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VARIATION OF THE DISPERSION FUNCTION, MOMENTUM COMPACTION FACTOR AND DAMPING PARTITION NUMBERS WITH PARTICLE ENERGY DEVIATION*

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ABSTRACT

Exact analytical expressions for the periodic solution and transportation of the dispersion function and its perturbation with energy deviation are derived and found to be valid for any order of the expansion; the solution depends on two specific integrals only. These integrals are related to the driving terms of the particular perturbations and are exactly solved for the first and second order expansions, including magnetic elements up to second order. The same method could be used to evaluate higher order perturbations of the dispersion function.

The variation with particle energy of two particularly important ring parameters, the momentum compaction factor and the damping partition numbers, is analyzed and their dependance on the perturbation of the dispersion function is emphasized. These results are then applied to a typical machine to illustrate the importance of the effects due to magnets with small bending radius and due to sextupoles. As the results demonstrate, none of the contributing terms should be neglected.

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1. Introduction

Recently, there has been increasing interest in the study of off–momentum particle behavior to enable a better understanding of the dynamic aperture in machines with large momentum acceptance or large circumference. Moreover the desire for localized dispersion function (for example to minimize the equilibrium beam emittance in electron machines) implies strong localized sextupole components for chromaticity correction. They perturb the motion of the off–momentum particles and consequently affect the dynamic acceptance.

The dispersion function, which defines the ideal closed orbit of the particle with energy deviation, is particularly important for the machine acceptance and several other parameters. The momentum compaction factor[1,2] and the damping partition numbers[3,4], (whose variation to first order in energy deviation are recalled in chapter 3 and 4,) are only two of them. Taking into account non–linear effects, a large contribution to their variation is found to come from the perturbation of the dispersion function itself. This perturbation is usually calculated either numerically from second order transfer matrices[5,6] or analytically from simplified[7–10] differential equations of motion. These simplifications can be very dangerous as they eliminate contributing terms which cannot always be neglected, especially in machines with small bending radius.

The complete differential equations describing the successive orders of perturbation of the dispersion function with energy deviation are all of the same form, namely that of an harmonic oscillator with corresponding driving functions. The periodic solution of this general equation is solved in chapter 2 using Green's function integrals; this leads to an exact analytical expression. This expression was already derived[7] for the periodic solution of the first order perturbation of the dispersion function but, is in fact valid for any order of the expansion. Furthermore, the periodic solution as well as the solution for the transportation of the different perturbations of the dispersion function are both completely determined by two particular integrals running over the elements which drive the corresponding perturbations. Thus, solving these two integrals in the elements under consideration (APPENDIX I and APPENDIX II) is the key to the determination of the exact perturbation of the dispersion function[11] to any order of its expansion. The observation of the driving functions immediately shows that the first order perturbation of the dispersion function is determined not only by all first order magnetic elements but also by the sextupole components. On the other hand, it is not affected by the octupoles which act only on the second order perturbation of the dispersion function.

2. Dispersion function

2.1 DIFFERENTIAL EQUATION FOR THE DISPERSION FUNCTION

Using the general equation of motion[12], the differential equation of particle motion in the horizontal plane to third order in the variables x, y and their derivatives is

$$\begin{aligned} & x'' - h(1+hx) - x'\left[x'x'' + y'y'' + (1-hx)(hx' + xh')\right] \\ & = (1 - \delta + \delta^2 - \delta^3)\Big\{ -h + (k - 2h^2)x + (2hk - h^3 + \frac{1}{2}r)x^2 - \frac{1}{2}hx'^2 \\ & \quad + \frac{1}{2}(h'' - hk - r)y^2 + h'yy' - \frac{1}{2}hy'^2 + (h^2k + hr + \frac{1}{6}q)x^3 \\ & \quad - \frac{1}{2}(h^2k + 3hr + q + h'^2 + k'')xy^2 - k'xyy' + \frac{1}{2}kx(x'^2 + y'^2)\Big\} . \end{aligned} \tag{2.1}$$

