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COUPLED VIBRATIONAL MODES OF THE SLC ARC MAGNET AND SUPPORT SYSTEM*

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1. INTRODUCTION

The magnet support system for the SLC Arcs will be a long series of pedestals with each pedestal supporting the ends of two adjacent magnets. It has been pointed out by several authors[1,2] that random magnet vibrations in the Arc with amplitudes larger than 0.1 μm rms are potentially harmful for the SLC operation. In order to assess the vibrational behavior of the Arc magnet system, we need to understand: (1) the sources and characteristics of the ground disturbances, (2) the coupled vibrational modes of the composite pedestal-magnet system and, (3) the response of the system to ground disturbance. This note is an attempt to study item (2).

Because the relevant vibration frequency for SLC is between 1 to 40 Hz,[2,3] in the following effort to identify the normal modes of the magnet-pedestal support system, we will pay particular attention to the frequency range from 1 to 40 Hz.

It has been estimated that[4] a single pedestal loaded by a single magnet (W = 1200 lb) will vibrate at a frequency of $f_{p,H}$ = 74.3 Hz and $f_{p,V}$ = 330 Hz. In addition one finds that a welded Arc magnet (model number EM 4004), which is 2.5 m long and weighs 1200 lb), when simply supported rigidly at both ends, sags under its own weight by the amount of δ_H = 12.9 mils and δ_V = 9.3 mils.[4] Therefore, according to Eq. (13) the transverse flexural vibrational frequencies of the magnet can be estimated to be $f_{M,H}$ = 31.02 Hz and $f_{M,V}$ = 36.54 Hz. With the vibrational frequencies of single magnet and pedestal known, we will calculate the coupled vibrational modes of a string of magnets and pedestals.

2. MAGNET AS A SIMPLY SUPPORTED UNIFORM BEAM

It is well known[5] that the differential equation governing the transverse vibration of an uniform beam is the Euler equation,

$$EI\,\frac{\partial^4 y}{\partial x^4} = -\rho\,\frac{\partial^2 y}{\partial t^2} \quad . \tag{1}$$

where ρ is the line mass density, E is the Young's modulus, and I is the moment of inertia of the beam cross section about the beam centerline. Note that we have assumed the cross section of the beam remains rigid under deflection. The treatment therefore excludes those modes in which the magnet cross section deforms. Those modes presumably are not strongly driven by ground disturbances.

Consider the whole beam vibrating at mode frequency ω, then Eq. (1) becomes

$$EI\,\frac{d^4 y}{dx^4} - \rho\,\omega^2 y = 0 \tag{2}$$

or

$$\frac{d^4 y}{dx^4} - \beta^4 y = 0 \quad , \quad \beta^4 = \frac{\rho\omega^2}{EI} \quad . \tag{3}$$

The general solution of Eq. (3) can be expressed in the following form:

$$y(x) = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x \ . \tag{4}$$

The vibrational frequency ω is related to the constant β, and the coefficients A, B, C and D are determined by the prescribed boundary conditions (BC).

For SLC, the magnet will be close to that of a simply-supported system, the BC requires that at both ends of the magnet y and its second derivative should be zero. The only possible solution is

$$A = B = C = 0 \quad \text{and} \quad \sin\beta\ell = 0 \quad . \tag{5}$$

Therefore, the normal modes are given by

$$y(x) = D\sin\beta_n x \quad , \tag{6}$$

where $\beta_n \ell = n\pi$, or

$$\omega_n = \sqrt{\frac{EI}{\rho}}\,\frac{n^2\pi^2}{\ell^2} \tag{7}$$

It is important to be aware of the fact that the frequencies are proportional to n^2. This is very different from the modes of a string under tension for which ω_n is proportional to the mode number n. This quadratic behavior of ω_n on n comes from the fact that the equation of motion, Eq. (1), is fourth order.

Now we are in a position to find the relationship between the vibration frequency and the sag of the magnet. For a magnet under its own weight, the Euler's equation (1) becomes

$$EI\,\frac{\partial^4 y}{\partial x^4} = \rho\,g \tag{8}$$

where g is the gravitation constant. Now the general solution can be put in the form:

$$y = A + Bx + Cx^2 + Dx^3 + \frac{1}{24}\frac{\rho g}{EI}x^4 \quad . \tag{9}$$

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The simply supported BC gives

$$y = \frac{1}{24}\,\frac{\rho g}{EI}\,\ell^4 \left(\frac{x}{\ell}\right)\left[1-\left(\frac{x}{\ell}\right)\right]\left[1+\left(\frac{x}{\ell}\right)-\left(\frac{x}{\ell}\right)^2\right] \qquad (10)$$

which implies the sag at the middle of the magnet:

$$\delta = y\left(\frac{\ell}{2}\right) = \frac{5}{384}\,\frac{\rho g}{EI}\,\ell^4 \quad . \qquad (11)$$

Combining Eqs. (11) and (7), we can relate the lowest mode vibrational frequency to the sag as:

$$f \doteq \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}\left(\frac{5\pi^4}{384}\right)^{1/2} \quad . \qquad (12)$$

The frequencies derived from Eq. (12) using the measured sag as input agree with the measured frequencies to about 10%.[4]

3. A SINGLE MAGNET ON TWO PEDESTAL

So far we have discussed the transverse vibrational properties of magnets on rigid supports. But, as mentioned in the introduction, the pedestal itself could vibrate with its own frequency. The problem now is to calculate the modes of the combined system.

