SLAC – PUB – 3590
February 1985
T

# Is the Chiral Angle Related to the Vacuum Charge? A Study in One-Dimension*

I. A. SCHMIDT†

*Universidad Federico Santa Maria, Valparaiso, Chile*

and

D. BOYANOVSKY AND R. BLANKENBECLER

*Stanford Linear Accelerator Center*

*Stanford University, Stanford, California, 94305*

Submitted to *Physical Review D*

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.
† Work supported by CONYCIT-Chile

## ABSTRACT

A resolution of ambiguities that arise in relating the chiral angle to the vacuum charge is presented; their relationship is studied and clarified. A recently developed formulation is applied to a simple one-dimensional example with fractional charge. This example provides a setting in which the breakdown of the adiabatic approximation and the winding of the chiral angle by multiples of $2\pi$ can be studied. We find that the above situations do not a priori imply a change in the vacuum charge nor any ambiguity in its definition. However, a mathematical and physical criterion for the occurence of level crossing, leading to an integer change in the charge of the vacuum, is provided. The results are given and analyzed from a physical point of view and their generality is discussed.

# 1. Introduction

The physics of fractional quantum numbers of the vacuum in theories with fermions that interact with topologically non-trivial background fields (b.f.) is by now well established and well studied. The remarkable work by Goldstone and Wilczek[1] indicated that the vacuum charge (density) can be a transcendental function of the b.f. This work was based on the seminal paper by Jackiw and Rebbi[2] who discovered the phenomena of fractional charge in the charge conjugate case. These papers sparked great interest in the subject and many authors have reproduced their results using a variety of techniques.[3-8]

Using the adiabatic method, which is an expansion in the ratio of the derivative of the chiral angle to the position dependent mass, the vacuum charge in one spatial dimension was found in Ref.1 to be given by

$$Q = \frac{\Delta\theta}{2\pi} , \tag{1.1}$$

where $\Delta\theta = \theta(x = +\infty) - \theta(x = -\infty)$ is the total net change in the chiral angle between the infinite spatial boundaries.

However, application of this result to specific examples leads immediately to several questions. What happens if the local mass term in the Hamiltonian is zero in a finite region of space? In this region the chiral angle is not defined and the adiabatic expansion certainly breaks down. Does a winding of $\Delta\theta$ by $2\pi n$ indicate the presence of $n$ extra units of charge in the ground state?

*Charge, Spectrum and Spectral Flow:*

In Ref.5 it was shown that the ground state charge is a property of the fermionic spectrum that is given by

$$Q = -\frac{1}{2}\eta \, , \tag{1.2}$$

where the spectral asymmetry $\eta$ is the asymmetry between the number of positive and negative states of the spectrum which consists in general of both bound state and continuum parts.

Also in this reference, it was shown that the quantity $\eta$ can be written in the form

$$\eta = \eta_V + \eta_{SF} \, , \tag{1.3}$$

where $\eta_V$ is the spectral asymmetry obtained from the *simplest* Hamiltonian $H_V$ that incorporates the same asymptotic behavior as the true b.f. The important quantity $\eta_{SF}$ measures the "spectral flow", *i.e.* the net number of energy levels that cross zero in the process of deforming the b.f. in $H_V$ to those of the Hamiltonian of interest $H$ (this is a local change). The "relative" asymmetry $\eta_{SF}$ was shown to be an even integer or zero in Ref.5 .

The ground state charge associated with $\eta_V$ is given by

$$Q_V = \frac{\Delta\theta_V}{2\pi} \qquad -\pi \leq \Delta\theta_V \leq +\pi. \tag{1.4}$$

(Note that $Q_V$ lies between $-\frac{1}{2}$ and $+\frac{1}{2}$.)