The prime indicates the derivative with respect to the azimuth s and

x	is the horizontal coordinate,
y	is the vertical coordinate,
s	is the arc length along the reference orbit,
$h(s) = \frac{1}{\rho}$	is the curvature of the reference orbit,
$k(s) = -\frac{1}{B\rho}\frac{\partial B_y}{\partial x}\Big\vert_{x=0}$	is the normalized quadrupole strength

$r(s) = -\frac{1}{B\rho}\frac{\partial^2 B_y}{\partial x^2}\Big|_{x=0}$ is the normalized sextupole strength

$q(s) = -\frac{1}{B\rho}\frac{\partial^3 B_y}{\partial x^3}\Big|_{x=0}$ is the normalized octupole strength

$B\rho$ is the particle rigidity,

$\delta = \frac{\Delta p}{p}$ is the particle energy deviation.

The horizontal motion x of a particle with energy deviation δ can be written in the form

$$x = \bar{x} + D\delta \,, \tag{2.2}$$

where $\bar{x}$ is the betatron oscillation and $D\delta$ is the closed orbit of this particle. The complete differential equation for the dispersion function $D(s)$ (also called α_p or η in accelerator theory) is deduced by inserting eq. (2.2) with $\bar{x} \equiv 0$ into eq. (2.1). To second order in δ one gets

$$\begin{aligned} D'' + (h^2 - k)D = h & \\ + \Big[-h + (2h^2 - k + h'D')D + (2hk - h^3 + \frac{1}{2}r)D^2 + \frac{1}{2}hD'^2 \Big]\delta & \\ + \Big[h - (2h^2 - k)D + (\frac{1}{2}k - h^2)DD'^2 + (\frac{1}{2}h + D'')D'^2 & \\ - (2hk - h^3 + \frac{1}{2}r + hh'D')D^2 + (h^2k + hr + \frac{1}{6}q)D^3 \Big]\delta^2 \,. & \end{aligned} \tag{2.3}$$

The expansion of the dispersion function in the form

$$D = D_0 + D_1\delta + D_2\delta^2 \tag{2.4}$$

enables to solve eq. (2.3) successively for each power of δ. The dispersion function $D_0(s)$ is the periodic solution of the well known differential equation

$$D_0'' + (h^2 - k)D_0 = h \,. \tag{2.5}$$

The differential equations for the perturbation to the first order, $D_1(s)$, and to second order, $D_2(s)$, in δ are the periodic solutions of

$$D_1'' + (h^2 - k)D_1 = -h + (2h^2 - k + h'D_0')D_0 + (2hk - h^3 + \frac{1}{2}r)D_0^2 + \frac{1}{2}hD_0'^2 \quad (2.6)$$

and

$$\begin{aligned} D_2'' + (h^2 - k)D_2 = h &- (2h^2 - k)D_0 + (\frac{1}{2}k - h^2)D_0 D_0'^2 + (\frac{1}{2}h + D_0'')D_0'^2 \\ &- (2hk - h^3 + \frac{1}{2}r + hh'D_0')D_0^2 + (h^2 k + hr + \frac{1}{6}q)D_0^3 \\ &+ (4hk - 2h^3 + r)D_0 D_1 + (2h^2 - k + h'D_0')D_1 \\ &+ (h'D_0 + hD_0')D_1' \end{aligned} \quad (2.7)$$

respectively.

In a series of reports[7–10] eq. (2.6) for D_1 has been derived for machines with large bending radius ρ where higher order terms in h as well as the terms coming from combined function magnets (hk) and from the variation of the dispersion function with azimuth (D_0') were disregarded. For small machines these terms cannot be neglected.

The corresponding differential equations for higher order perturbations of the dispersion function can be derived in a similar way. As demonstrated by the differential equation for $D_2(s)$, eq. (2.7), the number of terms involved will increase.