A magnet supported by two pedestals can be analyzed by the model as shown in Fig. 1. The pedestals are represented as springs with spring constant k. The solution can still be expressed by Eq. (4), but now the boundary condition becomes

$$\begin{cases} M(0) = 0\,, \quad \text{and} \quad M(\ell) = 0 \\ F(0) = ky(0) = -EI\,y'''(0) \\ F(\ell) = ky(\ell) = EI\,y'''(\ell) \end{cases} \qquad (13)$$

where $F(x)$ and $M(x)$ are the force and moment acting on the element at position x.

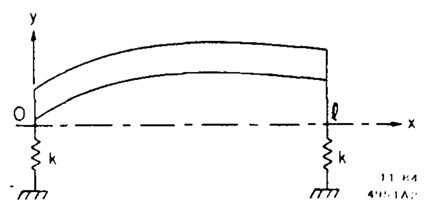

Fig. 1. A magnet supported by two pedestals.

Equation (13) implies that

$$\begin{cases} A - C = 0 \\ A\cosh\beta\ell + B\sinh\beta\ell - C\cos\beta\ell - D\sin\beta\ell = 0 \\ A + C = -\alpha(B-D) \\ A\cosh\beta\ell + B\sin\beta\ell + C\cos\beta\ell + D\sin\beta\ell \\ \quad = \alpha(A\sinh\beta\ell + B\cosh\beta\ell + C\sin\beta\ell - D\cos\beta\ell) \end{cases} \qquad (14)$$

where

$$\alpha = \frac{EI}{k}\,\beta^3 \quad . \qquad (15)$$

From Eq. (14) we can first express A and C in terms of B and D, then for the system to have a unique solution the determinant formed by the coefficients for B and D should be zero. It is then found that the normal mode frequency satisfies

$$\det = 2\sin\beta\ell\sinh\beta\ell - 2\alpha(\sin\beta\ell\cosh\beta\ell - \cos\beta\ell\sinh\beta\ell)$$
$$- \alpha^2(\cos\beta\ell\cosh\beta\ell - 1) = 0 \quad . \qquad (16)$$

It is interesting to see that Eq. (16) agrees with Eq. (5) for the simply support case ($\alpha = 0$).

4. THE M-MAGNET AND $(M+1)$ PEDESTAL SYSTEM

Now let us consider the M–magnet–$(M+1)$–pedestal system as shown in Fig. 2. Let the general solution to the i^{th} magnet be

$$y_i(x) = A_i\cosh\beta x + B_i\sinh\beta x + C_i\cos\beta x + D_i\sin\beta x \qquad (17)$$

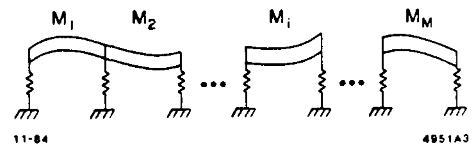

Fig. 2. M–magnet–$(M+1)$–pedestal system.

where x is measured from the left end of each magnet, and y stands for the transverse deflection of the magnet at position x. Applying the boundary conditions to be satisfied for simply supported ends, we obtain $4M$ coupled algebraic equations. Again, the A_i's and C_i's can be expressed in terms of the B_i's and D_i's, as

$$A_i = C_i = \frac{D_i\sin\beta\ell - B_i\sinh\beta\ell}{\cosh\beta\ell - \cos\beta\ell}\,, \quad i = 1, 2, \ldots, M \qquad (18)$$

and the equations for the B_i's and D_i's can be summarized in the matrix form:

$$K\,\overline{Y} = 0 \qquad (19)$$

where $\overline{Y}$ is a state vector, and when transposed is

$$\overline{Y}^T = (\,B_1, D_1, B_2, D_2, \ldots\, B_M, D_M\,) \qquad (20)$$

and K is the $2M \times 2M$ coefficient matrix. For complete derivation and expressions of the matrix K please see Ref. 3.

In order to have non-trivial solutions to Eq. (19), the determinant of the matrix K should be equal to zero, *i.e.*,

$$\det K = 0 \quad . \qquad (21)$$

The discrete values of ω_n which satisfy Eq. (21) are the normal mode frequencies of the composite system.

5. SOLUTIONS OF THE NORMAL MODES

To find the proper value ω to satisfy Eq. (21), we have to know all the constants needed in the matrix K. The basic input parameters required are the constants α and β. From Eq. (3), β is related to ω by

$$\beta = \left(\frac{\rho}{EI}\right)^{1/4}\omega^{1/2} \qquad (3)$$

and α is related to ω through Eq. (15),

$$\alpha = \frac{EI}{k}\,\beta^3 = \left(\frac{EI}{\rho}\right)\frac{\rho}{k}\,\beta^3 \quad . \qquad (15)$$

For the model magnet EM 4004, we have δ_h = 12.9 mils, ρ = 0.03182 lb/in, and ℓ = 97.6 in, which give

$$\beta\ell = 0.2251\,\omega^{1/2} \qquad (22)$$

$$\alpha = 0.00002039\ \omega^{3/2} \tag{23}$$

After the K matrix is constructed, we use the drive routine DGEFDI of LINPACK to find the determinant of the matrix K. We then numerically look for the zeros of the determinant by scanning ω in order to find the normal mode frequencies.

