SLAC – PUB – 3601
March 1985
(A)

THE PLASMA WAKE FIELD ACCELERATOR*

PISIN CHEN†
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

and

J. M. DAWSON
Department of Physics
University of California, Los Angeles, California 90024

ABSTRACT

A new scheme of electron acceleration, employing relativistic electron bunches in a cold plasma, is analyzed. The wake field of a leading bunch is derived in a single-particle model. We then extend the model to include finite bunch length effect. In particular, we discuss the relation between the charge distributions of the driving bunch and the energies transformable to the trailing electrons. It is shown that for symmetric charge distribution of the driving bunches, the maximum energy gain for a driven electron is $2\gamma_0 mc^2$. This limitation can be overcome by introducing asymmetric charge distributions,[12] in which case energy gains up to $\sqrt{1+(1-\frac{\pi}{2}+k_p|\varsigma_0|)^2}\,\gamma_0 mc^2$ are possible.

I. INTRODUCTION

The main theme of this Workshop is concerned with new acceleration mechanisms that employ lasers in certain ways. Since the first Workshop, there has been tremendous progress towards further understanding of the plasma beat-wave accelerators[1,2] both theoretically and experimentally, as was revealed during this Workshop.[3] However, it is also clear that in order to realize the plasma beat-wave accelerator at a scale beyond laboratory test-of-principle experiments, significant advances in laser technology are needed. For example, the beat-wave acceleration scheme requires fine tuning[4] between the plasma frequency ω_p and the beat-wave frequency of the laser in order for the wake plasma wave excited by the laser beat-wave to grow linearly. This in turn either puts constraints on

* Work supported in part by the Department of Energy, contract DE-AC03-76SF00515 and by the National Science Foundation, Grant PHY 83-04641.

† Permanent address: Department of Physics, University of California, Los Angeles, California 90024.

Presented at the 2nd International Workshop on Laser Acceleration of Particles, Los Angeles, California, January 7–18, 1985

the uniformity of the plasma density and the linearity of the plasma oscillation, or relies on very high power lasers to shorten the time of growth. In addition, it may be necessary to deliver the laser energy in a pulse shorter than 10 picoseconds in order to avoid competing instabilities.[5] Questions of laser efficiencies are also of considerable concern.

It turns out that if one replaces the lasers by high energy electron bunches traversing the plasma, large energy gradients can still be attained. The idea is to inject a sequence of bunched high energy electrons into a cold plasma. As in the two stream instability, the streaming electrons lose energy to the background plasma by exciting a wake plasma wave. If a late coming electron bunch rides on the wave at a proper phase, it will be boosted to a higher energy due to the longitudinal electric field in the wave.

Chen, Huff and Dawson[6] first studied this scheme using a single particle model under the electrostatic approximation. Later this model was improved by taking full electromagnetic effects into account.[7] Ruth *et al.*,[8] on the other hand, made an important contribution by recognizing the similarity between this scheme and the wake field acceleration scheme using EM cavities. Once this is seen, the "fundamental theorem of beam loading"[9] known in accelerator physics can be readily applied to the "plasma wake field accelerator." Indeed, computer simulations[6,7] have indicated that the maximum energy gain for driven electrons cannot exceed $2\gamma_0 mc^2$, in full agreement with the theorem. But is this really the upper limit of energy gain using wake field acceleration?

In this paper we will review the single particle model with care taken in its detailed derivation, which is a generalization of the nonrelativistic electrostatic method given by Kruer.[10] We then discuss the wake field generated by bunches with finite length. Attention is paid to the relation between the charge distribution of the driving bunch and the maximum energy transformable from the driving bunch to a test charge. We show that the energy gain limitation described above can be surpassed and large energy transforms are possible.

Since the plasma wake field accelerator makes use of already existing accelerators as the source for providing electron bunches, the technical barrier may be lower than that of the plasma beat-wave accelerators. In addition, the accessible free energy of electron beams is comparable to that of the most powerful laser beams. It is thus reasonable to hope that this scheme can meet the more immediate needs of the particle physics community.

II. SINGLE PARTICLE MODEL

Consider a system in which a relativistic electron bunch with initial $\beta_0 = v_b/c \lesssim 1$ streams through a cold, uniform plasma along the z-axis. Assuming that the size of the bunch is much smaller than λ_p^3, where λ_p is the plasma wavelength $(2\pi v_b/\omega_p)$, we can then treat the whole bunch containing q particles as a single particle with charge $Q = eq$.

