SLAC–PUB–3698
HUTP–85/A042
May 1985
(T)

HOLONOMY ANOMALIES*

JONATHAN BAGGER†
Lyman Laboratory of Physics
Harvard University, Cambridge, Massachusetts, 02138

DENNIS NEMESCHANSKY AND SHIMON YANKIELOWICZ‡
Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

Nonlinear sigma models and low-energy effective Lagrangians play an important role in modern particle physics. They were first discussed in the context of chiral dynamics, where they were used to describe the nonlinear interactions of Goldstone bosons. More recently, they have been used to investigate non-perturbative effects in gauge theories, to give a geometrical interpretation to matter couplings in supergravity theories, and even to provide a convenient framework for analyzing the possible compactifications of superstring theories.

In this talk we discuss a new type of anomaly that afflicts certain non-linear sigma models with fermions [1–4]. This anomaly is similar to the ordinary gauge and gravitational anomalies since it reflects a topological obstruction to the reparametrization invariance of the quantum effective action. However, the sigma model anomaly is different in one important respect – it can sometimes be cancelled by a set of local counterterms [3,4]. We will show that these counterterms have a simple topological interpretation, and that the anomaly cancellation requirements can easily be understood by a suitable generalization of 't Hooft's anomaly matching conditions [5].

In the first half of this talk, we construct nonlinear sigma models based on homogeneous spaces G/H. Following Callan, Coleman, Wess and Zumino [6], we add fermions to these models, where the fermions transform in various representations ρ_H of H.

* Work supported by the Department of Energy, contract DE-AC03-76SF00515, and by the National Science Foundation, contract NSF-PHY-82-15249.

† On leave of absence from Stanford Linear Accelerator Center, Stanford University, Stanford, California, 94305.
‡ On leave of absence from the Department of Physics, Tel-Aviv University, Israel.

Presented at the Symposium on Anomalies, Geometry and Topology
Argonne, Illinois, March 28-30, 1985

Anomalies arise when these fermions are chiral. We shall show that these anomalies can sometimes be cancelled by Chern-Simons terms.

In the second half of this talk we consider nonlinear sigma models based on general Riemannian manifolds $\mathcal{M}$. We now take the fermions to live in the tangent space of $\mathcal{M}$. As before, the sigma model anomalies can sometimes be cancelled by appropriate Chern-Simons terms.

Sigma models on homogeneous spaces G/H describe the interactions of the Goldstone bosons that arise by spontaneously breaking a group G down to a subgroup H. The Goldstone bosons can interact with fermions χ^A. The fermions form representations of H, and realize the full G-symmetry nonlinearly [6]. The sigma model Lagrangian is given by

$$\mathcal{L} \;=\; -\,\frac{1}{2}\,g_{ij}(\phi)\,\partial_\mu\phi^i\partial^\mu\phi^j \;-\; \frac{i}{2}\,\overline{\chi}_{LA}\gamma^\mu D_\mu\chi_L{}^A\;, \tag{1}$$

where the covariant derivative $D_\mu\chi^A$ is given by $D_\mu\chi^A = \partial_\mu\chi^A + \partial_\mu\phi^i\,\omega_i{}^A{}_B\chi^B$, and the connection $\partial_\mu\phi^i\,\omega_i{}^A{}_B$ is the pull-back to spacetime of an appropriate connection on G/H.

The most instructive way to think of the manifold G/H is as a *section* of a fiber bundle $\mathcal{E}$, with total space G and fiber H. This section is parametrized by the group element $g = \exp i\phi^i T^i$, where the T^i denote the generators of G that are not in H. In these coordinates, the metric $g_{ij} = \mathrm{Tr}[(g^{-1}\partial_i g)|_K(g^{-1}\partial_j g)|_K]$, where $g^{-1}\partial_i g$ is projected onto $K = G/H$. The fermions χ^A also have a bundle interpretation. They should be thought of as sections of a vector bundle $\mathcal{F}$ associated to $\mathcal{E}$. The fermion connection $\omega_i{}^A{}_B$ is given by the associated H-connection $\omega_i{}^A{}_B = \rho_H[(g^{-1}\partial_i g)|_H]^A{}_B$.