Since the charge is only a property of the counting of states in the spectrum, it will be modified only when states cross zero. The reader might expect that when the (position dependent) mass becomes zero in a finite region in space *or* when $\Delta\theta$ winds by $2\pi$ , such level crossing may be induced. However we shall see that this is not the case in general. Indeed, if the mass (hereafter denoted by $\rho(x)$) vanishes or if $\Delta\theta$ changes by $2\pi$ in a distance $d$, the spectrum will not be significantly changed unless $d$ becomes sufficiently large, typically of the order of the inverse of the local mass (in our later example $\kappa_0$).

In this paper we will study a simple example of a problem in one spatial dimension that will (hopefully) illuminate the physics that is occurring by showing that the *ambiguities* due to the vanishing of $\rho(x)$ or to the branches of $\Delta\theta$ do not affect the ground state charge, which is always well defined. In a sense, we will try to choose the most ambiguous example possible that is at the same time mathematically simple.

*General Formulation:*

Following the spirit and discussion of Ref.5 , we introduce the fundamental determinantal ratio

$$B(E) = \det\left[\frac{H+E}{H-E}\right] . \tag{1.5}$$

From this ratio, which is a direct comparison of the positive and negative spectrum, auxiliary quantities follow directly:

$$G_e = \frac{1}{2}Tr\left[\frac{1}{H+E}+\frac{1}{H-E}\right] = \frac{1}{2}\,\frac{d}{dE}\,\ell n\; B(E) \tag{1.6}$$

and

$$\rho_{\text{odd}} = \frac{1}{2\pi}\,\text{Im}\,G_e(E+i\eta)\ , \tag{1.7}$$

where $\rho_{\text{odd}}$ is the odd part of the density of states. The spectral asymmetry is then given by

$$\eta = 2 \int_0^\infty \rho_{\text{odd}}(E) dE \ . \tag{1.8}$$

## 2. Model Hamiltonian and Its Spectral Asymmetry

The one-dimensional Hamiltonian describing the interaction of a fermion with a b.f. that we propose to analyze is

$$H = -i\sigma_2 \frac{d}{dx} + \sigma_1 \phi(x) + \sigma_3 \kappa(x) \ , \tag{2.1}$$

with

$$H \psi = E \psi \ . \tag{2.2}$$

By writing $\phi(x) = \rho\cos\theta, \kappa(x) = \rho\sin\theta$ , and performing a chiral rotation $\psi = e^{i\sigma_2\theta/2}\chi$ , the Hamiltonian is transformed as

$$H = e^{i\sigma_2\theta/2} H_\chi e^{-i\sigma_2\theta/2} \ , \tag{2.3}$$

with

$$H_\chi = -i\sigma_2 \frac{d}{dx} + \sigma_1 \rho(x) + \frac{1}{2} \frac{d\theta}{dx} \ . \tag{2.4}$$

It was shown in Ref.5 that

$$B(E) = \frac{T(E)}{T(-E)} \ , \tag{2.5}$$

where $T(E)$ is the transmission coefficient of the scattering states of $H_\chi$ with conventional boundary conditions of a unit incoming wave from the right and a purely outgoing wave to the left.

*Spectral Asymmetry:*

Above thresholds, we write

$$B(E) = |B(E)|\, e^{i\delta(E)} , \tag{2.6}$$

then the asymmetry is given by

$$\eta = N^{+} - N^{-} + \frac{1}{\pi}[\delta(\infty) - \delta(0)], \tag{2.7}$$

where $N^{\pm}$ is the number of positive or negative energy bound states. The quantity $\delta(E)$ is the relative scattering phase shift between the positive and negative energy sectors, and $\delta(0)$ is this phase evaluated at thresholds (see Ref.5 for details). This expression for the asymmetry can be also interpreted as an extension of Levinson's theorem.[9–11]

At large distances (where $\theta' = 0$ ) we see that $\sigma_3$ maps positive energy states onto negative energy states and vice versa. Thus when $\theta' \neq 0$, positive and negative energy scattering states interact with the same b.f. but with an opposite sign. If one interaction is attractive, the other is repulsive. This is the origin of the spectral asymmetry.