To the linear approximation the equation of motion and the equation of continuity for the cold, nonrelativistic background plasma are

$$\partial_t \vec{v}_{p1} = -\frac{e}{m}\,\vec{E}_1 \ , \tag{1}$$

and

$$\partial_t n_{p1} + n_{p0}\nabla\cdot\vec{v}_{p1} = 0 \ , \tag{2}$$

where $\vec{E}_1$ is the total electric field contributed from the plasma and the beam: $\vec{E}_1 = \vec{E}_{p1} + \vec{E}_{b1}$, and where the plasma velocity $\vec{v}_p = \vec{v}_{p0} + \vec{v}_{p1}$, $\vec{v}_{p0} = 0$ and the plasma density $n_p = n_{p0} + n_{p1}$, $n_{p0} \gg n_{p1}$ are assumed. The charge and current densities of our beam-plasma system are

$$\rho_1(\vec{x}) = -en_{p1}(\vec{x}) - Q\delta(\vec{x}-\vec{x}_0) \ , \tag{3}$$

and

$$\vec{J}_1(\vec{x}) = -en_{p0}\,\vec{v}_{p1}(\vec{x}) - Q\vec{v}_b\,\delta(\vec{x}-\vec{x}_0) \ , \tag{4}$$

respectively, where $\vec{x}_0$ is the instantaneous position of the beam: $\vec{x}_0 \equiv v_b t\,\mathrm{e}_3$, and $\vec{x} = \rho\mathrm{e}_1 + z\mathrm{e}_3$, in cylindrical coordinates.

We are interested in the wake field $\vec{E}_1$ excited by the beam in the plasma. Our approach is to solve for the scalar potential ϕ_1 and the vector potential $\vec{A}_1$ first. In what follows it is more convenient to introduce a new variable $\varsigma \equiv z - v_b t$ which measures the distance behind the bunch. For the case of an ultra-relativistic electron beam where $\beta_0 \approx 1$, it is a good approximation to take v_b constant over many plasma wavelengths, even though a substantial amount of energy can be transferred to the plasma wave. Under this assumption we put $\partial_t = -v_b\partial_\varsigma$ and $\partial_z = \partial_\varsigma$.

In the Coulomb gauge we have a Poisson equation for the scalar potential,

$$\nabla^2\phi_1 = -4\pi\rho_1 \ , \tag{5}$$

and an inhomogeneous wave equation for the vector potential,

$$\nabla^2\vec{A}_1 - \frac{1}{c^2}\,\partial_t^2\vec{A}_1 = -\frac{4\pi}{c}\,\vec{J}_1 + \frac{1}{c}\,\nabla\partial_t\phi_1 \ . \tag{6}$$

In terms of the new variable ς, and neglecting the term involving the factor $(1-\beta_0^2)$, Eq. (6) can be reduced to

$$\nabla_\perp^2\vec{A}_1 = -\frac{4\pi}{c}\,\vec{J}_1 - \beta_0\,\nabla\,\partial_\varsigma\,\phi_1 \ , \tag{7}$$

where $\nabla_\perp^2$ is the two dimensional Laplacian in the transverse direction, and the symbol β_0 in the last term is saved for the purpose of clarity even though we had assumed $\beta_0 \approx 1$.

First we solve for ϕ_1 in Eq. (5). Taking ς-derivative twice and combining with Eqs. (1), (2) and (3) we get, with the gauge condition $\nabla \cdot \vec{A}_1 = 0$,

$$\nabla^2 \left(\partial_\varsigma^2 + k_p^2\right) \phi_1 = 4\pi Q \, \partial_\varsigma^2 \, \delta\left(\vec{x} - \vec{x}_0\right) \ , \tag{8}$$

where $k_p \equiv \omega_p / v_b = (4\pi n_{p0} e^2 / m v_b^2)^{1/2}$. Since $4\pi\delta(\vec{x} - \vec{x}_0) = -\nabla^2(1/|\vec{x} - \vec{x}_0|)$, the solution of this equation requires that we solve

$$\left(\partial_\varsigma^2 + k_p^2\right) \phi_1 = -Q \, \partial_\varsigma^2 \, \frac{1}{|\vec{x} - \vec{x}_0|} \ . \tag{9}$$

One may wonder whether by dropping the Laplacians from both sides of Eq. (5) we risk omitting the homogeneous solutions that satisfy either $\nabla^2 \Lambda(\vec{x}) = 0$ or $(\partial_\varsigma^2 + k_p^2)\Lambda(\vec{x}) = 0$. Actually if we assume that the plasma is quiescent before the bunch entered in the infinite past, then $\Lambda(\vec{x}) = 0$ identically, so no problem arises.