The symmetries of $\mathcal{L}$ are associated with the isometries of the manifold G/H. They are generated by global G-rotations. These transformations rotate points in the bundle $\mathcal{E}$. In particular, they take elements g of G/H into elements kg of $\mathcal{E}$. The transformed elements kg are not necessarily in the section G/H. They must be projected back by field-dependent H-transformations $h(g,k)$, such that $kgh \in G/H$. Under these H-rotations $\omega_i{}^A{}_B$ transforms like a connection. The Lagrangian $\mathcal{L}$ is invariant provided the fermions χ^A transform like tensors, $\chi^A \to \rho_H(h)^A{}_B\chi^B$.

Since the induced H-rotations are local, anomalies arise at the quantum level if the fermions χ^A transform in anomalous representations of H. In four dimensions, the variation of the effective action is given by [7]

$$\begin{aligned}\delta\Gamma \;=\; & \frac{1}{24\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\;\mathrm{Tr}\,\epsilon\,\partial_i\left[\omega_j\partial_k\omega_\ell \;+\; \frac{1}{2}\,\omega_j\omega_k\omega_\ell\right] \\ & \times\;\partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k\partial_\sigma\phi^\ell\;,\end{aligned} \tag{2}$$

where $h = 1+\epsilon$, and the trace is over the fermion representation ρ_H. The anomaly (2) is similar to an ordinary anomaly since it obstructs the invariance of the quantum effective action. In a gauge or gravitational theory, such an obstruction is fatal – unitarity is lost and unphysical degrees of freedom begin to propagate. In a sigma model, the anomaly is more subtle. This is because the gauge fields are composite – they are functions of the scalar fields ϕ^i. Sigma model anomalies do not create any new degrees of freedom. They merely break some of the symmetries and the geometrical interpretation associated with the classical action. For the case at hand, the anomaly breaks all symmetries in G that are not in H.

Sigma models with anomalies are unacceptable for physical reasons. Therefore we would like to know when – if ever – the anomaly can be cancelled. One way to cancel the anomaly is well-known from gauge theories: One simply adds extra spinors so that the fermionic determinant is well-defined. In practical terms, this means that the fermions must transform in an anomaly-free representation of H. In a sigma model, there is a second approach. One can add local counterterms to the effective action in just such a way that the anomaly is cancelled. Both the effective action and the counterterms transform anomalously under H, but the sum remains invariant.

What counterterms must one add to cancel the sigma model anomaly? The answer is obvious: In four dimensions, one simply adds an integral over the five-dimensional Chern-Simons term $\Omega_{ijk\ell m}(\omega)$ [8],

$$
\begin{aligned}
I &= -\,2\pi \int\limits_D d^5y\ \epsilon^{ijk\ell m}\,\Omega_{ijk\ell m}(\omega) \\
&= -\,\frac{1}{120\pi^2}\int\limits_D d^5y\ \epsilon^{ijk\ell m}\,\mathrm{Tr}\left[\omega_i\partial_j\omega_k\partial_\ell\omega_m + \frac{3}{2}\,\omega_i\omega_j\omega_k\partial_\ell\omega_m\right. \\
&\quad \left. + \frac{3}{5}\,\omega_i\omega_j\omega_k\omega_\ell\omega_m\right] .
\end{aligned} \tag{3}
$$