Nevertheless, the differential equations for D_0, D_1 and D_2, eqs. (2.5, 2.6, 2.7), are all of the form

$$D_i'' + (h^2 - k)D_i = f_i \qquad \text{with} \quad i = 0, 1, 2 \quad (2.8)$$

where

$$
\begin{aligned}
f_0 &= h\,, \\
f_1 &= -h + (2h^2 - k + h'D_0')D_0 + (2hk - h^3 + \frac{1}{2}r)D_0^2 + \frac{1}{2}hD_0'^2\,, \\
f_2 &= h - (2h^2 - k)D_0 + (\frac{1}{2}k - h^2)D_0D_0'^2 + (\frac{1}{2}h + D_0'')D_0'^2 \\
&\quad - (2hk - h^3 + \frac{1}{2}r + hh'D_0')D_0^2 + (h^2k + hr + \frac{1}{6}q)D_0^3 \\
&\quad + (4hk - 2h^3 + r)D_0D_1 + (2h^2 - k + h'D_0')D_1 + (h'D_0 + hD_0')D_1'\,.
\end{aligned}
\tag{2.9}
$$

Equation (2.8) is also valid for higher order perturbations of the dispersion function with more complicated driving functions f_i, $i \geq 3$ on the r.h.s.

2.2 Periodic solution

Assuming that the Twiss parameters (β, α) and the phase advance (μ) along the central trajectory are known, and using the usual variable transformations[13]

$$
\begin{aligned}
\phi(s) &= \int \frac{ds}{Q\beta}\,, \\
E_i(\phi) &= \frac{D_i}{\sqrt{\beta}}
\end{aligned}
$$

where Q is the horizontal tune (the number of horizontal betatron oscillations per turn around the machine), the general differential equation (2.8) for the dispersion function can be transformed into the equation of a forced harmonic oscillator:

$$
\frac{d^2E_i}{d\phi^2} + Q^2E_i = Q^2\beta^{3/2}f_i(\phi) \tag{2.10}
$$

where f_i are the functions defined in eq. (2.9). By means of Green's function, the periodic solution of this equation (2.10) can be written in the form[13]

$$E_i(\phi) = \frac{Q}{2\sin\pi Q} \int\limits_{\phi}^{2\pi+\phi} \beta^{3/2} f_i(\psi)\cos[Q(\pi+\phi-\psi)]\, d\psi\,. \tag{2.11}$$

Introducing the original variables

$$\begin{aligned} d\sigma &= Q\beta\, d\psi\,, \\ \mu(s) &= Q\phi\,, \\ D_i(s) &= E_i\sqrt{\beta}\,, \end{aligned} \tag{2.12}$$

into eq. (2.11) gives

$$D_i(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q} \int\limits_{s}^{s+L} \sqrt{\beta(\sigma)} f_i(\sigma)\, \cos[Q\pi + \mu(s) - \mu(\sigma)]\, d\sigma \tag{2.13}$$

with the functions f_i defined in eq. (2.9) and with L as the length (circumference) of the reference orbit. Differentiating eq. (2.13) with respect to s leads to the periodic solution for $D_i'(s)$:

$$D_i'(s) = -D_i(s)\frac{\alpha(s)}{\beta(s)} - \frac{1}{2\sqrt{\beta(s)}\sin\pi Q} \int\limits_{s}^{s+L} \sqrt{\beta(\sigma)} f_i(\sigma)\, \sin[Q\pi + \mu(s) - \mu(\sigma)]\, d\sigma\,. \tag{2.14}$$

Assuming hard edge approximation, where the magnetic field rises abruptly from zero outside the magnetic element to a constant value inside it, one can solve the integral on the r.h.s. of the eqs. (2.13), (2.14) analytically.