The solution of Eq. (9) is (see Ref. 10)

$$\phi_1(\rho, \varsigma) = -Q \int_\varsigma^\infty d\varsigma' \ k_p^{-1} \ \sin k_p(\varsigma' - \varsigma) \cdot \partial_{\varsigma'}^2 \ \frac{1}{\sqrt{\rho^2 + \varsigma'^2}} \ , \tag{10}$$

where $|\vec{x} - \vec{x}_0| = \sqrt{\rho^2 + \varsigma^2}$ has been used. Integrating by parts twice we get

$$\phi_1(\rho, \varsigma) = Q \left\{ -\frac{1}{\sqrt{\rho^2 + \varsigma^2}} + k_p \int_\varsigma^\infty d\varsigma' \ \frac{\sin k_p(\varsigma' - \varsigma)}{\sqrt{\rho^2 + \varsigma'^2}} \right\} \ . \tag{11}$$

Next we turn to the vector potential $\vec{A}_1$ in Eq. (7). Taking the ς-derivative on both sides of the equation and invoking the equation of motion for the current term, we obtain

$$\partial_\varsigma \left(\nabla_\perp^2 - \beta_0^2 \, k_p^2\right) \vec{A}_1 = -\beta_0 \, \nabla \left(\partial_\varsigma^2 + k_p^2\right) \phi_1 + 4\pi Q \, \vec{\beta}_0 \, \partial_\varsigma \, \delta\left(\vec{x} - \vec{x}_0\right) \ . \tag{12}$$

Combining with Eq. (8), the above equation decouples entirely from the scalar potential. Removing the ς-derivative common to each term, the equation further reduces to a inhomogeneous modified Helmholtz equation in two dimensions for each component of $\vec{A}_1$:

$$\left(\nabla_\perp^2 - \beta_0^2\, k_p^2\right) \vec{A}_1 = Q \left\{ \beta_0\, \nabla \partial_\varsigma\, \frac{1}{|\vec{x} - \vec{x}_0|} + 4\pi Q\, \vec{\beta}_0\, \delta\left(\vec{x} - \vec{x}_0\right) \right\} . \tag{13}$$

When concentrating on the longitudinal component of $\vec{A}_1$, we get

$$\left(\nabla_\perp^2 - \beta_0^2\, k_p^2\right) A_{1z} = -Q\beta_0\, \nabla_\perp^2\, \frac{1}{|\vec{x} - \vec{x}_0|} . \tag{14}$$

We are actually interested in the wake field trailing behind the bunch on the z-axis, i.e., at position $\vec{x} = z\mathbf{e}_3$. In that case

$$\phi_1(\varsigma) = -\frac{2\pi Q}{\lambda_p} \left\{ \frac{1}{k_p|\varsigma|} + k_p \int_\varsigma^\infty d\varsigma'\, \frac{\sin\, k_p(\varsigma' - \varsigma)}{k_p|\varsigma'|} \right\} , \tag{15}$$

where $\lambda_p = 2\pi k_p^{-1}$, and the corresponding potential in Eq. (14) reads

$$A_{1z}(\varsigma) = -\frac{2\pi Q}{\lambda_p}\, \beta_0^2 \int_0^\infty d\rho'\ K_1(\beta_0 k_p \rho') \cdot \frac{\rho'^2}{[\rho'^2 + \varsigma^2]^{3/2}} , \tag{16}$$

where K_1 is the modified Bessel function of order one.

Plots of ϕ_1 and A_{1z} as functions of $|\varsigma|$ are shown in Fig. 1. Notice that A_{1z} diminishes monotonically whereas ϕ_1 remains oscillatory. The longitudinal electric field is computed by taking the ς-derivative since $E_{1z} = \partial_\varsigma(\beta_0 A_{1z} - \phi_1)$.