The integral I runs over a five-dimensional disk D whose boundary ∂D is the image of spacetime in $\mathcal{E}$ [9]. The variation of I exactly cancels the sigma model anomaly,

$$
\begin{aligned}
\delta I &= -\,\frac{1}{120\pi^2}\int\limits_D d^5y\,\epsilon^{ijk\ell m}\partial_i\ \mathrm{Tr}\left[\epsilon\,\partial_j\left(\omega_k\partial_\ell\omega_m + \frac{1}{2}\,\omega_k\omega_\ell\omega_k\right)\right] \\
&= -\,\frac{1}{24\pi^2}\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\ \mathrm{Tr}\,\epsilon\,\partial_i\left[\omega_j\partial_k\omega_\ell + \frac{1}{2}\,\omega_j\omega_k\omega_\ell\right] \\
&\quad \times\ \partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k\partial_\sigma\phi^\ell .
\end{aligned} \tag{4}
$$

The coefficient $-1/120\pi^2$ was chosen to cancel the anomaly (2). It is also precisely the right coefficient to ensure that the effective action is independent of D.

The counterterm (3) is written as an integral over a disk D. Lagrangian mechanics, however, requires that an action be written as an integral over spacetime. Equation (3) can be pulled back to an integral over spacetime whenever the Chern-Simons term is closed, $d\Omega = 0$. Then $\Omega = d\alpha$ locally, and

$$
\begin{aligned}
-\,\frac{1}{2\pi}\,I &= \int\limits_D d^5y\ \epsilon^{ijk\ell m}\ \Omega_{ijk\ell m} \\
&= 5\int\limits_{\partial D} d^4y\ \epsilon^{ijk\ell}\ \alpha_{ijk\ell} \\
&= 5\int d^4x\,\epsilon^{\mu\nu\rho\sigma}\ \alpha_{ijk\ell}\,\partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k\partial_\sigma\phi^\ell .
\end{aligned} \tag{5}
$$

It is easy to show that $d\Omega = \mathrm{Tr}\,R^3$, where the curvature $R = d\omega + \omega^2$. We have shown that the sigma model anomaly can be cancelled by local counterterms whenever

$\mathrm{Tr}\, R^3 = 0$. This result can be generalized to any even dimension. In $2n-2$ dimensions, the anomaly is cancelled by the $(2n-1)$-dimensional Chern-Simons term, provided $\mathrm{Tr}\, R^n = 0$.

There are two cases when this condition is automatically satisfied. The first is when $\dim \mathcal{E} \leq 2n-1$. Then $\mathrm{Tr}\, R^n$ automatically vanishes since there are no $2n$-forms on $\mathcal{E}$. The second is when $\mathrm{Tr}\, R^n$ vanishes in any fermionic representation. In two dimensions this happens for symmetric spaces, as can be seen from the commutation relations of the Lie algebras.

If $\mathrm{Tr}\, R^n \neq 0$, all is not lost, for the connection used to construct the Chern-Simons term is not unique. One can always add a tensor τ to the connection ω. Since τ is a tensor, $\omega' = \omega + \tau$ is a connection, and τ is called torsion. To see how this works, let us restrict ourselves to two dimensions. Instead of the H-connection $\omega = \rho_H[(g^{-1}dg)|_H]$, let us consider the G-connection $\omega' = \rho_G[g^{-1}dg]$. This connection is only defined when the fermion representations ρ_H form G-representations ρ_G. Since $\rho_G[g^{-1}dg] = \rho_G[(g^{-1}dg)|_K] + \rho_G[(g^{-1}dg)|_H]$, we see that the torsion $\tau = \rho_G[(g^{-1}dg)|_K]$. The G-connection clearly satisfies the trace condition since $R' = d\omega' + \omega'^2 = 0$. The corresponding integrated Chern-Simons term takes a very simple form,

$$\begin{aligned} I &= -\,2\pi \int_D d^3y\ \epsilon^{ijk}\,\Omega_{ijk} \\ &= \frac{i}{12\pi}\int_D d^3y\ \epsilon^{ijk}\ \mathrm{Tr}\left(\omega'_i\omega'_j\omega'_k\right)\ . \end{aligned} \tag{6}$$