*Constant $\kappa$:*

The most common case treated in the literature corresponds to $\kappa(x) = constant$ . However as we shall see this case is very special. In particular, for constant $\kappa$ there exists an operator $U$ that anticommutes with H,

$$U = (\sigma_3 H - \kappa) . \tag{2.8}$$

This operator ensures that if there is a bound state with positive energy $|E|$, which is different from $|\kappa|$, then there is another at $-|E|$ . This pairing property then implies that when $\phi(x)$ is varied, these states move in *pairs* with equal and opposite energy. Therefore there cannot be any net spectral flow. Note that the pairing argument fails for the bound states of energy $E = \pm\kappa$ since they are annihilated by $U$ .

These states of energy $\pm\kappa$ are eigenstates of $(H - \sigma_3\kappa)$ with zero eigenvalue. They are the Jackiw and Rebbi "zero modes" in the charge conjugate case[2] $(\kappa = 0)$. These zero modes are eigenstates of $\sigma_3$ and exist when $\phi(x)$ has a kink-type profile.

By *carefully* following the branches of the function $\theta(x) = \tan^{-1}(\kappa/\phi(x))$ the reader can be convinced that if the signs of $\phi_+$ and $\phi_-$ are the same, where $\phi_\pm = \phi(x = \pm\infty)$, then $\theta_+$ and $\theta_-$ are on the same branch, and hence $\Delta\theta$ lies between $\pm\frac{\pi}{2}$. However if the signs are *opposite* , then $\theta_+$ and $\theta_-$ are on consecutive branches. In the case of constant $\kappa$ one always has the limits

$$-\pi \leq \Delta\theta \leq +\pi. \tag{2.9}$$

Therefore the charge must lie in the range between minus one-half and plus one-half. The charge conjugate limit ($\kappa \to 0\pm$, $Q = \pm\frac{1}{2}$) is reached from $|Q| < \frac{1}{2}$ in both cases. These features are the direct consequence of the fact that for constant $\kappa$ there can be no spectral flow. Indeed, the above analysis of branches indicates that to obtain $|\Delta\theta| \geq 2\pi$, $\kappa(x)$ must change sign.

*Ambiguities and the Example:*

In order to understand the physics in the case when there are ambiguities in the definition (or branches) of $\Delta\theta$, we propose to study the following simple problem:

$$\kappa(x) = \begin{cases} \kappa \\ \kappa_0 \\ \kappa \end{cases} \qquad \phi(x) = \begin{cases} \phi & x > d \\ 0 & -d < x < d \\ -\phi & x < -d \end{cases} \tag{2.10}$$

with $\kappa > 0$ , and $\phi > 0$.

We are interested in studying the behavior of the spectrum of $H$ as $\kappa_0$ is allowed to approach and cross zero, and to evaluate the ground state charge using Eqns. (1.2) and (2.7). When $\kappa_0$ vanishes, then the mass term is zero over a finite region of width $2d$, however the gap in the spectrum is $\rho_1 = [\kappa^2 + \phi^2]^{\frac{1}{2}}$ and is nonzero. With the b.f. as given by eqn.(2.10), the three regions will be denoted by 1 for $x > d$, by 2 for $-d < x < d$, and by 3 for $x < -d$ .

Still following Ref.5 , the transmission coefficient can be obtained by matching the boundary conditions $(i = 1, 2)$ at $x = \pm d$ respectively:

$$\chi_{i+1} = e^{i\sigma_2\eta_i}\chi_i \ , \tag{2.11}$$

where the wave $\chi_i$ is a linear combination of right(+) and left(-) going plane waves with spinors given by

$$\chi_j^{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \frac{\rho_j \pm ik_j}{E} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{\pm i\alpha_j} \end{pmatrix} \tag{2.12}$$

where $k_j = [E^2 - \rho_j^2]^{\frac{1}{2}}$ , $\rho_1 = \rho_3$ , and $\rho_2 = |\kappa_0|$. Also,

$$\eta_j = \frac{1}{2}(\theta_j - \theta_{j+1}) \qquad \frac{1}{2}\Delta\theta = \eta_1 + \eta_2 \ . \tag{2.13}$$

