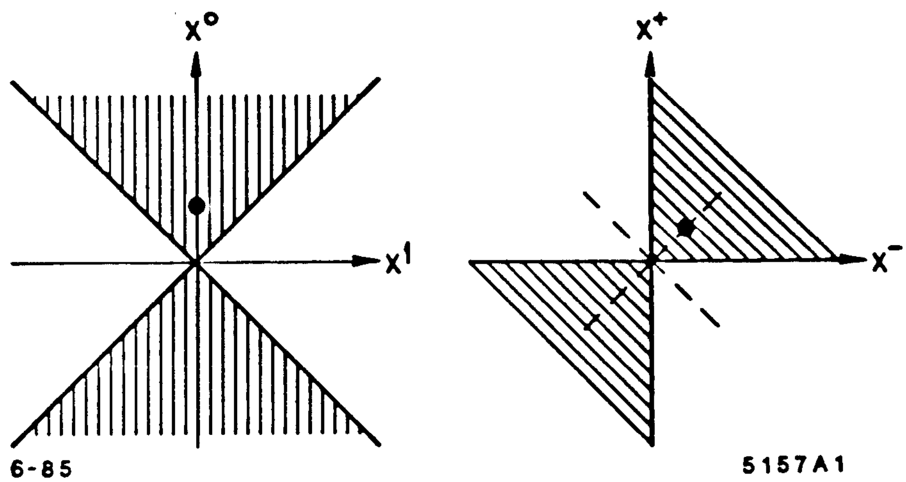

Fig. 1

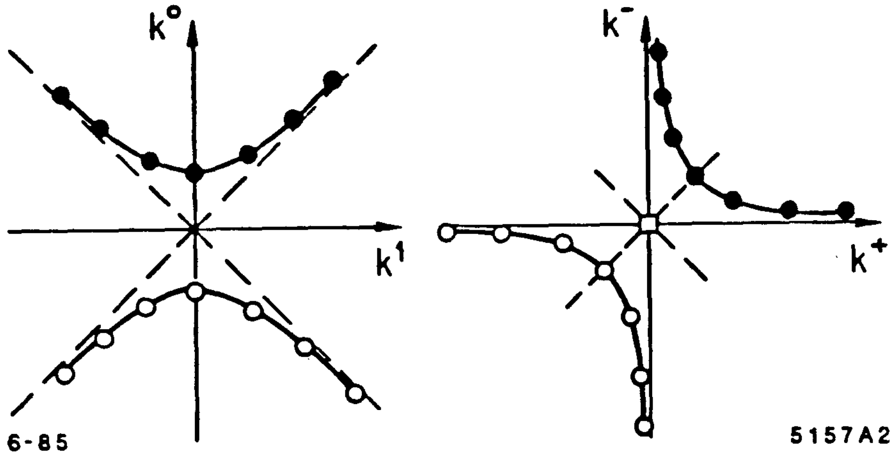

Fig. 2

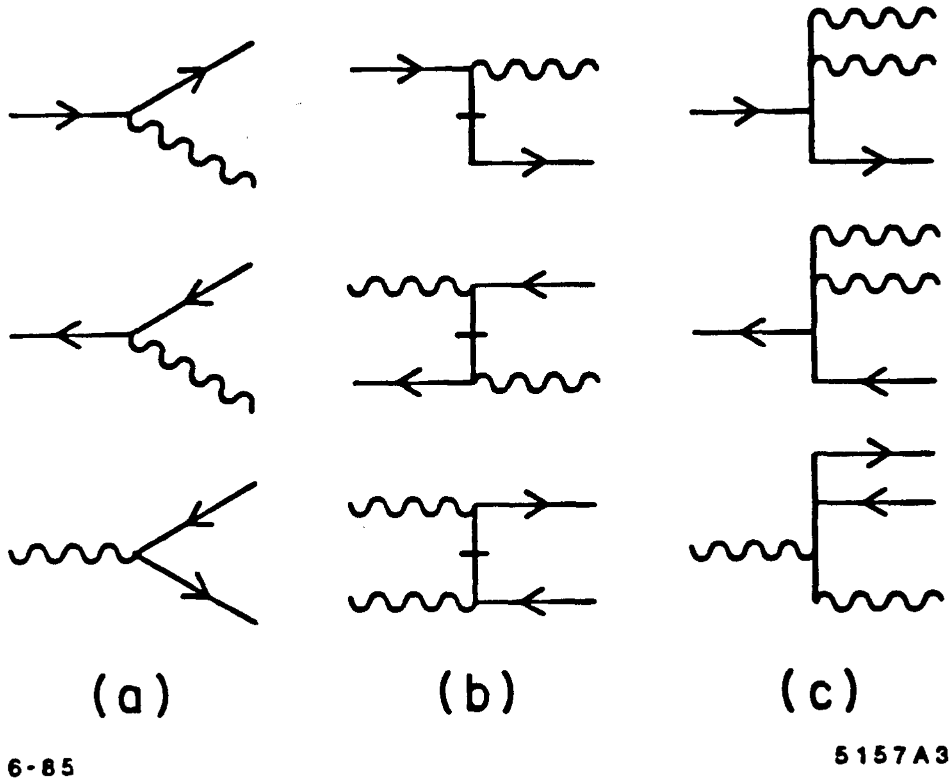

Fig. 3