Its variation is precisely the two-dimensional anomaly,

$$\begin{aligned} \delta I &= -\,\frac{i}{12\pi}\int_D d^3y\,\epsilon^{ijk}\partial_i\ \mathrm{Tr}\left(\,\epsilon\,\partial_j\omega'_k\right) \\ &= -\,\frac{i}{12\pi}\int_D d^3y\,\epsilon^{ijk}\partial_i\ \mathrm{Tr}\left(\,\epsilon\,\partial_j\omega_k\right) \\ &= -\,\frac{i}{4\pi}\int d^2x\,\epsilon^{\mu\nu}\ \mathrm{Tr}\left(\,\epsilon\,\partial_i\omega_j\right)\ \partial_\mu\phi^i\partial_\nu\phi^j\ , \end{aligned} \tag{7}$$

where we have used the fact that ϵ belongs to H. Therefore (6) cancels the anomaly of the effective action.

In higher dimensions, the G-connection can also be used to cancel the sigma model anomaly. The variation of the Chern-Simons term cancels the anomaly up to local counterterms [4]. These extra counterterms do not appear in two dimensions. The anomaly matching conditions, however, remain unchanged.

Actually, it is not necessary for the fermion representations ρ_H to form complete G-representations ρ_G. All that is necessary is for the two representations to give the *same* anomalous variation of I. A general representation ρ_G of G decomposes into a sum of representations of H. To cancel the anomaly, these representations must include the fermion representations ρ_H. The other representations must be anomaly-free under H.

This condition for anomaly cancellation is precisely the 't Hooft matching condition [6]. It implies that the nonlinear sigma model can be thought of as a low-energy effective Lagrangian corresponding to an underlying preonic theory. The fermions in the underlying theory form representations ρ_G of a global symmetry group G. If G is spontaneously broken to H, the low-energy theory is a nonlinear sigma model on the manifold G/H, with the fermions transforming in representations ρ_H of H. The fermions in the two theories are related by a chiral G-rotation. This change of variables gives rise to a Jacobian that is precisely the Chern-Simons term. Since the preonic theory is globally G-invariant, the H-anomalies of the nonlinear sigma model cancel between the Chern-Simons term and the low-energy fermions [3,4].

If the 't Hooft condition cannot be satisfied, there is no connection ω' such that $\mathrm{Tr}\, R'^n = 0$. The sigma model anomaly cannot be cancelled, and the nonlinear model does not correspond to any underlying preonic theory. Since the curvature R generates the holonomy group of G/H, we say that such a sigma model suffers from a *holonomy anomaly.*

Within the context of chiral dynamics, it is natural for fermions to transform in representations ρ_H of H. Other choices, of course, are possible. In supersymmetric models, for example, fermions are sections of the tangent bundle $\mathcal{T}$ to $\mathcal{M} = G/H$. More generally, fermions can be sections of vector bundles $\mathcal{U}$ associated to $\mathcal{T}$. In this case the fermions form representations $\rho_{\mathcal{H}}$ of the structure group $\mathcal{H}$ of the tangent bundle $\mathcal{T}$.

If the fermions are to transform in *any* representation of $\mathcal{H}$, it is necessary to introduce an orthonormal frame $e_i{}^a$ on G/H. The orthonormal frame, or vielbein, depends on the coordinates ϕ^i. It gives an orthonormal basis in the tangent space at each point of G/H. The Lagrangian for the sigma model is again given by equation (1). The only difference is that the connection $\omega_i{}^A{}_B$ is valued in the Lie algebra of the structure group $\mathcal{H}$, rather than in the Lie algebra of the isotropy group H. The connection $\omega_i{}^A{}_B$ is just the spin connection in the fermion representation $\rho_{\mathcal{H}}$.

The classical sigma model action is invariant under general coordinate transformations ξ^i, and under local frame rotations L^{ab}. The coordinate transformations are analogous to global G-transformations, and the frame rotations correspond to the compensating H-projections. This can be seen by choosing a frame in which the vielbeins are symmetric in i and a. In this frame, general coordinate transformations must be accompanied by gauge-restoring local frame rotations,

$$\delta\, e_i{}^a \;=\; \partial_i \xi^j\, e_j{}^a \;+\; L^{ab}\, e_{ib}\ , \tag{8}$$

where $L^{ab} = \frac{1}{2} e^{ja} e^{kb} (\partial_j \xi_k - \partial_k \xi_j)$. The frame rotation L^{ab} induces field-dependent frame transformations on the spinors χ and on the connection ω. If the fermions are in anomalous representations of $\mathcal{H}$, the effective action is not invariant under diffeomorphisms.