After some straightforward algebra, the transmission coefficient is found to be:

$$\frac{e^{-2i(k_1-k_2)d}}{T(E)} = \left[C + i\frac{ES}{k_1}\right] - \frac{1}{2}\left(e^{iz} - 1\right)F \tag{2.14}$$

with

$$F = C\left(\frac{E^2}{k_1k_2} - 1\right) + iES(\frac{1}{k_2} - \frac{1}{k_1}) - \cos(\eta_1 - \eta_2)\ \frac{\rho_1\rho_2}{k_1k_2} \tag{2.15}$$

and

$$C = \cos\left(\frac{\Delta\theta}{2}\right) \qquad S = \sin\left(\frac{\Delta\theta}{2}\right) \qquad z = 4k_2d\ . \tag{2.16}$$

The next step of the computation involves the evaluation of the angles $\theta_i$ involved in (2.16). Here is where we encounter the first ambiguity because $\kappa(x)$ and $\phi(x)$ change at the same point.

To understand the nature of this ambiguity, let $\phi(x)$ change either slightly *before* or *after* $\kappa(x)$, and such that $\phi(x) = \phi_0$ for $-d \pm \epsilon < x < d \pm \epsilon$. Then we shall take the limit $\phi_0 \to 0$, $\epsilon \to 0$.

Following *carefully* the branches of the function $\theta(x) = \tan^{-1}(\kappa(x)/\phi(x))$ the reader can be convinced that these different limits yield results for $\Delta\theta/2$ and $(\eta_1 - \eta_2)$ that differ by $\pm\pi$ ($\eta_1 - \eta_2$ is either zero or $\pm\pi$). Therefore these ambiguities can yield an overall minus sign in front of $1/T(E)$. The only remaining ambiguity that can arise when $\kappa_0 = 0$ is the overall sign of C and S (see below) which depends on the sign of the limit. Since $\rho_2$ is zero at this point, this is again an overall sign ambiguity for the transmission coefficient.

Hence the ratio $B(E) = T(E)/T(-E)$ is *unambiguous*. Furthermore, since the properties of interest in the spectrum only depend on the difference of phases, the ambiguity (being a constant phase) does *not* influence the spectrum. These ambiguities have no physical relevance for the charge.

For the purpose of calculation we compute the angles taking $\phi(x) = 0$ for $-d + |\epsilon| < x < d - |\epsilon|$ (however the answer will not depend on this definition as was discussed above). Then we find

$$\eta_1 = \eta_2 \qquad\qquad \frac{\Delta\theta}{2} = \tan^{-1}\left(\frac{\kappa}{\phi}\right) - \frac{\pi}{2}\,\mathrm{sign}(\kappa_0)\ . \tag{2.17}$$

The expression for $1/T(E)$, Eq. (2.14), has very interesting features. In the infinite energy limit any dependence on the value of $d$ disappears and

$$\frac{1}{T(\pm E)} \xrightarrow[E\to\infty]{} C \pm iS = e^{\pm i(\Delta\theta/2)}\ . \tag{2.18}$$

Therefore from (2.6) we find

$$\frac{\delta(\infty)}{\pi} = -\frac{\Delta\theta}{\pi}\ . \tag{2.19}$$

The phase shift $\delta(\infty)$ knows about local details through the sign of $\kappa_0$ in (2.17) and this dependence survives no matter how *small* $d$ is. Therefore, even for $d$ *infinitesimally* small, $\delta(\infty)$ changes by $2\pi$ when $\kappa_0$ crosses zero. However, on physical grounds, one *does not* expect any energy level to cross zero just because $\kappa_0$ does, if $d$ is sufficiently small. Indeed, the resultant percentage change in any energy eigenvalue is small, and vanishes with $d$. This point will be further clarified in our later discussions. For $d = 0$, the problem is the same as the $\kappa =$ constant case and $\Delta\theta/2 = \tan^{-1}(\kappa/\phi) - (\pi/2)$ . Therefore for $\kappa_0 < 0$ and for any $d \neq 0$ there is a $2\pi$ difference in $\Delta\theta$ when compared to its $d \equiv 0$ value.