We first show that the expressions we get in Eqs. (15) and (16) give the correct physical limit when the background plasma is "turned off." To see this we examine a point right behind the bunch, i.e., $k_p|\varsigma| \ll 1$. In that case

$$A_{1z}(\varsigma) \simeq -\frac{Q\beta_0}{|\varsigma|} + Q\beta_0^3\, k_p^2|\varsigma| \left[\gamma + \ell n\left(\beta_0 k_p|\varsigma|\right) - 1 + \frac{\pi}{2}\, \frac{1}{\beta_0 k_p|\varsigma|} \right] , \tag{17}$$

where γ is the Euler's constant. When turning off the plasma by taking the limit k_p (or ω_p) $\rightarrow 0$, only the first terms in Eqs. (15) and (17) survive, i.e.

$$\lim_{k_p \to 0}\ E_{1z}(\varsigma) = \left(1 - \beta_0^2\right)\, \frac{Q}{|\varsigma|^2} \simeq 0 . \tag{18}$$

Thus we recover the well-known expression for the longitudinal electric field of a relativistic charge moving in vacuum with speed $\beta_0 c$. The remaining terms thus correspond to the plasma response to the presence of the relativistic beam.

Figure 2 shows a plot of E_{1z} without making the $k_p|\varsigma| \ll 1$ approximation. It can be seen that E_{1z} is maximum at $|\varsigma| \simeq (n+\frac{1}{2})\lambda_p$, where n is any non-negative integer, and the contribution to the maximum comes predominantly from the scalar potential. If the separation between the driven bunch and the driving bunch is such that $|\varsigma| \simeq (n+\frac{1}{2})\lambda_p$, the energy gradient attainable for each electron in the driven bunch is

$$G = -eE_{1z} \simeq \frac{8\pi^2 eQ}{5\lambda_p^2} . \tag{19}$$

As an example, consider a plasma of density $n_{p0} = 10^{16}\ \mathrm{cm}^{-3}$ (which corresponds to $\lambda_p \simeq 0.33$ mm). If the driving bunch consists of $q = 5 \times 10^{10}$ particles, Eq. (19) shows that $G \simeq 4.8$ GeV/m. Note that this treatment ignores nonlinear plasma effects and self-consistent effects that act to slow the driving bunch. It is only valid if the electric field does not approach the cold plasma wave-breaking amplitude, and if the electric energy is small compared to the free energy of the driving bunch. The first condition provides an upper limit on the maximum allowed energy gradient: $G_{\max} \simeq \sqrt{n_{p0}}$ eV/cm = 10 GeV/m. Comparing with $G \simeq 4.8$ GeV/m, our linear theory is probably still reasonable. The second condition requires that $(E_{1z}^2/8\pi) \cdot L < q\gamma_0 mc^2/\mathrm{Area}$, where L is the allowable length of the beam-plasma acceleration. Taking the area to be $\pi c^2/\omega_p^2$ and solving for L for the above case gives $L \simeq 0.125\gamma_0$ cm. For $\gamma_0 = 10^5$ (50 GeV) L equals 125 m, so that our constant velocity assumption is extremely well satisfied.

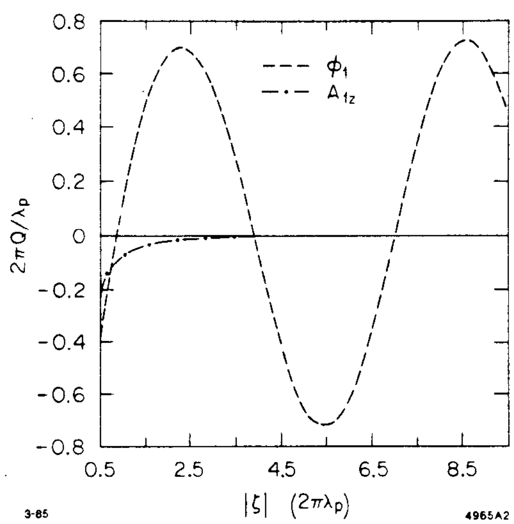

Fig. 1. Potentials as functions of distance behind the driving bunch.

