SLAC – PUB – 3588
February 1985
T

ANOMALY CONSTRAINTS ON NONLINEAR SIGMA MODELS*

JONATHAN BAGGER, DENNIS NEMESCHANSKY

and

SHIMON YANKIELOWICZ†

Stanford Linear Accelerator Center

Stanford University, Stanford, California, 94305

ABSTRACT

Anomalies in nonlinear sigma models can sometimes be cancelled by local counterterms. We show that these counterterms have a simple topological interpretation, and that the requirements for anomaly cancellation can be easily understood in terms of 't Hooft's anomaly matching conditions. We exhibit the anomaly cancellation on homogeneous spaces G/H and on general Riemannian manifolds $\mathcal{M}$. We include external gauge fields on the manifolds and derive the generalized anomaly cancellation conditions. Finally, we discuss the implications of this work for superstring theories.

Submitted to *Nuclear Physics B*

⋆ Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.

† On leave of absence from the Department of Physics, Tel-Aviv University, Israel.

1. Introduction

The past few years have witnessed a resurgence of interest in anomalies. Much progress has been made in understanding the topological origins of the abelian and nonabelian gauge field anomalies [1]. New perturbative and nonperturbative anomalies have also been found [2,3], and their topological origins have been clarified as well [4].

In this paper we will discuss a new anomaly that afflicts certain nonlinear sigma models with fermions [5–7]. This anomaly is similar to the ordinary gauge and gravitational anomalies since it reflects a topological obstruction to the reparametrization invariance of the quantum effective action. However, as emphasized by Manohar, Moore and Nelson [7], the sigma model anomaly is different in one important respect – it can sometimes be cancelled by a set of local counterterms. We will show that these counterterms have a simple topological interpretation, and that the anomaly cancellation requirements can be easily understood by a suitable generalization of 't Hooft's anomaly matching conditions [8].

In the rest of this paper we explain these ideas. In Section 2 we start with a general discussion of anomalies in sigma models. We give a general condition for anomaly cancellation, based on the existence of local counterterms. In Section 3 we specialize to homogeneous spaces G/H. Following Callan, Coleman, Wess and Zumino [9], we include fermions in various representations of H. We show that the anomalies can be cancelled by an appropriate Chern-Simons terms whenever the 't Hooft conditions are satisfied. In Section 4 we discuss sigma models based on Riemannian manifolds $\mathcal{M}$. We now take the fermions to transform in the tangent space of $\mathcal{M}$. As in Section 3, we show that the anomalies can sometimes be cancelled by Chern-Simons terms. (Similar results have also been found in Reference [10].) Finally, in Section 5 we introduce external gauge fields on the manifold $\mathcal{M}$. Now both "gauge" and "gravitational" anomalies can appear. We derive the conditions under which these anomalies can be cancelled by local counterterms. We apply these conditions to strings in general, and to heterotic strings, in particular. For heterotic strings, anomaly cancellation seems to demand that the spin connection be embedded in the external gauge group [11].

2. General Considerations

Nonlinear sigma models are theories whose scalar fields $\phi^i(x)$ lie on a Riemannian manifold $\mathcal{M}$. The Lagrangian has the following form,

$$\mathcal{L} \;=\; -\,\frac{1}{2}\, g_{ij}(\phi)\,\partial_\mu\phi^i\partial^\mu\phi^j \;, \tag{1}$$

where $g_{ij}(\phi)$ is the metric on $\mathcal{M}$. The Lagrangian (1) is manifestly invariant under the diffeomorphisms of the manifold $\mathcal{M}$. It describes an interacting scalar field theory whose classical solutions are harmonic maps into $\mathcal{M}$.

It is easy to add fermions to the sigma model (1). The spinors χ^A should be thought of as *sections* of a vector bundle $\mathcal{B}$ over $\mathcal{M}$. In this case, the generalized sigma model Lagrangian is as follows,

$$\mathcal{L} \;=\; -\,\frac{1}{2}\, g_{ij}(\phi)\,\partial_\mu\phi^i\partial^\mu\phi^j \;-\; \frac{i}{2}\,\overline{\chi}_A\gamma^\mu D_\mu\chi^A \;, \tag{2}$$

where the covariant derivative $D_\mu\chi^A$ is given by $D_\mu\chi^A = \partial_\mu\chi^A + \partial_\mu\phi^i\,\omega_i{}^A{}_B\chi^B$, and the connection $\partial_\mu\phi^i\,\omega_i{}^A{}_B$ is the pull-back to spacetime of the appropriate connection $\omega_i{}^A{}_B$ on $\mathcal{B}$. The covariant derivative ensures that the Lagrangian (2) is invariant under bundle reparametrizations, where

$$\begin{aligned} \delta\,\chi^A &= L^A{}_B\,\chi^B \\ \delta\,\omega_i{}^A{}_B &= -\,\partial_i\,L^A{}_B \,-\, \omega_i{}^A{}_C\,L^C{}_B \,+\, L^A{}_C\,\omega_i{}^C{}_B\,, \end{aligned} \tag{3}$$

and $L^A{}_B$ is a function of the scalar field ϕ^i.

To discuss the anomaly, let us restrict our attention to four (Euclidean) spacetime dimensions, and write all spinors as left-handed Weyl spinors. Furthermore, let us integrate out the spinors and construct the effective action Γ for the scalar fields ϕ^i,

$$\begin{aligned} \Gamma[\phi,\,g,\,\omega] \;=\; -\,\log\int [d\chi_L{}^A]\,[d\overline{\chi}_{LB}] \\ \exp\int d^4x\left[-\,\frac{1}{2}\,g_{ij}\,\partial_\mu\phi^i\partial^\mu\phi^j \,-\, \frac{i}{2}\,\overline{\chi}_{LA}\gamma^\mu D_\mu\chi_L{}^A\right]\,. \end{aligned} \tag{4}$$

In the absence of anomalies, Γ remains invariant under the bundle reparametrizations (3). Under such transformations, the entire change in Γ comes from the variation of the connection,

$$\delta\Gamma \;=\; \int d^4x\,\frac{\delta\Gamma}{\delta\omega_i{}^A{}_B}\,\delta\omega_i{}^A{}_B \;=\; \int d^4x\,L^A{}_B(\phi)\,D_\mu\left\langle J^\mu{}_A{}^B\right\rangle\,. \tag{5}$$

Equation (5) implies that the effective action is invariant if and only if the induced current $J^\mu{}_A{}^B = \overline{\chi}_{LA}\gamma^\mu\chi_L{}^B$ is covariantly conserved.

