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Fractional Charge in a Nut-Shell [1]

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Abstract

We study the physics of charge frationalization using simple methods. The strategy is to count the number of states of the theory with solitons relative to the same theory with trivial background fields. The interplay between high and low energy contributions is exposed and the topological properties clarified.

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We study fermions interacting with static external background fields with non-trivial topology. The physics of the problem is simple. The spatial variations of the background fields (b.f.) (gradients) act as "localized" scattering potentials. The spectrum of the free problem is modified. Bound states may be formed (some of them of topological origin), and there is a change in the density of continuum states with respect to the free problem (constant or trivial fields).

Since the spatial variations of the (b.f.) are localized, the change in the density of states is finite. The continuum wave functions are phase shifted near the region of the spatial variations. We study the case when the theory is not CP-invariant, where there is no ambiguity arising from states at zero energy.

The vacuum charge is related to the asymmetry between the positive and negative part of the spectrum (Atiyah-Patodi-Singer invariant)

$$Q = -\frac{1}{2}\eta \tag{1}$$

As a measure of the asymmetry in the spectrum of the Hamiltonian H we introduce the fundamental quantity $B(E)$ given by

$$B(E) = \det\left[\frac{H+E}{H-E}\right] \tag{2}$$

$B(E)$ compares the positive and negative parts of the spectrum. We also introduce the even part of the resolvent and odd part of the density of states.

$$G_e = \frac{1}{2}T_r\left[\frac{1}{H+E} + \frac{1}{H-E}\right] = \frac{1}{2}\frac{d}{dE}\,ln\,B(E) \tag{3}$$

$$\rho_{odd} = \frac{1}{2\pi} Im\, G_e(E+i\eta) \tag{4}$$

Then

$$\eta = 2\int_0^\infty \rho_{odd}(E)dE = \int_0^\infty [\rho(E) - \rho(-E)]dE \tag{5}$$

Regulators are not necessary: the localized spatial variations induce finite changes in the density of states (DOS) and B(E) is a direct comparison of the positive and negative energy spectrum of the full H. We write above the continuum thresholds

$$B(E) = |B(E)|e^{i\delta(E)} \tag{6}$$

where $\delta(E) = \delta^+(E) - \delta^-(E)$(difference of phase shifts for $E > 0$ and $E < 0$). Therefore

$$\eta = N^+ - N^- + \frac{1}{\pi}[\delta(\infty) - \delta(E_T)] \tag{7}$$

where $N^\pm$ is the number of positive and negative energy bound states and E_T = threshold energy.

The most general Hamiltonian of interest in 1-spatial dimension is

$$H = -i\sigma_2 \tfrac{d}{dx} + \sigma_1 \phi(x) + \sigma_3 K(x) \tag{8}$$

$$H\chi = E\chi \tag{9}$$

Writing

$$\phi(x) = \rho(x)\cos\theta(x) \qquad K(x) = \rho(x)\sin\theta(x) \tag{10}$$

and performing a chiral rotation

$$\psi(x) = e^{i\sigma_2\theta(x)}\chi(x), \tag{11}$$

we find

$$B(E) = \frac{T_\chi(E)}{T_\chi(-E)} \tag{12}$$

where T_χ is the transmission coefficient of the spinors χ that are eigenspinors of the transformed Hamiltonian

$$H_\chi = -i\sigma_2 \frac{d}{dx} + \frac{1}{2}\frac{d\theta}{dx} + \rho(x)\sigma_1 \tag{13}$$

The spinors χ are, asymptotically, eigenstates of charge conjugation (σ_3). Positive and negative energy spinors interact with opposite sign with the potential ($\theta'(x)$); this is the basic reason for the asymmetry in the spectrum (notice that the spectrum of $H_\chi(\theta' = 0)$ is symmetric). We have studied the cases:

$$\phi(x) = \begin{cases} -\phi & x < 0 \\ \phi(>0) & x > 0 \end{cases} \qquad K(x) = \begin{cases} K & x < -d \\ K_0 & -d < x < d \\ K & x > d \end{cases} \tag{14}$$

and also

$$\phi(x) = \begin{cases} -\phi & x < -d \\ 0 & -d < x < d \\ \phi & x > d \end{cases} \qquad K(x) = \begin{cases} K & x < -d \\ K_0 & -d < x < d \\ K & x > d \end{cases} \tag{15}$$

These examples provide a setting in which the adiabatic approximation breaks down and also the chiral angle $\theta(x)$ winds by 2π, allowing us to understand whether these ambiguities have any influence on the vacuum charge.

For $d = 0$ these examples correspond to the K = constant case, where we find that there is an operator U such that $\{H, U\} = 0$. The existence of U ensures that the net number of levels crossing $E = 0$ in the process of deforming $\phi(x)$ is zero (zero spectral flow). We find for this case

$$\delta(\infty) = \Delta\theta \equiv \theta(x = +\infty) - \theta(x = -\infty), \quad -pi \le \Delta\theta \le \pi \tag{16}$$

$$N^+ - N^- = \frac{1}{2} sign(K)[sign(\phi_\pm) - sign(\phi_-)]; \quad \phi_\pm = \phi(x = \pm\infty) \tag{17}$$

and

$$\frac{\delta(0)}{\pi} = N^+ - N^- \tag{18}$$

Therefore $\delta(0)/\pi$ cancels the bound states contribution (Levinson's) theorem, but only when there is no spectral flow. When there is spectral flow, $\delta(0)/\pi$ adds to $N^+ - N^-$. The threshold phase shift $\delta(0)$ changes by π (or $-\pi$) when a new bound state emerges from the positive or negative continuum; therefore η does not change when new bound states appear.