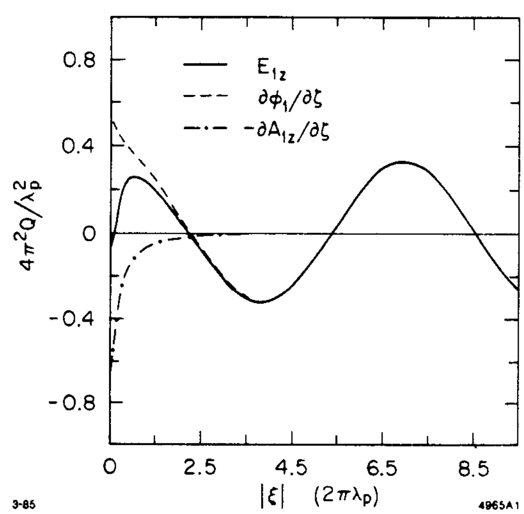

Fig. 2. Longitudinal electric field as a function of distance behind the driving bunch.

III. THE TRANSFORMER RATIO

A. WAKE FUNCTION AND TRANSFORMER RATIO

In the previous section, we have calculated the longitudinal electric field in the wake plasma wave excited by a point charge. It is obvious that the electric field per unit charge is a characteristic of the beam-plasma system. Following the analogous situation in the wake field acceleration in metallic cavities, we shall call it the (longitudinal) "plasma wake function":

$$W_z(\varsigma) \equiv -Q^{-1} E_{1z}(\varsigma) , \qquad \varsigma \leq 0 . \tag{20}$$

For a bunch with finite length, the current density associated with the bunch is in general a function of ς, and each charge in the bunch leaves behind it a wake field characterized by $W_z(\varsigma)$. Physically, a trailing charge will either gain or lose energy according to its phase relationship, $\varphi = k_p\varsigma$, with the leading charge that generates the wake. On the other hand, the trailing charge in addition will lose energy by exciting its own wake. The net electric field generated by a longitudinally finite size bunch is a convolution integral of W_z and the current density $I(\varsigma)$:

$$\mathcal{E}(\varsigma) = \int_{\varsigma}^{\infty} W_z(\varsigma - \varsigma')\, I(\varsigma')\, d\varsigma' , \qquad \varsigma \leq 0 , \tag{21}$$

where $W_z(\varsigma - \varsigma')$ acts as a Green's function.

Let the bunch extend from $\varsigma = 0$ to $\varsigma = \varsigma_0$. If $|\varsigma_0|$ is sufficiently large compared to the plasma wavelength, λ_p, then in general the electric field inside the bunch acts to retard some particles and accelerate others but on the average is retarding in the bunch, and $\mathcal{E}(\varsigma)$ behind the bunch is oscillatory. Let the maximum retarding electric field inside the bunch be $\mathcal{E}_m^-$ and the maximum accelerating electric field induced behind the bunch be $\mathcal{E}_m^+$. The ratio $R \equiv \mathcal{E}_m^+/\mathcal{E}_m^-$ is called the transformer ratio. The physical implication of the transformer ratio is as follows: if a monoenergetic driving bunch with particles of initial energy $\gamma_0 mc^2$ excites a plasma wake field, and if within the length L where the particles in the bunch that experience the maximum retarding field $\mathcal{E}_m^-$ come to a stop, i.e., $\gamma_0 mc^2 = eL\mathcal{E}_m^-$, then the maximum possible energy gain for a test charge behind the bunch will be $R\gamma_0 mc^2$ in the distance L.

It is well known in accelerator physics[9] that the transformer ratio for a point charge is equal to two. This fact has been called the fundamental theorem of beam loading. One can also prove that,[11] assuming only one mode in the cavity, $R \leq 2$ for all finite length bunches with symmetric current distribution. Thus it has been the worry that there is a fundamental limitation on the driven electron energy gain in collinear wake field acceleration.

This limitation is also observed in the computer simulation[6,7] of plasma wake field accelerations. Using a one-and-two-halves dimensional (x, v_x, v_y, v_z) relativistic particle code (physically this corresponds to a one dimensional beam-plasma system where both the beam and the plasma extend infinitely in the transverse directions), it was found that, for the driving beam with current density profile $I(x) \sim 1 + \sin kx$, the driven beam gains energy only up to $\Delta U \lesssim 2\gamma_0 mc^2$ (see Fig. 3).