The critical distance (for fixed $\kappa_0$) at which a bound state crosses $E = 0$ can be found from (E is below threshold)

$$\frac{1}{T(E=0)} = 0 = \frac{1}{2}\left[1 + C\right] - \frac{e^{-4|\kappa_0|d}}{2}\left[1 - C\right]\ . \tag{2.20}$$

Notice that $C$ must be negative in (2.20) in order to have a solution. From (2.17) we see that for this to happen $\kappa_0$ must be negative (however small). But the solution for the critical $d$ from (2.20) indicates that as $\kappa_0$ crosses zero and becomes negative, a bound state will cross zero for $d = d_c$, where $d_c \sim \mathcal{O}(1/|\kappa_0|)$. Thus $d$ has to be very large for this to happen for small $\kappa_0$. This is in agreement with our physical intuition, since for $\kappa_0 < 0$ but $d$ very small, the situation is essentially that of the $d = 0$ case where there is a bound state at $E = \kappa > 0$. As $d$ becomes larger the potential widens and the bound state energy decreases; the uncertainty principle prevents the bound state from moving significantly in energy for very small $d$ and $\kappa_0$ and thus there are no energy levels crossing zero discontinuously even when $\Delta\theta$ jumps by $2\pi$ with $\kappa_0 < 0$ .

In order to analyze what happens to the index $\eta$ when $d$ and $\kappa_0$ are varied, we must evaluate the phase-shifts and the bound state contributions in Eqs. (2.7) and (2.14).

The bound state energies $E_b$ are obtained from the equation $1/T(E_b) = 0$. From the expression for $1/T(E)$ we can also read off the threshold phase shifts. The phase of $B(E)$ in the infinite energy limit is given by (2.19) .

*The $d = 0$ Case:*

At $d = 0$ the case is the same as $\kappa =$ constant:

$$\frac{\Delta\theta}{2} = \frac{\Delta\theta_0}{2} \equiv \tan^{-1}\left(\frac{\kappa}{\phi}\right) - \frac{\pi}{2}\,, \quad S < 0\,, \quad C > 0\,, \tag{2.21}$$

where $\Delta\theta_0$ is the chiral angle evaluated for the case $d \equiv 0$ and

$$B(E) = \frac{C - iES/k}{C + iES/k}\,. \tag{2.22}$$

From (2.14) at $d = 0$ we see that there is a bound state at $E = \rho C = \kappa$ ($\kappa > 0$);

the phase of $B(E)$ is given by

$$\delta(E) = -2 \tan^{-1}\left(\frac{SE}{Ck_1}\right) . \tag{2.23}$$

Following the phase $\delta(E)$ from $E = \infty$ to threshold ($E = \rho_1$) we find $\delta(\rho_1) = \pi$, and therefore for $d = 0$:

$$\eta = 1 + \frac{1}{\pi}\left[-\Delta\theta_0 - \pi\right] = -\frac{\Delta\theta_0}{\pi} \qquad -\pi \leq \Delta\theta_0 \leq \pi . \tag{2.24}$$

The phase shift at threshold cancels the bound state contribution because this bound state came from the positive continuum.