For $K_0 < 0$ and $d \neq 0$, $\Delta\theta$ jumps by 2π; however for $d \ll 1/|K_0|$ not bound state crosses $E = 0$ (no spectral flow). This trivial 2π is compensated by a change of 2π in $\delta(0)$ and is associated to the definition of the branches of $\delta(E)$ and not to new bound states. Therefore $[\delta(\infty) - \delta(0)]$ stays invariant ($\delta(\infty) = \Delta\theta$).

When $d > 1/|K_0|$, one bound state crosses $E = 0$ but $\delta(\infty) - \delta(0)$ remains constant; η jumps by $+2$, and spectral flow has occured. Although the winding of $\Delta\theta$ by 2π may indicate spectral flow and a change in the vacuum charge Q by one unit, this does not occur unless this change in fields takes place over a distance $d \gg \rho(x)$ ($\rho(x)$ = local mass term). The integral values of the charge depend on this spectral flow and local details of the b.f. For the charge conjugate case ($K \to 0^+$) with a soliton profile

$$\begin{gathered} N^+ - N^- = 1; \quad \delta(\infty) = \pi; \quad \delta(0) = \pi \\ \eta = 1 + \tfrac{1}{\pi}[\pi - \pi] \end{gathered} \tag{19}$$

The bound state (at E=0) is the Jackiw-Rebbi-Schrieffer "zero mode." Expression exposes the interplay between high and low energy contributions.

We have also studied the vacuum charge in 2+1 dimensions for the case

$$H = i \not{D} + m\sigma_3 \qquad i \not{D} = i\not{\delta} - \not{A}, \tag{20}$$

where A_μ corresponds to a static vortex. For constant m, there is no spectral flow as in 1 + 1 dimension (for K = constant). In Ref. (3) we showed that the phase shift at infinite energy is

$$\frac{\delta(\infty)}{\pi} = -sign(m) \int d^2x \epsilon_{ij} F_{ij} \tag{21}$$

this phase shift at infinite energy is completely determined by the axial anomaly in 1 + 1 - Euclidean dimensions where the vortex plays the role of an instanton.

We also found that because of the long range nature of the gauge fields there are $\delta(\infty)/\pi$ states at threshold $E = \pm m$ and the vacuum charge is

$$Q = \frac{1}{2} sign(m) \int d^2x \epsilon_{ij} F_{ij} \tag{22}$$

The relation between the chiral anomaly in 1 + 1 dimensions and the vacuum charge in 2 + 1 dimensions is easily understood. The operator $i \not{\delta} - \not{A}$ in H is the 1 + 1 - dimensional Dirac operator; with topologically non-trivial A_μ, this operator has "zero modes." These zero modes are eigenstates of σ_3 therefore they are eigenstates of H with energies $E = \pm m$, i.e. threshold states. The sign of the energy depends on the sign of the mass and the flux of the vortex. The sign (m) factor in the expression for the charge plays the same role as the $\pm$ in the 1 - dimensional case; these states are filled or empty depending on sign (m).

Some of these states are bound states, but there are also resonant states (non-normalizable). For a sphericaly symmetric vortex the "zero mode" wavefunction behaves as

$$\psi_0 \underset{r\to\infty}{\sim} r^{J-F}$$

where J=angular momentum
F=vortex flux

In general we write the η-invariant as

$$\eta = \eta_{SF} + \eta_V, \tag{23}$$

Where η_{SF} is a local quantity, i.e. it has information about details of the b.f. It only has contributions from low energy $(\delta(0), N^{\pm})$ and is an even integer or zero. It corresponds to spectral flow (energy levels crossing zero).

In contrast η_V is a topological invariant and is completely determined by $\delta(\infty)$ i.e. is a high energy property of the theory. Since it is a high energy property it can be easily computed, and directly related to the anomaly in one less dimension and topological indices. (Callias, Atiya-Singer, etc).

There is a very important physical aspect that relates fractionalization to anomalies. The phase shift at infinite energies is independent of the mass scale ρ in the Hamiltonian. A non-zero phase shift at large energies implies that the energy levels are shifted relative to the non-interacting system. Since the large eigenvalues are shifted equally, the density of states (the energy derivative of the phase shifts) goes to zero as $1/E^3$ as $E \to \infty$.

The total deficity of continuum states cannot depend on the mass scale. It is given by $\delta(\infty) - \delta(0)$ and $\delta(0) \sim n\pi$ (n = integer). The mass and the potential range R set the scale of the density of states as a function of energy. As the mass $\to 0$ and $R \to \infty$, the continuum relative density of states becomes a δ-function at E=0.

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