The evaluation of $\delta\Gamma$ can be done in many ways. One can compute the anomaly diagrammatically [12,13], as in Figure 1. One can also use Fujikawa's method, where the anomaly shows up as a nontrivial Jacobian associated with the fermionic functional integral [14]. Or, for nonabelian symmetries, one can compute the anomaly from the six-dimensional Pontrjagin class [1,4]. No matter how one proceeds, one finds the same expression for the sigma model anomaly [13],

$$\begin{aligned}\delta\Gamma &= \int d^4x\, L^A{}_B(\phi)\, D_\mu \left\langle J^\mu{}_A{}^B \right\rangle \\ &= \frac{1}{24\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\, L\, \partial_i \left[\omega_j \partial_k \omega_\ell \,+\, \frac{1}{2}\,\omega_j\omega_k\omega_\ell\right] \\ &\quad\times\; \partial_\mu\phi^i \partial_\nu\phi^j \partial_\rho\phi^k \partial_\sigma\phi^\ell \,.\end{aligned} \tag{6}$$

This is the unique form of the anomaly that satisfies the Wess-Zumino consistency condition [15]. Because of the pull-backs $\partial_\mu\phi^i$, the anomaly vanishes for all manifolds of dimension less than four.

The anomaly (6) is similar to an ordinary anomaly since it obstructs the invariance of the quantum effective action. In a gauge or gravitational theory, such an obstruction is fatal – unitarity is lost and unphysical degrees of freedom begin to propagate [16]. In a sigma model, the anomaly is more subtle. This is because the gauge fields are composite – they are functions of the scalar fields ϕ^i. Sigma model anomalies do not create any new degrees of freedom. They merely break some of the symmetries associated with the classical action.

Sigma models with anomalies can be unacceptable for physical reasons. In that case one would like to know when – if ever – the anomaly can be cancelled. One way to cancel the anomaly is well-known from gauge theories: One simply adds extra spinors so that the fermionic determinant is well-defined. In practical terms, this means that the fermions must transform in an anomaly-free representation of the structure group of $\mathcal{B}$. In a sigma model, there is a second approach. One can add local counterterms to the bare action in just such a way that the anomaly is cancelled. Of course, with this approach, the classical action is not invariant under bundle reparametrizations. The classical "anomaly" exactly cancels the quantum anomaly, leaving the full quantum theory reparametrization invariant.

What counterterms must one add to cancel the sigma model anomaly? The answer is obvious: In four dimensions, one simply adds an integral over the five-dimensional Chern-Simons term $\Omega_{ijk\ell m}(\omega)$ [17],

$$\begin{aligned} I &= -2\pi \int_D d^5y \; \epsilon^{ijk\ell m} \, \Omega_{ijk\ell m}(\omega) \\ &= -\frac{1}{120\pi^2} \int_D d^5y \; \epsilon^{ijk\ell m} \; \mathrm{Tr} \left[\omega_i \partial_j \omega_k \partial_\ell \omega_m + \frac{3}{2} \omega_i \omega_j \omega_k \partial_\ell \omega_m \right. \\ &\quad \left. + \frac{3}{5} \omega_i \omega_j \omega_k \omega_\ell \omega_m \right] . \end{aligned} \tag{7}$$

The integral I runs over a five-dimensional disk D whose boundary ∂D is the image of spacetime in $\mathcal{M}$. The variation of I exactly cancels the sigma model anomaly,

$$\begin{aligned} \delta I &= -\frac{1}{120\pi^2} \int_D d^5y \, \epsilon^{ijk\ell m} \partial_i \; \mathrm{Tr} \left[L \, \partial_j \left(\omega_k \partial_\ell \omega_m + \frac{1}{2} \omega_k \omega_\ell \omega_k \right) \right] \\ &= -\frac{1}{24\pi^2} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \; \mathrm{Tr} \, L \, \partial_i \left[\omega_j \partial_k \omega_\ell + \frac{1}{2} \omega_j \omega_k \omega_\ell \right] \\ &\quad \times \; \partial_\mu \phi^i \partial_\nu \phi^j \partial_\rho \phi^k \partial_\sigma \phi^\ell . \end{aligned} \tag{8}$$

The coefficient $-1/120\pi^2$ was chosen to cancel the anomaly (6). It is also precisely the right coefficient to ensure that the effective action is independent of D [18].

The counterterm (7) is written as an integral over a disk D. Lagrangian mechanics, however, requires that an action be written as an integral over spacetime. Equation (7) can be pulled back to an integral over spacetime whenever the Chern-Simons term is closed, $\partial_{[i}\Omega_{jk\ell mn]} = 0$. Then $\Omega_{ijk\ell m} = \partial_{[i}\alpha_{jk\ell m]}$, and

$$\begin{aligned} -\frac{1}{2\pi}\,I &= \int_D d^5y\; \epsilon^{ijk\ell m}\; \Omega_{ijk\ell m} \\ &= 5\int_{\partial D} d^4y\; \epsilon^{ijk\ell}\; \alpha_{ijk\ell} \\ &= 5\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\; \alpha_{ijk\ell}\, \partial_\mu\phi^i\partial_\nu\phi^j\partial_\rho\phi^k\partial_\sigma\phi^\ell\ . \end{aligned} \tag{9}$$

The integral I can be pulled back to spacetime whenever $\partial_{[i}\Omega_{jk\ell mn]} = 0$. It is easy to show that $\partial_{[i}\Omega_{jk\ell mn]} = \mathrm{Tr}\, R_{[ij}R_{k\ell}R_{mn]}$ where $R_{ij}{}^A{}_B = \partial_{[i}\omega_{j]}{}^A{}_B + \omega_{[i}{}^A{}_C\omega_{j]}{}^C{}_B$ is the curvature of the bundle $\mathcal{B}$. In the language of differential forms, this becomes $d\Omega = \mathrm{Tr}\, R^3$, where d is the exterior derivative, and all wedge products are implied. We have shown that the sigma model anomaly can be cancelled by local counterterms whenever $\mathrm{Tr}\, R^3 = 0$.

When the trace of R^3 does not vanish, the story is more complicated. This is because the connection ω is not quite uniquely defined. One can always shift ω by a tensor τ,

$$\omega' = \omega + \tau\ . \tag{10}$$

Since τ is a tensor, ω' is a connection, and τ is called torsion.