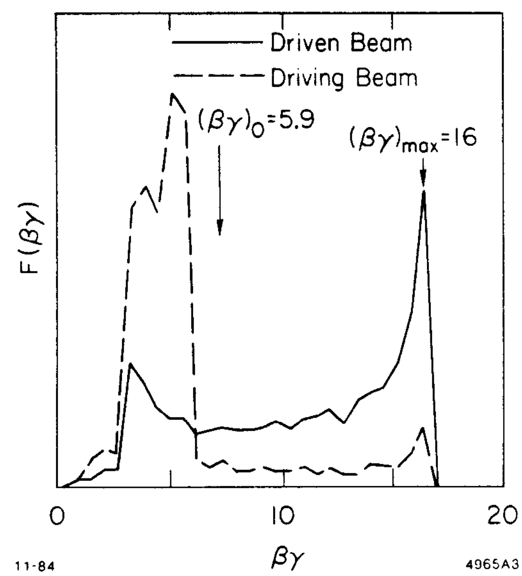

Fig. 3. Momentum distribution of the driving and driven electron beams when the latter has attained its maximum upper limit. The density profiles for both driving and driven electron beams are $1+\sin kx$, 180° out of phase.

B. The Optimal Transformer Ratio[12]

Can this limitation of energy gain be overcome? It turns out that this can be done. If the current distribution is asymmetric with respect to the mid-point of the bunch length, then the generalized fundamental theorem of beam loading (i.e. $R \leq 2$) for finite length symmetric bunches can be evaded. A simple physical way to look at how this can be accomplished is as follows. If the plasma electrons can move out of the way as the bunch charge builds up, the fields within the bunch can be kept very small. If the bunch charge is suddenly terminated the plasma finds itself very non-neutral just behind the bunch, and a large plasma oscillation exists.

To illustrate this issue further let us consider a one dimensional beam-plasma system. For an infinite thin disk moving with speed $\beta_0 c$ in the normal direction the vector potential $\vec{A}_1$ in Eq. (16) vanishes and[8]

$$E_{1z}(\varsigma) = -\partial_\varsigma \phi_1(\varsigma) = \begin{cases} 0 & \varsigma > 0 \\ 2\pi e\sigma & \varsigma = 0 \\ 4\pi e\sigma \cos k_p\varsigma & \varsigma < 0 \ . \end{cases} \tag{22}$$

The corresponding plasma wake function is thus $W_z(\varsigma) = -(e\sigma)^{-1}E_{1z}(\varsigma) = -4\pi\cos k_p\varsigma$. Consider a triangular bunch with current distribution $I(\varsigma)$ rising linearly at the head of the bunch and cut off at the tail, i.e.

$$I(\varsigma) = \begin{cases} 0 & \varsigma > 0 \\ I_0 k_p|\varsigma| & 0 \geq \varsigma \geq \varsigma_0 \ , \quad I_0 > 0 \\ 0 & \varsigma_0 > \varsigma \ . \end{cases} \tag{23}$$

Let the bunch length be $|\varsigma_0| = 2\pi N/k_p$, then it is straight forward to show, from Eq. (21), that inside the bunch,

$$\mathcal{E}^-(\varsigma) = 4\pi k_p I_0 \int_{\varsigma}^{0} \varsigma' \cos k_p(\varsigma - \varsigma')\ d\varsigma' = \frac{4\pi I_0}{k_p}\ (\cos k_p\varsigma - 1)\ , \quad 0 \geq \varsigma \geq \varsigma_0 \tag{24}$$

whereas behind the bunch,

$$\mathcal{E}^+(\varsigma) = 4\pi k_p I_0 \int_{\varsigma_0}^{0} \varsigma' \cos k_p(\varsigma - \varsigma')\ d\varsigma' = -\frac{8\pi^2 I_0 N}{k_p}\ \sin k_p\varsigma\ , \quad \varsigma_0 > \varsigma\ . \tag{25}$$

Thus

$$R = \frac{\mathcal{E}_m^+}{\mathcal{E}_m^-} = \pi N\ . \tag{26}$$

This simple calculation was checked by a computer simulation[13] (Fig. 4) which agrees very well with the prediction. Notice that R is proportional to the number of ripples, N, of $\mathcal{E}^-$. This can be easily understood because the smoother the $\mathcal{E}^-$, the more particles experience $\mathcal{E}_m^-$ which bring them to a stop. Thus it allows more energy to be transformed to the driven electrons. The question naturally arises as to whether the triangular bunches give the best transformer ratios.