Defining an element j of length l_j by its transfer matrix

$$\mathbf{R_j}(\sigma) = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

and using the values for the functions β_j, α_j and μ_j at the entrance of this element, the variation of the phase μ and the β–function are given by

$$\begin{aligned}\mu(\sigma) &= \arctan\left(\frac{R_{12}(\sigma)}{\beta_j R_{11}(\sigma) - \alpha_j R_{12}(\sigma)}\right) ,\\ \beta(\sigma) &= \beta_j R_{11}^2(\sigma) - 2\alpha_j R_{11}(\sigma) R_{12}(\sigma) + \frac{1+\alpha_j^2}{\beta_j} R_{12}^2(\sigma) .\end{aligned} \tag{2.15}$$

Taking the origin $s = 0$ as the starting point for the phase $\mu(s)$ one obtains

$$\begin{aligned}\sqrt{\beta(\sigma)}\cos[\pi Q + \mu(s) - \mu_j - \mu(\sigma)] &= \left[\sqrt{\beta_j} R_{11}(\sigma) - \frac{\alpha_j}{\sqrt{\beta_j}} R_{12}(\sigma)\right] \cos\Delta\mu_j \\ &\quad + \frac{R_{12}(\sigma)}{\sqrt{\beta_j}} \sin\Delta\mu_j ,\\ \sqrt{\beta(\sigma)}\sin[\pi Q + \mu(s) - \mu_j - \mu(\sigma)] &= \left[\sqrt{\beta_j} R_{11}(\sigma) - \frac{\alpha_j}{\sqrt{\beta_j}} R_{12}(\sigma)\right] \sin\Delta\mu_j \\ &\quad - \frac{R_{12}(\sigma)}{\sqrt{\beta_j}} \cos\Delta\mu_j ,\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}\Delta\mu_j &= \pi Q + \mu(s) - \mu_j \qquad \text{for} \quad \mu(s) < \mu_j ,\\ \Delta\mu_j &= -\pi Q + \mu(s) - \mu_j \qquad \text{for} \quad \mu(s) > \mu_j .\end{aligned}$$

Thus eqs. (2.13), (2.14) can be written in the following form

$$D_i(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q} \sum_j \left\{ \sqrt{\beta_j} \cos\Delta\mu_j \int_0^{l_j} f_i(\sigma) R_{11}(\sigma)\, d\sigma \right.$$

$$\left. - \frac{1}{\sqrt{\beta_j}} (\alpha_j \cos\Delta\mu_j - \sin\Delta\mu_j) \int_0^{l_j} f_i(\sigma) R_{12}(\sigma)\, d\sigma \right\} ,$$

$$D_i'(s) = -D_i(s)\frac{\alpha(s)}{\beta(s)} - \frac{1}{2\sqrt{\beta(s)}\sin\pi Q} \sum_j \left\{ \sqrt{\beta_j} \sin\Delta\mu_j \int_0^{l_j} f_i(\sigma) R_{11}(\sigma)\, d\sigma \right.$$

$$\left. - \frac{1}{\sqrt{\beta_j}} (\alpha_j \sin\Delta\mu_j + \cos\Delta\mu_j) \int_0^{l_j} f_i(\sigma) R_{12}(\sigma)\, d\sigma \right\} , \tag{2.17}$$

where j runs over the elements of the machine. These expressions were already derived[7] for the perturbation D_1. In fact, eqs. (2.17) are valid for any order, i, of the perturbation of the dispersion function. Their periodic solutions are completely determined by the following integrals

$$\int_0^{l_j} f_i(\sigma) R_{11}(\sigma)\, d\sigma \qquad \text{and} \qquad \int_0^{l_j} f_i(\sigma) R_{12}(\sigma)\, d\sigma \tag{2.18}$$

over the elements for which the functions f_i are different from zero.

Eqs. (2.9) show that the function D_0 is only driven by the bending magnets ($h \neq 0$) while the function D_1 is driven by the combined function bending magnets ($hk \neq 0$), their edges ($h' \neq 0$) and by the quadrupoles ($k \neq 0$) and the sextupoles ($r \neq 0$). The function D_2 is driven by the same elements as well as by the octupoles ($q \neq 0$).

The computation of the above integrals corresponding to the functions D_0 and D_1 is derived in APPENDIX I for all elements up to second order by replacing the transfer matrix elements by their corresponding values. The same method could be applied to higher order perturbations of the dispersion function.