The ability to shift ω is very important. It leads to an entire family of connections ω'. Each connection ω' gives rise to a corresponding curvature

$R' = d\omega' + \omega'^2$ and Chern-Simons term $\Omega(\omega')$. If the trace of R'^3 vanishes for some connection ω', the integral

$$I' = -2\pi \int_D \Omega(\omega') \tag{11}$$

can be pulled back to spacetime.

When $\omega \neq \omega'$, the integral I' does not quite cancel the anomaly (6). One still needs to add a set of local counterterms. To find these terms, we first note that the characteristic classes $\mathrm{Tr}\, R^3$ and $\mathrm{Tr}\, R'^3$ differ by a total derivative,

$$\mathrm{Tr}\, R'^3 - \mathrm{Tr}\, R^3 = dQ(\omega, \omega') . \tag{12}$$

Using Cartan's homotopy operator, it is not hard to show that[♯1] [19]

$$Q(\omega, \omega') = 3 \int_0^1 dt \; \mathrm{Str}\left(\tau, R_t{}^2\right) , \tag{13}$$

where "Str" denotes symmetrized trace, $\omega_t = \omega + t\,\tau$ and $R_t = d\omega_t + \omega_t{}^2$. Since τ is torsion, ω_t is a connection, and R_t is a curvature.

Since $d\Omega(\omega) = \mathrm{Tr}\, R^3$, equation (12) implies that $\Omega(\omega') - \Omega(\omega) - Q(\omega, \omega')$ is a total derivative,

$$\Omega(\omega') - \Omega(\omega) - Q(\omega, \omega') = d\gamma(\omega, \omega') . \tag{14}$$

A second application of the homotopy operator gives

$$\gamma(\omega, \omega') = \frac{1}{2} \mathrm{Tr} \left\{ [\omega, \omega'](d\omega' + d\omega) - (\omega\omega'^3 - \omega'\omega^3) - \frac{1}{2} (\omega'\omega)^2 \right\} . \tag{15}$$

Both $Q(\omega, \omega')$ and $\gamma(\omega, \omega')$ are globally well-defined.

♯1 This can also be found using coboundary operators, see Reference [20].

Since $Q(\omega, \omega')$ is manifestly gauge invariant, the gauge variation of $\Omega(\omega)$ can be written in terms of the variations of $\Omega(\omega')$ and $\gamma(\omega, \omega')$,

$$\delta\,\Omega(\omega) \;=\; \delta\,\Omega(\omega') \;-\; d\,\delta\,\gamma(\omega, \omega')\;. \tag{16}$$

Therefore

$$I \;=\; -\,2\pi \int_D \Omega(\omega') \;+\; 2\pi \int_{\partial D} \gamma(\omega, \omega') \tag{17}$$

precisely cancels the sigma model anomaly. Since $\mathrm{Tr}\,R'^3 = 0$, this expression is local in the fields ϕ^i.

We are now able to state the general conditions for anomaly cancellation. In four dimensions, the sigma model anomaly can be cancelled if there exists a connection $\omega' = \omega + \tau$ such that $\mathrm{Tr}\,R'^3 = 0$. It is easy to generalize this result to any even dimension. In $2n-2$ dimensions, the anomaly is cancelled, up to local counterterms, by the $(2n-1)$-dimensional Chern-Simons term, provided $\mathrm{Tr}\,R'^n = 0$ for some connection ω'.

The anomaly cancellation condition presented here is a *local* condition. It must hold at each point of the manifold $\mathcal{M}$. A *global* condition can be found by integrating $\mathrm{Tr}\,R^n$ over a $2n$-dimensional submanifold of $\mathcal{M}$. If $\int \mathrm{Tr}\,R^n \neq 0$, it is impossible to cancel the sigma model anomaly by local counterterms. This is the global condition found by Moore and Nelson [5].

3. Homogeneous Spaces G/H

In the remainder of this paper, we examine the sigma model anomaly in various types of theories. We first consider theories based on homogeneous spaces $\mathcal{M} = G/H$. These theories describe the interactions of the Goldstone bosons that arise by spontaneously breaking a group G down to a subgroup H. The Goldstone bosons can interact with fermions χ^A. The fermions form representations ρ_H of H, and realize the full G-symmetry nonlinearly [9].

The symmetries of $\mathcal{L}$ are associated with the isometries of the manifold $\mathcal{M}$. The group G is the full isometry group of $\mathcal{M}$, and H is its isotropy subgroup. On manifolds $\mathcal{M} = G/H$, standard coordinates can always be found such that the isometries in H leave the origin invariant [9]. These coordinates are parametrized by the group element $g = \exp i\phi^i T^i$, where the T^i denote the generators of G that are not in H.

In standard coordinates, global G-rotations are implemented by left group multiplications, $g \to kg$. The elements kg can be brought to standard form by right H-rotations, $kg \to kgh^{-1}$. The combined transformations take standard forms to standard forms,

$$g \;\to\; g' \;=\; kgh^{-1}\ , \tag{18}$$

where $g' = \exp i\phi'^i T^i$, and h is a function of g and k. Equation (18) leads immediately to the transformations of the coordinates ϕ^i and the fermions χ^A. The transformations of the coordinates are given by $g \to g'$, while the fermions transform under the right H-rotations, $\chi^A = \rho_H(h)^A{}_B\, \chi^B$.

Standard coordinates are useful because the symmetries in $H \subseteq G$ are represented linearly on the fields:

$$\begin{aligned} \delta \phi^i &= \epsilon^a f^{aij} \phi^j \\ \delta \chi^A &= i \epsilon^a T^{aA}{}_B \chi^B . \end{aligned} \tag{19}$$

Here we have adopted the convention that indices $(a, b, c...)$ run over the generators of H, while indices $(i, j, k...)$ span the other generators of G. In equation (19), the f^{aij} are structure constants of G, and the $T^{aA}{}_B$ are the generators of H in the representation ρ_H. Note that the transformations (19) do not change the origin of $\mathcal{M}$. They correspond to unbroken symmetries, and leave the S-matrix invariant.