*The $d \neq 0$ Case:*

For $d \neq 0$ and $\kappa_0 > 0$, which leads to $C > 0$, the condition for a level crossing zero, Eq. (2.20) , is <u>never</u> fulfilled. However new bound states peel off from the positive and negative energy continuum, $E = \pm\rho_1$, and the critical values of $d$ at which this occurs is obtained from $0 = 1/T(\pm\rho_1)$. The solution is easily found:

$$\frac{\tan(2k_2 d)}{k_2} = \pm\left(\frac{S}{C\rho_1 - \rho_2}\right) . \tag{2.25}$$

From the same condition we see that the threshold phase $\delta(\rho_1)$ increases (decreases) by $\pi$ whenever a bound state peels off from the positive (negative) continuum.

As $d$ increases from zero (with $\kappa_0 > 0$) the bound state originally at $E = \kappa$ moves in energy but ***never*** crosses zero. For $d > d_1$, the critical distance for a new positive energy bound state, there are now *two* positive energy bound states,

$\delta(\rho_1) = 2\pi$ , $\Delta\theta = \Delta\theta_0$ because $\kappa_0 > 0$ , and the asymmetry is

$$\eta = 2 + \frac{1}{\pi}\left[-\Delta\theta_0 - 2\pi\right] = -\frac{\Delta\theta_0}{\pi} \; . \tag{2.26}$$

The phase shifts at threshold know that the new bound state came from the positive continuum and increased by $\pi$. This is as expected from Levinson's theorem of potential scattering.[12]

When $d > d_2$ a new bound state peels off from the negative continuum. The threshold phase shift drops by $\pi$. Now we have two positive energy and one negative energy bound states, but

$$\eta = 2 - 1 + \frac{1}{\pi}\left[-\Delta\theta_0 - 2\pi + \pi\right] = -\frac{\Delta\theta_0}{\pi} \; . \tag{2.27}$$

The continuum reacts through the threshold phase shift whenever a new bound state appears but $\eta$ and the charge remains invariant. The ordering of the above events depend, of course, on the parameters in the b.f.

*Negative* $\kappa_0$*:*

For $\kappa_0 < 0$ the situation is different, and $\Delta\theta$ is no longer equal to $\Delta\theta_0$ . As soon as $d \neq 0$, $\Delta\theta/2$ changes by $\pi$ because of (2.17) and (2.21) ; now $C < 0$, $S > 0$ and the condition (2.20) can be fulfilled, but for fixed $|\kappa_0|$ a bound state can cross zero only when $d$ is large enough. No bound state abruptly crosses zero when $\kappa_0$ goes through zero if $d \neq 0$. There is always a critical minimal distance d. For $d$ small the bound state wave function is forced to vary too rapidly; it cannot lower its energy to zero unless $d$ becomes sufficiently large. For $d \approx 0$ we can neglect the terms proportional to $d$ in (2.14) and see that for $\kappa_0 < 0$ and $d \neq 0$ but very small,

$$\delta(d,E) = \delta(0,E) - 2\pi \ , \tag{2.28}$$

where $\delta(0,E)$ is given by (2.23) with $S < 0$ and $C > 0$ (they are evaluated at $\Delta\theta_0$). Therefore we now have

$$\begin{aligned} \delta(d,\infty) &= -\Delta\theta = -\Delta\theta_0 - 2\pi \\ \delta(d,\rho_1) &= \delta(0,\rho_1) - 2\pi \ . \end{aligned} \tag{2.29}$$

Now we see that for $\kappa_0 < 0$, but $d$ very small, $\eta$ has the same *value* as the $d = 0$ case. No level had crossed zero as $d$ moved away from zero and no new bound state had appeared. The shift in $\delta(\infty)$ of $2\pi$ is compensated by the same shift in $\delta(\rho_1)$. The branch independent quantity $(\delta(\infty) - \delta(\rho_1))$ stays fixed. Once the branch of $\delta(\infty)$ is chosen, $\delta(\rho_1)$ is obtained by following the function $\delta(E)$ to threshold and there are *no* ambiguities. Therefore this $2\pi$ difference does not imply a change in the charge by one unit as the naive formula (1.1) suggests. However the $2\pi$ does anticipate the possibility of spectral flow, which indeed will occur if $d$ is sufficiently large.