To look for a better current distribution, it is more convenient to turn the convolution integral of Eq. (21) inside out. If we specify the desired $\mathcal{E}^-(\varsigma)$ and $\mathcal{E}^+(\varsigma)$ in the entire domain, the corresponding current density $I(\varsigma)$ can be obtained. This can be done by first making a Laplace transform of Eq. (21). The convolution (Faltung) theorem says that the Laplace transform of a convolution integral is equal to the product of Laplace transforms of the functionals in the integrand, i.e.

$$\mathcal{L}\left\{\mathcal{E}(\varsigma)\right\} = \mathcal{L}\left\{\int_{\varsigma}^{0} W_z(\varsigma - \varsigma')\ I(\varsigma')\ d\varsigma'\right\} = \mathcal{L}\left\{W_z(\varsigma)\right\} \cdot \mathcal{L}\left\{I(\varsigma)\right\}\ . \tag{27}$$

Furthermore, it can be proved that the inverse Laplace transform of $\mathcal{L}\{I(\varsigma)\}$,

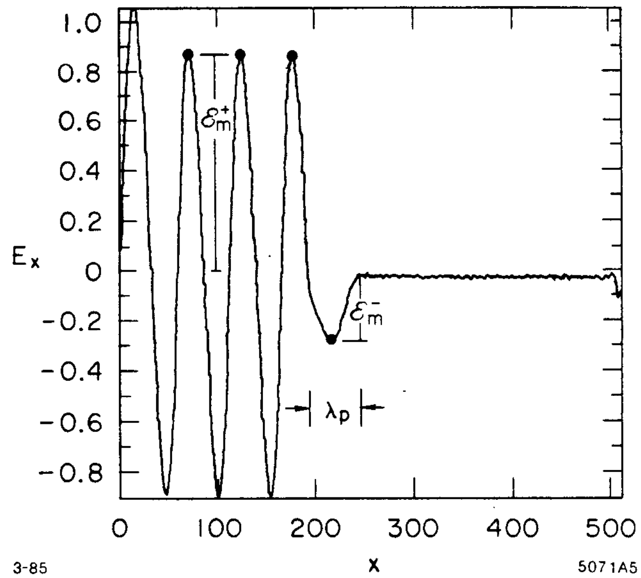

Fig. 4. The electric field generated by a bunch with triangular density profile moving to the right of the picture. The bunch is one wavelength long, and the transformer ratio is $R = \mathcal{E}_m^+/\mathcal{E}_m^- \simeq \pi$.

i.e. the $I(\varsigma)$ itself, can be expressed in the complex plane as

$$I(\varsigma) = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} \frac{\mathcal{L}\{\mathcal{E}(\varsigma)\}}{\mathcal{L}\{W_z(\varsigma)\}} e^{s\varsigma} \, ds \ , \tag{28}$$

where γ is some negative quantity which avoids the integration to be carried along the imaginary axis. To find I, our ansatz is that there exists some smooth function $\mathcal{E}^-$ inside the bunch and a sinusoidal function $\mathcal{E}^+$ behind the bunch.

Ideally, one would like to have a constant $\mathcal{E}^-$ such that all particles experience the same retarding field and stop at the same distance. But it can be shown by carrying out the calculations in Eq. (28) that the solution of $I(\varsigma)$ for such a situation does not exist. This is actually not too surprising because we had insisted that $\mathcal{E}^-$ be the same nonzero value at $\varsigma = 0$, and this is impossible to prepair. Notice that even a delta-function current at the head of the bunch can only provide $\mathcal{E}^-(0) = \frac{1}{2}\mathcal{E}^-(0^+)$ [cf. Eq. (22)].