In the same coordinates, the isometries in G that are not in H shift the origin of $\mathcal{M}$:

$$\begin{aligned} \delta \phi^i &= \epsilon^i - \frac{1}{2} \epsilon^j \phi^k f^{ijk} + \frac{1}{4} \epsilon^j \phi^k \phi^\ell f^{jka} f^{a\ell i} \\ \delta \chi^A &= -\frac{i}{2} \epsilon^i \phi^j f^{ija} T^{aA}{}_B \chi^B . \end{aligned} \tag{20}$$

These transformations are realized nonlinearly on the fields. They correspond to symmetries spontaneously broken by the vacuum. The classical action, however, is invariant under all the isometries in G.

The preceding construction has a natural interpretation in the language of fiber bundles. From this point of view, manifolds G/H should be thought of as *sections* of fiber bundles $\mathcal{E}$, with total space G and fiber H. These sections are parametrized by the group elements $g = \exp i\phi^i T^i$. As shown in Figure 2, global G-rotations take elements g of G/H into elements kg of $\mathcal{E}$. The transformed elements kg are not necessarily in the section G/H. They must be projected back by field-dependent H-transformations, such that $g' = kgh^{-1} \in G/H$. Projecting back to the section G/H is equivalent to restoring the standard coordinates $g' = \exp i\phi'^i T^i$.

The Lagrangian for the sigma model is given by equation (1). For manifolds $\mathcal{M} = G/H$, we must specify the metric g_{ij} and the connection $\omega_i{}^A{}_B$. Following Callan, Coleman, Wess and Zumino [9], we take the metric to be $g_{ij} = \mathrm{Tr}[(g^{-1}\partial_i g)|_K (g^{-1}\partial_j g)|_K]$, where the Maurer-Cartan form $g^{-1}dg$ is projected onto $K = G/H$. In a similar fashion, we take the fermion connection to be the associated H-connection $\omega_i{}^A{}_B = \rho_H[(g^{-1}\partial_i g)|_H]^A{}_B$, where $g^{-1}dg$ is now projected onto H.

To show that $\mathcal{L}$ is invariant under the transformation $g \to kgh^{-1}$, we need to find the variation of the Maurer-Cartan form $g^{-1}dg$. Since k is a global element of G, $g^{-1}dg$ is manifestly k-invariant. Only local h-transformations contribute to the variation:

$$g^{-1}dg \;\rightarrow\; h(g^{-1}dg)h^{-1} \;+\; hdh^{-1}\;. \tag{21}$$

Since hdh^{-1} is valued in the Lie algebra of H, the metric g_{ij} is invariant under the transformations (18). The connection, however, transforms as follows:

$$\omega_i \;\rightarrow\; \rho_H(h)\,\omega_i\,\rho_H(h^{-1}) \;+\; \rho_H(h\partial_i h^{-1})\;. \tag{22}$$

The Lagrangian is invariant provided the fermions transform like tensors, $\chi^A \;\rightarrow\; \rho_H(h)^A{}_B\chi^B$.

Since the induced H-rotations are local, anomalies arise at the quantum level if the fermions χ^A transform in anomalous representations of H. In the presence of the anomaly, the quantum effective action is not invariant under the full isometry group G – it is only invariant under the isotropy subgroup H.

The anomaly implies that the quantum effective action has a smaller group of symmetries than the manifold itself.

For spaces G/H, the sigma model anomaly can be understood as follows. In standard coordinates, the symmetries in $K = G/H$ give rise to nonlinear transformations of the Goldstone bosons ϕ^i. These nonlinear transformations induce local ϕ-dependent transformations of the Fermi fields via equation (20). In the presence of the anomaly, the quantum effective action is not invariant under local rotations of the spinors. Therefore the anomaly explicitly breaks all symmetries in G that are not in H. The fields ϕ^i are not true Goldstone bosons – because of the anomaly, they gain mass at the quantum level (see Figure 3) [21]. The symmetries in H, however, are unaffected by the anomaly. They are represented linearly on the fields ϕ^i, and give rise to rigid rotations of the Fermi fields χ^A. Rigid rotations of spinors do not give anomalous contributions to the effective action, so Γ is invariant under the isometries in H.

In the previous section we showed how to cancel the anomaly using the appropriate Chern-Simons term. The entire argument can be carried over to the present case, provided we replace the general fermion connection by the associated H-connection $\omega = \rho_H[(g^{-1}dg)|_H]$. As before, the anomaly can be cancelled if there exists a connection $\omega' = \omega + \tau$ such that $\mathrm{Tr}\, R'^n = 0$. In the rest of this section, we will show that such a connection exists whenever the fermion representation ρ_H satisfies 't Hooft's anomaly matching conditions.[♮2]

To see how this works, let us restrict ourselves to two dimensions. Instead of the H-connection $\omega = \rho_H[(g^{-1}dg)|_H]$, let us consider the G-connection $\omega' = \rho_G[g^{-1}dg]$. This connection is only defined when the fermion representations ρ_H

♮2 This result was first obtained by E. Witten (unpublished).

form G-representations ρ_G. Since $\rho_G[g^{-1}dg] = \rho_G[(g^{-1}dg)|_K] + \rho_G[(g^{-1}dg)|_H]$, we see that the torsion $\tau = \rho_G[(g^{-1}dg)|_K]$. The G-connection clearly satisfies the trace condition since $R' = d\omega' + \omega'^2 = 0$. The corresponding integrated Chern-Simons term takes a very simple form,

$$\begin{aligned} I &= -2\pi \int_D d^3y\, \epsilon^{ijk}\, \Omega_{ijk} \\ &= \frac{i}{12\pi} \int_D d^3y\, \epsilon^{ijk}\, \mathrm{Tr}\left(\omega'_i \omega'_j \omega'_k\right) . \end{aligned} \tag{23}$$

Its variation is precisely the two-dimensional anomaly,

$$\begin{aligned} \delta I &= -\frac{i}{12\pi} \int_D d^3y\, \epsilon^{ijk} \partial_i\, \mathrm{Tr}\left(\eta\, \partial_j \omega'_k\right) \\ &= -\frac{i}{12\pi} \int_D d^3y\, \epsilon^{ijk} \partial_i\, \mathrm{Tr}\left(\eta\, \partial_j \omega_k\right) \\ &= -\frac{i}{4\pi} \int d^2x\, \epsilon^{\mu\nu}\, \mathrm{Tr}\left(\eta\, \partial_i \omega_j\right)\, \partial_\mu \phi^i \partial_\nu \phi^j . \end{aligned} \tag{24}$$

where $h = 1 - \eta$. Equation (23) cancels the anomaly of the effective action, and is local in the fields ϕ^i.