*Scenario for Increasing d:*

With the above discussion, and using the conditions (2.25) and (2.20) , we have the following scenario for the $\kappa_0 < 0$ case. New bound states can peel off from the positive and negative continuum at the critical values $d = d_i$, which depend upon the parameters $\kappa, \phi$, and $\kappa_0$. At $d = d_0$, given by (2.20) , one bound state crosses zero. For $d < d_1$, the value at which a new positive energy bound state appears, there is only one bound state with positive energy and

$$\eta = 1 + \frac{1}{\pi}\left[-\Delta\theta_0 - \pi\right] = -\frac{\Delta\theta_0}{\pi} \qquad -\pi \le \Delta\theta_0 \le \pi \ . \tag{2.30}$$

The extra $2\pi$ in (2.28) is cancelled between $\delta(\infty)$ and $\delta(\rho_1)$ and the rest of $\delta(\rho_1)$ cancels the positive energy bound state. For $d > d_1$ a new positive energy bound state appears and $\delta(\rho_1)$ increases by $\pi$, but we still find

$$\eta = 2 + \frac{1}{\pi}\left[-\Delta\theta_0 - 2\pi\right] = -\frac{\Delta\theta_0}{\pi}\ . \tag{2.31}$$

For $d > d_2$ a negative energy bound state appears and $\delta(\rho_1)$ decreases by $\pi$. This new bound state arises from the negative continuum; however $\eta$ is again unchanged:

$$\eta = 2 - 1 + \frac{1}{\pi}\left[-\frac{\Delta\theta_0}{\pi} - \pi\right] = -\frac{\Delta\theta_0}{\pi}\ . \tag{2.32}$$

At $d = d_0$ one bound state crosses zero from positive to negative energy (this is the one originally at $E = \kappa$ that drops in energy as $d$ increases). The phase shifts *stay constant*, since no new bound states have appeared. Now there are two bound states with negative energy and one with positive energy:

$$\eta = 1 - 2 + \frac{1}{\pi}\left[-\Delta\theta_0 - \pi\right] = -\frac{\Delta\theta_0}{\pi} - 2 = -\frac{\Delta\theta}{\pi}\ . \tag{2.33}$$

Again we see that $\eta$ changes only when there is spectral flow, i.e. energy levels crossing zero. Although the possibility of levels crossing zero arises when $\kappa_0 < 0$, and even though for negative $\kappa_0$ the chiral angle winds by $2\pi$, spectral flow is not realized when this winding occurs over a very small distance. The winding *does not* by itself induce spectral flow. Only when the critical distance $d = d_0$ is reached can levels cross zero. Therefore we see the motivation for writing

$$\eta = \eta_V + \eta_{SF} \tag{2.34}$$

with

$$\eta_V = -\frac{\Delta\theta_0}{\pi} \qquad -\pi \le \Delta\theta_0 \le \pi \ . \tag{2.35}$$

Here $\eta_{SF}$ directly measures the spectral flow; this quantity jumps by $\pm 2$ whenever a bound state crosses zero, and depends on the local details of the b.f. On the other hand, the quantity $\eta_V$ is a topological invariant.

## 3. Discussions and Conclusions

In conclusion, the winding of the chiral angle by $2\pi$ does not necessarily mean that the vacuum charge has changed by one unit. Such a change is only produced by levels crossing zero. No spectral flow is induced when the angle winds by $2\pi$ in a very small distance. Level crossings will be induced when the winding of the chiral angle occurs over a sufficiently large distance in space. Very roughly, this will occur when the product of this distance times the local mass is of order one.[13] For our example, see Eqn.(2.20) . The fractional part of the charge, $Q_F$ , is *topologically invariant* and is given by the phase shifts at infinite energy on their principal branch:

$$Q_F = \frac{\Delta\theta}{2\pi} \qquad -\pi \le \Delta\theta \le \pi \ . \tag{3.1}$$

This fractional part of the vacuum charge is a high energy property of the theory and hence does not depend upon the mass $\rho(x)$.