As before, the holonomy anomaly can sometimes be cancelled a Chern-Simons term formed from the connection $\omega_i{}^A{}_B$. If $\operatorname{Tr} R^n = 0$, the Chern-Simons term can be pulled back to spacetime, and Lagrangian mechanics is well-defined. If $\operatorname{Tr} R^n \neq 0$, one must modify the connection by adding torsion.

On a group G one can always find a connection ω' such that $R' = 0$. This follows from the fact that every group manifold is parallelizable. The connection ω' is valued in the Lie algebra of the structure group $\mathcal{G}$ of G. Since G/H can be locally embedded in G, the connection ω' differs from ω by torsion. The connection ω' can be used to cancel the anomaly provided the fermion representations $\rho_{\mathcal{H}}$ can be completed to representations $\rho_{\mathcal{G}}$ of $\mathcal{G}$. This is the appropriate generalization of the 't Hooft matching condition to the present case.

It is now obvious how to cancel the anomaly for a general Riemannian manifold $\mathcal{M}$. As before, the fermions form representations $\rho_{\mathcal{H}}$ of the structure group $\mathcal{H}$ of the tangent bundle $\mathcal{T}$. If the fermions are anomalous, one must add a Chern-Simons term. The Chern-Simons term can be pulled back whenever $\mathrm{Tr}\, R^n = 0$. Otherwise one must modify the connection.

For manifolds $\mathcal{M} = G/H$, we were able to cancel the anomaly using the fact that there is a natural embedding of G/H in G. For a general Riemannian manifold $\mathcal{M}$, the story is almost the same [4]. We now use the fact that $\mathcal{M}$ can be isometrically embedded in a flat space of sufficiently high dimension d. As before, the spin connection ω on $\mathcal{M}$ can be extended to a connection ω' on $\mathbf{R}^d$ if the fermion representations $\rho_{\mathcal{H}}$ can be completed to representations $\rho_{\mathcal{G}}$ of $\mathcal{G} = O(d)$. Since $\mathbf{R}^d$ is flat, $R' = 0$, and the Chern-Simons term built from ω' can be pulled back to spacetime. It cancels the anomaly, up to local counterterms, and the sigma model on $\mathcal{M}$ is well-defined [4].

The purpose of this talk was to give a unified picture of sigma model anomalies. We have shown how the anomalies can be cancelled by Chern-Simons terms built out of appropriate connections ω'. When generalized 't Hooft conditions are satisfied, the Chern-Simons term can be pulled back to an integral over spacetime, and the effective action is well-defined.

---

[1] G. Moore and P. Nelson, Phys. Rev. Lett. 53 (1984) 1519; and Harvard preprint HUTP-84/A076.

[2] E. Cohen and C. Gomez, CERN-TH-4043 (1984); P. DiVecchia, S. Ferrara and L. Girardello, Phys. Lett. 151B (1985) 199.

[3] A. Manohar, G. Moore and P. Nelson, Phys. Lett. 152B (1985) 68.

[4] J. Bagger, D. Nemeschansky and S. Yankielowicz, SLAC-PUB-3588.

[5] C. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2247.

[6] G. 't Hooft, in *Recent Developments in Gauge Theories*, eds. G. 't Hooft *et al.*, (Plenum, New York, 1980).

[7] W. Bardeen, Phys. Rev. 184 (1969) 1848.

[8] S. Chern and J. Simons, Proc. Nat. Acad. Sci. 68 (1971) 791.

[9] E. Witten, Nucl. Phys. B223 (1983) 422.