In higher dimensions, the G-connection can also be used to cancel the sigma model anomaly. As shown in Section 2, the variation of the Chern-Simons term cancels the anomaly – up to local counterterms. These extra counterterms do not appear in two dimensions. The anomaly cancellation conditions, however, remain unchanged.

— Actually, it is not necessary for the fermion representations ρ_H to form *complete* G-representations ρ_G. All that is necessary is for the two representations to give the *same* anomalous variation of I. A general representation ρ_G of G decomposes into a sum of representations of H. To cancel the anomaly, these representations must include the fermion representations ρ_H. The other representations must be anomaly-free under H [7].

This condition for anomaly cancellation is precisely the 't Hooft matching condition. It implies that the nonlinear sigma model can be thought of as a low-energy effective Lagrangian corresponding to an underlying preonic theory. To verify this, let us imagine that we gauge a subgroup of the global symmetry group G. The Lagrangian (with the Chern-Simons term) is locally right h-invariant, so we only need consider transformations $g \to k(x)g$. Under such a transformation, the entire change in the action comes from the anomalous variation of the Chern-Simons term. This variation reproduces the anomaly of a linear theory with chiral fermions in representations ρ_G of G.

When 't Hooft's condition is satisfied, the sigma model corresponds to a preonic theory with global symmetry group G. When G is spontaneously broken to H, the low-energy fermions transform in representations ρ_H of H. They are related to the fundamental fermions by a chiral G-rotation. This change of variables gives rise to a Jacobian that is precisely the Chern-Simons term. Since the preonic theory is globally G-invariant, the H-anomalies of the nonlinear sigma model must cancel between the Chern-Simons term and the low-energy fermions. Furthermore, the chiral G-anomalies of the two theories must match.

If the 't Hooft condition cannot be satisfied, there is no connection ω' such that $\mathrm{Tr}\, R'^n = 0$. The sigma model anomaly cannot be cancelled, and the nonlinear model does not correspond to any underlying preonic theory. Since the curvature R generates the holonomy group of G/H, we say that such a sigma model suffers from a *holonomy anomaly.*

To illustrate the anomaly cancellation, we conclude this section with a simple example. Let us consider manifolds $\mathcal{M} = SU(3)/SU(2)$, in two spacetime dimensions. There are two such manifolds $\mathcal{M}$, corresponding to the two embeddings of $SU(2)$ in $SU(3)$. These embeddings are specified by the subgroup decomposition of the fundamental representation of $SU(3)$. One embedding corresponds to $3 \to 2+1$, while the other is given by $3 \to 3$. If we take our fermions to transform in a doublet of $SU(2)$, it is obvious that the H-anomaly can be cancelled when $3 \to 2+1$. When $3 \to 3$, however, the anomaly cannot be cancelled – no representation of $SU(3)$ contains a doublet of $SU(2)$. In this case, the sigma model suffers from a holonomy anomaly.

4. Riemannian Manifolds $\mathcal{M}$

Within the context of chiral dynamics, it is natural for fermions to form representations ρ_H of the isotropy subgroup H of the manifold G/H. For more general models, other choices are often necessary. In supersymmetric models, for example, fermions are sections of the tangent bundle $\mathcal{T}$. More generally, fermions can be sections of vector bundles $\mathcal{U}$ associated to $\mathcal{T}$. In this section we study sigma models based on general Riemannian manifolds $\mathcal{M}$. The fermions form representations $\rho_{\mathcal{H}}$ of the structure group $\mathcal{H}$ of the tangent bundle $\mathcal{T}$.

If the fermions are to transform in any representation of $\mathcal{H}$, it is necessary to introduce an orthonormal frame $e_i{}^a$ on $\mathcal{M}$. The orthonormal frame, or vielbein, depends on the coordinates ϕ^i, and is chosen to provide an orthonormal basis in $\mathcal{T}$ at each point of $\mathcal{M}$,

$$g^{ij}\, e_i{}^a e_j{}^b \;=\; \delta^{ab} \qquad \delta_{ab}\, e_i{}^a e_j{}^b \;=\; g_{ij}\ . \tag{25}$$

In this section, the letters $(i, j, k \ldots)$ represent coordinate frame indices, and the letters $(a, b, c \ldots)$ denote the orthonormal tangent frame. The vielbein and its inverse, $e^i{}_a = g^{ij}\delta_{ab}e_j{}^b$, allow one to pass between the two types of indices.

The structure group $\mathcal{H}$ must preserve the invariant tensors defined on $\mathcal{T}$. Therefore the structure group of an ordinary n-dimensional Riemannian manifold is $O(n)$ — it must preserve the metric δ^{ab}. Certain special manifolds have additional invariant tensors. A Kähler manifold, for example, is a $2n$-dimensional Riemannian manifold endowed with a parallel complex structure $I^a{}_b$, where $I^a{}_c I^c{}_b = -\delta^a{}_b$. Its structure group preserves both $I^a{}_b$ and δ_{ab}. The structure group of a Kähler manifold cannot be all of $O(2n)$, but rather is only $U(n)$.

Note that the relations (25) are invariant under local frame rotations, $\delta e_i{}^a = L^a{}_b(\phi)e_i{}^b$, where $L^a{}_b(\phi)$ is a field-dependent generator of the structure group $\mathcal{H}$. This tells us that theories with vielbeins are invariant — not only under isometries of the manifold $\mathcal{M}$ — but also under local frame rotations.

Since we would like the orthonormality of the vielbein to be preserved under parallel transport, we introduce a spin connection $\omega_i{}^a{}_b$, such that

$$\nabla_i e_j{}^a \;=\; \partial_i e_j{}^a \;+\; \omega_i{}^a{}_b e_j{}^b \;-\; \Gamma_{ij}{}^k e_k{}^a \;=\; 0\,. \tag{26}$$

The derivative ∇ is covariant with respect to diffeomorphisms as well as local frame rotations. The relation (26) allows us to solve for ω in terms of Γ,

$$\omega_i{}^a{}_b \;=\; e_j{}^a\partial_i e^j{}_b \;+\; e_j{}^a\Gamma^j{}_{ik}e^k{}_b\,. \tag{27}$$

For a general Riemannian manifold $\mathcal{M}$, the sigma model Lagrangian is again given by equation (2). The only difference is that the fermion connection $\omega_i{}^A{}_B$ is valued in the Lie algebra of the structure group $\mathcal{H}$, rather than in the Lie algebra of the isotropy group H. The connection $\omega_i{}^A{}_B$ is just the spin connection (27), in the fermion representation $\rho_{\mathcal{H}}$.