For example, let us look at the case of a single magnet. For a magnet simply-supported on rigid pedestals, the frequencies of the lowest two modes derived from the sag by Eq. (7) are $\omega_1 = 194.899$ (31.02 Hz) and $\omega_2 = 779.596$ (124.08 Hz). In comparison, for the rigid support the condition of determinants equal to zero by Eq. (16) gives $\omega_1 = 194.782$ (31.00 Hz) and $\omega_2 = 779.128$ (124.00 Hz), almost identical to the analytic calculation. It is especially interesting to see that the n^2 dependence of the frequency is correctly predicted. If we use the realistic stiffness constant of the pedestal, the frequencies are shifted to $\omega_1 = 188.155$ (29.95 Hz) and $\omega_2 = 676.036$ (107.59 Hz). This is no longer exactly $4 \times \omega_1$.

With ω known, the expansion coefficients A, B, C and D for the beam deflection can be found through Eq. (14), and the corresponding patterns are plotted in Fig. 3 for the first three modes. It is worth emphasizing that there are $(n-1)$ nodes in the n^{th} mode.

Next let us look at the case of ten magnets on eleven pedestals as an example to illustrate the behavior of the composite magnet and pedestal system. The requirements that the determinant equals zero gives ten ω's clustered around the fundamental mode $\omega_1 = 188.155$ and another ten ω's clustered around $\omega_2 = 676.036$. Specifically, the ten fundamental modes now range from 182.23 to 194.49. Again, the first three cases of the fundamental mode are plotted in Fig. 4.

For comparison, we list in Table 1 the results of the calculation of the first two modes of a single magnet and a coupled magnet-pedestal system.

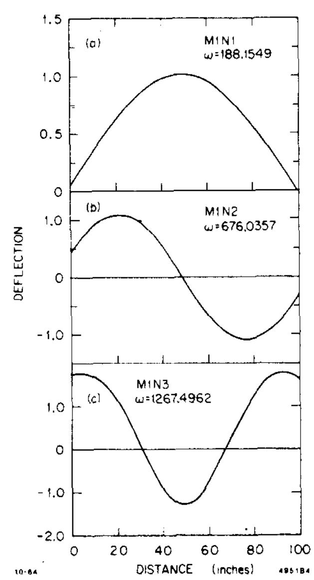

Fig. 3. The deflection pattern of the first three modes of a magnet.

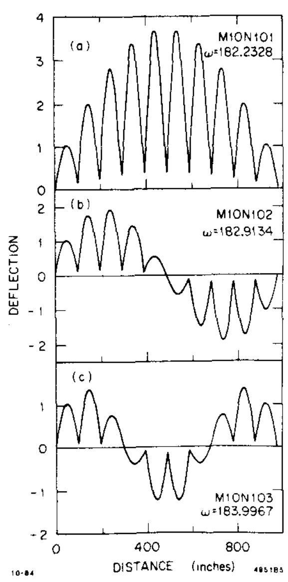

Fig. 4. First three deflection patterns of the first mode of the 10–magnet system.

Table 1. Summary of Normal Mode Frequencies

	Single Magnet ($M = 1$)		Magnet–Pedestal System ($M = 10$)	
	Rigid Support	On Pedestal	On Pedestal	
Mode	Numerical Eq. (16) $\alpha = 0$	Numerical Eq. (16) $\alpha \neq 0$	Numerical Eq. (21) $\alpha \neq 0$	
			29.00	30.07
			29.11	30.36
$n = 1$	31.00 Hz	29.95	29.28	30.61
			29.51	30.82
			29.78	30.95
			94.96	107.75
			96.16	119.20
$n = 2$	124.00 Hz	107.59	98.07	116.15
			100.71	120.07
			103.96	122.92

Since each magnet is 97.6 inches long, the pedestals are located at 97.6 inches intervals. It is interesting to see that the deflection of the pedestals form a pattern like that of the magnets, *i.e.*, there are $(\ell-1)$ nodes in the ℓ^{th} case of the $n = 1$ mode. However, the deflection of the magnets themselves in the $n = 1$ mode is always one with half sine waveform. The pattern formed by the pedestal will be of importance when the response of the system under the ground vibration is to be estimated.

In summary, we have proved that if the pedestals are stiff enough, the coupled system does not vibrate at any lower frequency than the single magnet frequency; therefore, we only have to concentrate on the lowest mode without worrying about the lowering of the vibration frequencies from the higher order modes through coupling. Any method to stiffen a magnet to raise the vibrational frequency beyond 40 Hz will make the coupled system vibrate at a higher frequency, as well.

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