We should thus relax our ansatz by allowing for some smoother fall of $\mathcal{E}^-$ at the head of the bunch. For instance, if we let

$$\mathcal{E}^-(\varsigma) = \begin{cases} \dfrac{4\pi I_0}{k_p} \sin k_p\varsigma \ , & 0 \geq \varsigma \geq -\dfrac{\pi}{2k_p} \ , \quad I_0 > 0 \\[2ex] -\dfrac{4\pi I_0}{k_p} \ , & -\dfrac{\pi}{2k_p} \geq \varsigma \geq \varsigma_0 \ , \end{cases} \tag{29}$$

it can be shown that the corresponding current distribution is

$$I(\varsigma) = \begin{cases} I_0 \,, & 0 \geq \varsigma \geq -\dfrac{\pi}{2k_p} \,, \\ \\ \dfrac{2}{\pi} I_0 \, k_p |\varsigma| \,, & -\dfrac{\pi}{2k_p} \geq \varsigma \geq \varsigma_0 \,. \end{cases} \tag{30}$$

The transformer ratio in this case becomes

$$R = \sqrt{1 + \left(1 - \frac{\pi}{2} + k_p |\varsigma_0|\right)^2} \,. \tag{31}$$

For $|\varsigma_0| = 2\pi N/k_p$, $R = \sqrt{1 + (1 - \frac{\pi}{2} + 2\pi N)^2}$. We see that the R in this case is larger than the R for the corresponding triangular current distribution.

At first thought it looks like the energy transformation can be indefinitely improved by increasing the bunch length. However, the absolute value of $\mathcal{E}_m^+$ will not increase proportionally unless the charge density in the bunch is kept constant. Thus one faces the technical limitation of a maximum possible peak current which one can provide near the tail of the bunch. The ultimate limit, however, comes from the cold plasma wave-breaking limit which $e\mathcal{E}_m^+$ cannot exceed. But before reaching this limit, nonlinear plasma effects have already set in, and the previous calculations have to be modified.

IV. DISCUSSION

The single particle model described in Section II is useful for finite size bunches since it gives Green's functions for them. The scalar and vector potentials can be obtained by integrating the Green's functions over the finite size of a bunch. This is exemplified by the discussion of finite length bunches and the corresponding transformer ratios in Section III, although for the sake of simplicity we studied the one-dimensional case. A more realistic situation to consider is a bunch with a finite cross-section. In that case the contribution of A_{1z} to W_z at small distance becomes important when we look for $\mathcal{E}^-(\varsigma)$ inside the bunch. We have not discussed the transverse plasma wake function in this paper, it is important for the study of beam dynamics and should be pursued further.

In summary, large energy gradients over long distances of acceleration are attainable in the plasma wake field accelerator. The study of beam-plasma interaction deserves more attention in the plasma beat-wave scheme as well — because it also affects the accelerated electron bunch and will play an essential part in its beam dynamics.

ACKNOWLEDGEMENTS

We thank Karl Bane and Perry Wilson for allowing us to use some of the yet to be published results in Ref. 12.

REFERENCES

1. T. Tajima and J. M. Dawson, Phys. Rev. Lett. 43, 267 (1979).
2. T. Katsouleas and J. M. Dawson, Phys. Rev. Lett. 51, 392 (1983).
3. See the articles related to the plasma beat-wave accelerator in this proceedings.
4. C. M. Tang, P. Sprangle and R. Sudan, Appl. Phys. Lett. 45, 375 (1984).
5. C. Joshi, W. B. Morri, T. Katsouleas, J. M. Dawson, J. M. Kindel and D. W. Forslund, Nature 311, 535 (1984).
6. P. Chen, R. W. Huff and J. M. Dawson, UCLA Report No. PPG-802, 1984, and Bull. Am. Phys. Soc. 29, 1355 (1984).
7. P. Chen, J. M. Dawson, R. W. Huff and T. Katsouleas, Phys. Rev. Lett. 54, 693 (1985).
8. R. D. Ruth, A. W. Chao, P. L. Morton and P. B. Wilson, SLAC-PUB-3374, 1984 (to be published in "Particle Accelerators").
9. A. W. Chao in *"Physics of High Energy Particle Accelerators"*, AIP Conf. Proc. No. 105 (Am. Inst. Phys., New York, 1983).
10. W. Kruer, Ph.D. Thesis, Princeton University, 1969.
11. K. L. F. Bane, P. B. Wilson and T. Weiland, SLAC-PUB-3528, 1984.
12. K. L. F. Bane, P. Chen and P. B. Wilson, *"Collinear Wake Field Acceleration"*, to be presented at the 1985 Particle Accelerator Conference, Vancouver, B.C., Canada, May 13-16, 1985.
13. P. Chen, J. J. Su, J. M. Dawson, P. B. Wilson and K. L. F. Bane, UCLA Report No. PPG-851, 1985.