It is easy to check that the classical Lagrangian is invariant under general coordinate transformations ξ^i, and under local frame rotations L^{ab}. The coordinate transformations are analogous to global G-transformations on homogeneous spaces G/H. Furthermore, the frame rotations correspond to the compensating H-rotations. To see this, let us use the local frame freedom to choose a gauge for the vielbein, $\delta e_i{}^a = \lambda^{ab} e_{ib}$. Since $\lambda^{ab} = -\lambda^{ba}$, we can gauge away the antisymmetric part of e^{ia}. However, once we have fixed this gauge, we must be careful to accompany general coordinate transformations ξ^i by gauge-restoring local frame rotations L^{ab},

$$\delta\, e_i{}^a \;=\; \partial_i \xi^j\, e_j{}^a \;+\; L^{ab}\, e_{ib} \,, \tag{28}$$

where $L^{ab} \;=\; \frac{1}{2}\, e^{ja} e^{kb} (\partial_j \xi_k \;-\; \partial_k \xi_j)$. Infinitesimal coordinate transformations induce infinitesimal frame rotations on the fermions and on the connection,

$$\begin{aligned} \delta\, \chi^A \;&=\; L^A{}_B\, \chi^B \\ \delta\, \omega_i{}^A{}_B \;&=\; -\, \partial_i\, L^A{}_B \;-\; \omega_i{}^A{}_C\, L^C{}_B \;+\; L^A{}_C\, \omega_i{}^C{}_B \,. \end{aligned} \tag{29}$$

Here $L^A{}_B$ generates the frame rotation in the representation $\rho_{\mathcal{H}}$. If the fermions are in anomalous representations of $\mathcal{H}$, the effective action is not invariant under diffeomorphisms ξ^i.

To cancel the anomaly, one must add a Chern-Simons term $\Omega(\omega)$. For an arbitrary Riemannian manifold, this term is built out of the connection $\omega_i{}^A{}_B$. As shown in Section 2, the Lagrangian is local whenever $\mathrm{Tr}\, R^n = 0$. If $\mathrm{Tr}\, R^n \neq 0$, one must add torsion τ to the connection ω.

The sigma model anomaly can always be cancelled for group manifolds G. This follows from the fact that group manifolds can always be parallelized by adding torsion τ. To find the parallelizing torsion, we work in the left-invariant basis, where $e_i{}^a = 2\,\mathrm{Tr}\,[(g^{-1}\partial_i g)\,T^a]$. In this basis, the Maurer-Cartan equation implies $\omega_i{}^a{}_b = -\frac{i}{2}\, e_i{}^c\, f_c{}^a{}_b$, so $R^a{}_{bcd} = -\frac{1}{4}\, f^a{}_{be}\, f^e{}_{cd}$. From the Maurer-Cartan equation, it is easy to see that the parallelizing torsion is given by $\tau_i{}^a{}_b = -\frac{i}{2}\, e_i{}^c\, f_c{}^a{}_b$, The connection $\omega' = \omega + \tau$ has vanishing curvature. This connection trivially satisfies $\mathrm{Tr}\, R'^n = 0$, so $I' = -2\pi \int \Omega(\omega')$ is local in the fields ϕ^i. The integral I' cancels the anomaly (up to local counterterms). Note that on group manifolds there are no matching conditions. The anomaly can be cancelled for *any* representation $\rho_{\mathcal{H}}$ of $\mathcal{H}$. All one must do is add parallelizing torsion to the connection ω – there is no need to enlarge the representation space. Similar results hold for the sphere S^7, since it too has parallelizing torsion.

For a general Riemannian manifold $\mathcal{M}$, how can the anomaly be cancelled? As before, if $\mathrm{Tr}\, R^n$ does not vanish, we must add torsion τ to the connection ω. In each of the previous cases, we added just enough torsion to flatten the curvature R'. For manifolds G/H with fermions in H, we used the fact that G/H can be locally embedded in G. We found that the anomaly could be cancelled whenever the fermion representation ρ_H could be completed to a representation ρ_G of G. For group manifolds (and the sphere S^7), the situation was even simpler. We could always find parallelizing torsion, so the anomaly could always be cancelled.

The generalization to an arbitrary Riemannian manifold is now clear. We must embed $\mathcal{M}$ into a space $\mathcal{N}$ with a flat connection $\hat{\omega}$. This can only be done if the fermion representation $\rho_{\mathcal{H}}$ can be completed to a representation $\rho_{\mathcal{G}}$ of $\mathcal{G}$, where $\mathcal{G}$ is the structure group of the space $\mathcal{N}$. If $\mathcal{M}$ cannot be embedded in a group manifold or S^7, it can always be isometrically embedded in a flat space of sufficiently high dimension d. If $\mathcal{M}$ is complex, it should not be embedded in $\mathbf{R}^d$, but rather $\mathbf{C}^{d/2}$, in order to preserve the complex structure $I^a{}_b$.

Since $\mathcal{H} \subseteq \mathcal{G}$, the flat connection $\hat{\omega}$ can be decomposed into pieces parallel and perpendicular to $\mathcal{H}$,

$$\hat{\omega} = \hat{\omega}|_{\mathcal{H}} + \hat{\omega}|_{\mathcal{K}} , \tag{30}$$

where $\mathcal{K} = \mathcal{G}/\mathcal{H}$. Restricted to the manifold $\mathcal{M}$, $\hat{\omega}|_{\mathcal{H}} = \omega$ and $\hat{\omega}|_{\mathcal{K}} = \tau$. The connection $\omega' = \omega + \tau$ is flat, so $I' = -2\pi \int \Omega(\omega')$ can be pulled back to spacetime. The integral I' also cancels the anomaly, up to local counterterms. For general Riemannian manifolds, the anomaly can be cancelled by local counterterms whenever the fermion representations $\rho_{\mathcal{H}}$ can be completed to representations $\rho_{\mathcal{G}}$ of $\mathcal{G}$. This is the appropriate generalization of the 't Hooft anomaly-matching condition.