2.3 TRANSPORT OF THE DISPERSION FUNCTION

Although eqs. (2.17) define the periodic solution of D_i and D_i' respectively everywhere in the machine, the transport of the dispersion function emphasizes its variation due to the different elements. Moreover, it facilitates the calculation of the perturbation of the dispersion function along transfer lines.

Supposing the values of the dispersion functions D_i and their derivatives D_i' are known at the point s_0 either by determination of the periodic solution in a ring using eqs. (2.17) or as given initial values for a transfer line, the transported values of the functions D_i and D_i' can then be determined at any point s of the transfer channel using the transfer matrix $\mathbf{M}$ defined as

$$\mathbf{M} = \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta(s_0)}}\,[\cos\Delta\mu + \alpha(s_0)\sin\Delta\mu] & \sqrt{\beta(s)\beta(s_0)}\,\sin\Delta\mu \\ \frac{\alpha(s_0)-\alpha(s)}{\sqrt{\beta(s_0)\beta(s)}}\cos\Delta\mu - \frac{1+\alpha(s_0)\alpha(s)}{\sqrt{\beta(s_0)\beta(s)}}\sin\Delta\mu & \sqrt{\frac{\beta(s_0)}{\beta(s)}}\,[\cos\Delta\mu + \alpha(s)\sin\Delta\mu] \end{pmatrix},$$

with $\Delta\mu = \mu(s) - \mu(s_0)$.

In fact, the general solution of eq. (2.10) can be written in the form

$$E_i(\phi) = A_i\cos[Q(\phi-\phi_0)] + B_i\sin[Q(\phi-\phi_0)] + Q\int_{\phi_0}^{\phi} \beta^{3/2} f_i(\psi)\sin[Q(\phi-\phi_0-\psi)]\,d\psi$$

where $\phi \equiv \phi(s)$ and $\phi_0 \equiv \phi(s_0)$ and with the coefficients A_i, B_i determined by the initial conditions:

$$A_i = E_i(\phi_0)\,, \qquad B_i = \frac{1}{Q}\frac{dE_i(\phi_0)}{d\phi}\,.$$

Using the original variables as defined in eq. (2.12), the perturbation of the dispersion functions at any point s are defined by

$$D_i(s) = D_i(s_0)M_{11}+D_i'(s_0)M_{12}+\sqrt{\beta(s)}\int_{s_0}^{s}\sqrt{\beta(\sigma)}f_i(\sigma)\sin[\mu(s)-\mu(s_0)-\mu(\sigma)]\,d\sigma, \tag{2.19}$$

Hence, the dispersion functions can be determined successively from the entry (beginning) $D_i(0) \equiv D_{ib}$ to the exit $D_i(l)$ of each element with $\mathbf{M} = \mathbf{R}(l)$ as the transfer matrix of this particular element. Introducing eq. (2.16) into eq. (2.19) simplifies the equations for the dispersion functions to

$$\begin{aligned} D_i(l) &= D_{ib}M_{11} + D_{ib}'M_{12} + M_{12}\int_0^l f_i(\sigma)R_{11}(\sigma)\,d\sigma - M_{11}\int_0^l f_i(\sigma)R_{12}(\sigma)\,d\sigma\,, \\ D_i'(l) &= D_{ib}M_{21} + D_{ib}'M_{22} + M_{22}\int_0^l f_i(\sigma)R_{11}(\sigma)\,d\sigma - M_{21}\int_0^l f_i(\sigma)R_{12}(\sigma)\,d\sigma\,. \end{aligned} \tag{2.20}$$

These relations are also general and valid for any order, i, of perturbation of the dispersion function. They depend on the same integrals (2.18) already defined for the calculation of the periodic solution.

Applying eq. (2.20) to the dispersion function D_0 leads to the well known formulae

$$D_0(l) = D_{0b}M_{11} + D_{0b}'M_{12} + M_{13}\,,$$

$$D_0'(l) = D_{0b}M_{21} + D_{0b}'M_{22} + M_{23}\,.$$

The complete expressions for the transport of the function D_1 through the different machine elements up to second order are summarized in APPENDIX II; they were obtained by replacing the integrals in eqs. (2.20) by their analytical solutions derived in APPENDIX I.