On the other hand, the integer part of Q is provided by spectral flow, which is a consequence of local properties (in our example it depends on $d$ and $\kappa_0$), and is *not directly* related to the winding of $\Delta\theta$.

$Q_F$ can be obtained from a very simple Hamiltonian with background fields with the same asymptotic values of $\kappa(x)$ and $\phi(x)$ as the Hamiltonian of interest but with no local details (minimal number of step functions). The integer in the vacuum charge arises when levels cross zero in the process of deforming the minimal $\kappa(x)$ and $\phi(x)$ to the profiles of interest. This is measured by $\eta_{SF}$ which was shown in Ref.5 to be given in terms of $B_R(E)$, a direct comparison of the determinantal ratios for $H$ and $H_V$ respectively,

$$B_R(E) = B(E)/B_V(E) \; . \tag{3.2}$$

The corresponding $G_e$, $\rho_{odd}$, and $\eta_{SF}$ are computed by substituting $B_R(E)$ into Eqns. (1.6) , (1.7) , and (1.8) .

We also note that the breakdown of the adiabatic expansion caused by the vanishing of the local mass does not signal the appearance of spectral flow. It is therefore not surprising that the final result obtained from this expansion gives the fractional part of the charge correctly but not the integer part.

We hope that this analysis and discussion illuminates the physics of fractional charge in one spatial dimension and clarifies its relation to the chiral angle and its winding. We have learned that ambiguities that arise from the branches of $\Delta\theta$ and those that arise when the local mass term changes sign or is zero over a finite region, do not introduce ambiguities in the vacuum charge.

ACKNOWLEDGEMENTS

We wish to thank Michael Stone for making the right comment and for asking the right question that lead to this investigation.

## REFERENCES


1. J. Goldstone and F. Wilczek , Phys. Rev. Lett. *47*, 968 (1981).
2. R. Jackiw and C. Rebbi, Phys. Rev. *D13*, 3398 (1976). See also R. Jackiw and J. R. Schrieffer, Nucl. Phys. *B190*,[FS3], 253 (1981).
3. R. Jackiw, Nuffield Lectures, MIT preprint CTP949 (1981); for the condensed matter version see M. J. Rice and F. Mele, Phys. Rev. Lett. *49*, 1455 (1982).
4. R. MacKenzie and F. Wilczek, Phys. Rev. *D30*, 2194 (1984). See also the thesis of R. MacKenzie, UCSB preprint (1984).
5. R. Blankenbecler and D. Boyanovsky, Phys. Rev. *D31*, in proof (1985), and the long list of references therein.
6. M. Stone, Illinois preprint ILL-(TH)-84-51 (1984). To be published.
7. L. Brown and M. Bernstein, U. of Washington preprint 40048-36-P4 (1984). To be published.
8. A. Z. Capri, R. Ferrari, and L. E. Picasso, Phys. Rev. *D30*, 2136 (1984).
9. See also Ref.6 for another discussion of Eqn.(2.7) .
10. H. Yamagishi, Phys. Rev. Lett. *50*, 458 (1983), Phys. Rev. it D28, 977 (1983).
11. B. Grossman, Phys. Rev. Lett. *50*, 464 (1983).
12. See for example, R. Newton, Scattering Theory of Waves and Particles, Springer Verlag (1982).
13. It should be straightforward to apply our approach to the radial equation in higher dimension, see for example D. Boyanovsky and R. Blankenbecler,

SLAC-PUB-3567, February (1985), and to discuss problems such as spectral flow in Skyrmion models. See, in th is connection, E. D'Hoker and E. Farhi, Nucl. Phys. *B241*, 109 (1984).