If the anomaly cannot be cancelled, one can sometimes adjust the fermion representations to ensure that the final theory is anomaly-free. There is one type of model where this is not possible — the supersymmetric nonlinear sigma models. In these models, supersymmetry relates the spinors χ^A to the scalars ϕ^i. The representations of the spinors are fixed by the manifold $\mathcal{M}$. We shall see that this leads to severe constraints on supersymmetric nonlinear sigma models.

In four spacetime dimensions, $N = 1$ supersymmetric sigma models require $\mathcal{M}$ to be Kähler, and the spinors χ^A to be sections of the complexified tangent bundle $\mathcal{T}_c$ over $\mathcal{M}$ [22]. For irreducible manifolds, this simply says that the structure group $\mathcal{H}$ of $\mathcal{T}_c$ is $U(n)$. Since $\delta\phi^i = e^i{}_a\, \bar{\epsilon}_R \chi_L{}^a$, supersymmetry restricts the fermion representation $\rho_{\mathcal{H}}$ to be the fundamental representation $\mathbf{n}$ of $U(n)$.

A particularly intriguing class of Kähler manifolds are the irreducible symmetric spaces G/H [23] listed in Table 1. As before, G is the full isometry group, and H is its isotropy subgroup. The fermions transform in the fundamental representation $\rho_{\mathcal{H}} = \mathbf{n}$ of $\mathcal{H} = U(n)$. Because of the $U(1)$ charges, the 't Hooft matching condition can never be satisfied. The supersymmetric sigma models associated with the manifolds of Table 1 all suffer from the holonomy anomaly.

5. External Gauge Fields – String Considerations

In this section we introduce a background gauge field $A_i(\phi)$ on the Riemannian manifold $\mathcal{M}$. This field couples to the fermions through the covariant derivative $\mathcal{D}_\mu \chi^a = D_\mu \chi^a + i\partial_\mu \phi^i A_i{}^a{}_b \chi^b$, where $D_\mu \chi^a$ is the derivative defined in (2), and $A_i{}^a{}_b = A_i^{(c)} T^{(c)a}{}_b$ is valued in the Lie algebra of the gauge group G. The fermionic part of the Lagrangian is now

$$\mathcal{L}_f = -\frac{i}{2} \overline{\chi}_a \gamma^\mu \mathcal{D}_\mu \chi^a . \qquad (31)$$

In equation (31), we have adopted the convention that lowercase indices $(a, b, c...)$ are gauge indices, and all Lorentz indices $(A, B, C...)$ are dropped from the spinors χ.

The Lagrangian (31) is manifestly invariant under coordinate transformations and frame rotations. It is also invariant under field-dependent gauge rotations, provided the fermions and the connection transform as follows:

$$\begin{aligned} \delta \chi^a &= i\,\epsilon^{(c)}\, T^{(c)a}{}_b\, \chi^b \\ \delta A_i^{(a)} &= -\,\partial_i \epsilon^{(a)} - i\, f^{abc}\, A_i^{(b)}\, \epsilon^{(c)} . \end{aligned} \qquad (32)$$

In the presence of anomalies, the quantum effective action is not invariant under these symmetries. As before, we must cancel the anomalies by Chern-Simons terms. Now, however, we need a Chern-Simons term for each of the connections ω and A. The integral

$$I = -2\pi \int_D \Omega(\omega) - 2\pi \int_D \Omega(A) \tag{33}$$

precisely cancels the sigma model anomalies. Since $d\Omega(\omega) = \mathrm{Tr}\, R^n$ and $d\Omega(A) = \mathrm{Tr}\, F^n$, where $F = dA + A^2$, the integral I can be pulled back to spacetime whenever

$$\mathrm{Tr}\, R^n + \mathrm{Tr}\, F^n = 0 . \tag{34}$$

In equation (34), the traces are over the appropriate fermion representations. An extra minus sign is implied for representations of right-handed fermions. If (34) is satisfied, the Lagrangian is local in the fields ϕ^i.

A trivial way to satisfy the condition (34) is for $\mathrm{Tr}\, R^n$ and $\mathrm{Tr}\, F^n$ to vanish identically. Then each of the two terms in I can be separately pulled back to spacetime. A more interesting possibility is when $\mathrm{Tr}\, R^n = -\mathrm{Tr}\, F^n \neq 0$. Then I can be pulled back even though neither Chern-Simons term is closed.

For the rest of the paper, we specialize to the case of two spacetime dimensions. This is the case that is relevant for string theories. The anomaly cancellation condition is now

$$\mathrm{Tr}\, R^2 + \mathrm{Tr}\, F^2 = 0 . \tag{35}$$

When (35) is satisfied, the anomaly is cancelled by a three-dimensional integral over the Chern-Simons terms $\Omega(\omega)$ and $\Omega(A)$.

Two-dimensional sigma models describe the propagation of strings in various background geometries [24]. The background geometries arise from condensations of infinite numbers of strings. The background fields are classical solutions to the string equations of motion. Quantum modes are found by expanding the string variables about the nontrivial vacuum solutions. To have a consistent, anomaly-free string theory, the condition (35) must be satisfied at all points on the string world sheet.

For ordinary two-dimensional sigma models, condition (35) requires $\mathrm{Tr}\, R^2 = -\,\mathrm{Tr}\, F^2$. For string theories, however, the consistency condition is a little stronger. This is because consistent strings require anomaly cancellation *and* conformal invariance in the full quantum theory. The conformal symmetry is typically spoiled by Chern-Simons terms in the sigma model Lagrangian.[♮3] Consistent string theories require anomaly cancellation without the introduction of Chern-Simons terms.

For the heterotic string [25], anomaly cancellation seems to require that the spin connection ω be embedded in the gauge group G [11]. This follows from the fact that the left- and right-handed fermions couple to different connections. The left-handed fermions live on a curved six-dimensional space. They give rise to a supersymmetric sigma model with a spin connection but no gauge fields. The right-handed fermions transform under $G = E(8) \times E(8)$ or $O(32)$, but live on a flat world sheet. Since Chern-Simons terms spoil conformal invariance, the anomalies of the left- and right-movers must cancel against each other. This requires that the spin connection be embedded in G.

♮3 Even if all β-functions can be arranged to vanish, the Chern-Simons terms also change the value of the central charge in the Virasoro algebra [25].

ACKNOWLEDGEMENTS

We would like to thank Luis Alvarez-Gaumé, Greg Moore, Phil Nelson, Michael Peskin, Marvin Weinstein, Edward Witten and Yong-Shi Wu for many helpful discussions. J.B. owes a special debt to Chong-Leong Ong for his collaboration during the early stages of this work.

REFERENCES

[1] B. Zumino, in *Relativity, Groups and Topology II*, eds. B. DeWitt and R. Stora, (North-Holland, Amsterdam, 1984); R. Stora, LAPP preprint LAPP-TH-94 (1983); Y.-S. Wu, A. Zee and B. Zumino, Nucl. Phys. B239 (1984) 477; L. Alvarez-Gaumé and P. Ginsparg, Nucl. Phys. B243 (1984) 449; W. Bardeen and B. Zumino, Nucl. Phys. B244 (1984) 421.

[2] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1984) 269.

[3] E. Witten, Phys. Lett. 117B (1982) 324.

[4] O. Alvarez, I. Singer and B. Zumino, Comm. Math. Phys. 96 (1984) 409; L. Alvarez-Gaumé and P. Ginsparg, Harvard preprint HUTP-84/A016 (1984).

[5] G. Moore and P. Nelson, Phys. Rev. Lett. 53 (1984) 1519; and Harvard preprint HUTP-84/A076 (1984).

[6] E. Cohen and C. Gomez, Nucl. Phys. B254 (1985) 235; P. DiVecchia, S. Ferrara and L. Girardello, Phys. Lett. 151B (1985) 199.

[7] A. Manohar, G. Moore and P. Nelson, Phys. Lett. 152B (1985) 68.

[8] G. 't Hooft, in *Recent Developments in Gauge Theories*, eds. G. 't Hooft *et al.*, (Plenum, New York, 1980).

[9] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2239; C. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2247.

[10] L. Alvarez-Gaumé and P. Ginsparg, Harvard preprint HUTP-85/A015 (1985).

[11] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Santa Barbara preprint NSF-ITP-84-170 (1984).

[12] S. Adler, Phys. Rev. 177 (1969) 2426; J. Bell and R. Jackiw, Nuovo Cimento 60A (1969) 47.

[13] W. Bardeen, Phys. Rev. 184 (1969) 1848.

[14] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195.

[15] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95.

[16] D. Gross and R. Jackiw, Phys. Rev. D6 (1972) 477.

[17] S. Chern and J. Simons, Proc. Nat. Acad. Sci. 68 (1971) 791.

[18] E. Witten, Nucl. Phys. B223 (1983) 422.

[19] B. Zumino, in Reference [1].

[20] Y. Wu, Phys. Lett. 153B (1985) 70.

[21] E. Witten, private communication.

[22] B. Zumino, Phys. Lett. 86B (1979) 279.

[23] C. Ong, Phys. Rev. D27 (1983) 911; E. Calabi and E. Vesentini, Ann. Math. 71 (1960) 472.

[24] C. Callan, E. Martinec, M. Perry and D. Friedan, Princeton preprint (1985); A. Sen, Fermilab preprint PUB-85/77 (1985).

[25] D. Nemeschansky and S. Yankielowicz, Phys. Rev. Lett. 54 (1985) 620; D. Friedan and S. Shenker (unpublished).

[26] D. Gross, J. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54 (1985) 502.

Table 1: Typical Kähler Manifolds G/H

G	H	$\mathcal{H}$	$\rho_{\mathcal{H}}$
$SU(p+q)$	$SU(p) \times SU(q) \times U(1)$	$U(pq)$	$\mathbf{pq}$
$SO(2p)$	$SU(p) \times U(1)$	$U(p(p-1)/2)$	$\mathbf{p(p-1)/2}$
$Sp(2p)$	$SU(p) \times U(1)$	$U(p(p+1)/2)$	$\mathbf{p(p+1)/2}$
$SO(p+2)$	$SO(p) \times U(1)$	$U(p)$	$\mathbf{p}$
$E(6)$	$SO(10) \times U(1)$	$U(16)$	$\mathbf{16}$
$E(7)$	$E(6) \times U(1)$	$U(27)$	$\mathbf{27}$

For Kähler manifolds G/H, the fermions transform in the fundamental representation ρ_H of the structure group $\mathcal{H}$.

FIGURE CAPTIONS

1. The anomalous triangle diagram in four dimensions. In $2d$ dimensions, the corresponding diagram has $d+1$ legs.

2. A fiber bundle $\mathcal{E}$, with total space G and fiber H. Left G-rotations must be returned to the section G/H by right H-transformations.

3. The lowest order graph that gives mass to the scalar fields ϕ^i.

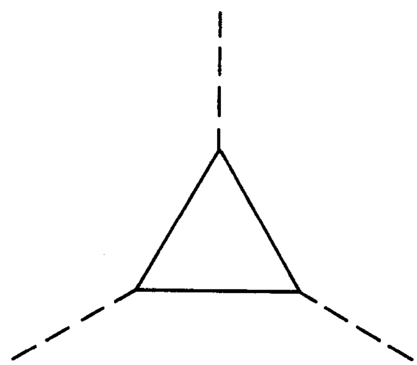

FIGURE 1

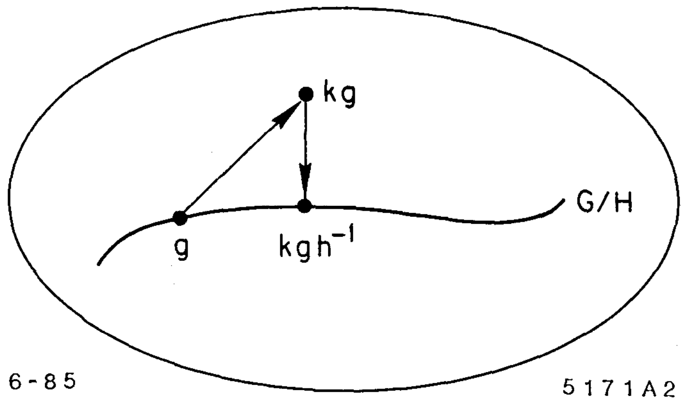

FIGURE 2

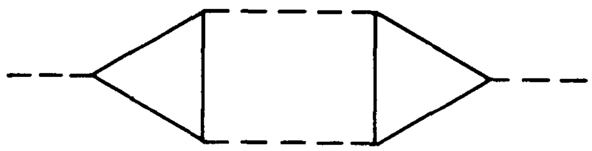

FIGURE 3