3. Momentum compaction factor

The momentum compaction factor represents the relative change of the orbit length with respect to the particle energy deviation. It can be defined in the form

$$\alpha = \frac{dL}{dp}\frac{p_0}{L_0} = \frac{d\left(\frac{\Delta L}{L_0}\right)}{d\delta} = \alpha_1 + 2\alpha_1\alpha_2\delta \ \ldots\ , \tag{3.1}$$

as given in[1] with L_0 as the length of the ideal orbit.

Simple geometrical considerations lead to

$$L = \int_0^{L_0} \sqrt{1 + x'^2}\,(1 + hx)ds\,. \tag{3.2}$$

Inserting eq. (2.2) with $\bar{x} \equiv 0$ into eq. (3.2), expanding the square root and introducing eq. (2.4) gives

$$L = L_0 + \delta \int_0^{L_0} hD_0 ds + \delta^2 \int_0^{L_0} \left(hD_1 + \frac{1}{2}D_0'^2\right) ds \ \ldots\ , \tag{3.3}$$

which together with eq. (3.1) provides the following relations[1,2]:

$$\begin{aligned} \alpha_1 &= \frac{1}{L_0}\int_0^{L_0} hD_0 ds\,, \\ \alpha_1\alpha_2 &= \frac{1}{L_0}\int_0^{L_0} \left(hD_1 + \frac{1}{2}D_0'^2\right) ds\,. \end{aligned} \tag{3.4}$$

Thus, the perturbation of the momentum compaction factor to first order in δ has contributions from the slope of the dispersion function, $D_0'(s)$, all along the ring and from the first order perturbation $D_1(s)$ in the bending magnets, better known as the *Johnsen effect*[1].

4. Damping partition numbers

4.1 Introduction

For circular machines whose closed orbit lies in a horizontal plane, the transverse and longitudinal damping partition numbers J_x, J_y and J_ϵ can be defined[3,4] in terms of the second (I_2) and fourth (I_4) synchrotron integral as

$$J_x = 1 - \frac{I_4}{I_2} \qquad J_y = 1 \qquad J_\epsilon = 2 + \frac{I_4}{I_2}$$

with

$$I_2 = \int h^2\, ds$$

and

$$I_4 = \int (h_0 h^2 - 2hk) D\, ds\,.$$

The index "0" refers to the reference orbit.

The variation of the longitudinal damping partition number J_ϵ with respect to an energy deviation δ is given by

$$\frac{dJ_\epsilon}{d\delta} = \frac{1}{I_{20}}\frac{dI_4}{d\delta} - \frac{I_{40}}{I_{20}^2}\frac{dI_2}{d\delta}\,, \tag{4.1}$$

where $I_{20} = I_2$ and $I_{40} = I_4$ for $\delta = 0$.

4.2 Expansion of the synchrotron integrals: I_2 and I_4

The dependence of I_2 and I_4 on δ is evaluated using the variation of the variables $h(s)$, $k(s)$ and $D(s)$ with δ. Expanding B_y in a Taylor series and substituting $x = D\delta$ gives

$$h(s) = \frac{B_y}{B\rho} = \frac{1}{(B\rho)_0(1+\delta)} \left(B\Big|_{x=0} + \frac{\partial B_y}{\partial x}\Big|_{x=0} D\delta \right)$$

$$= h_0 - (h_0 + k_0 D)\delta$$

and

$$k(s) = -\frac{1}{B\rho}\frac{\partial B_y}{\partial x} = -\frac{1}{(B\rho)_0(1+\delta)} \left(\frac{\partial B_y}{\partial x}\Big|_{x=0} + \frac{\partial^2 B_y}{\partial x^2}\Big|_{x=0} D\delta \right)$$

$$= k_0 - (k_0 - r_0 D)\,\delta\,.$$

Using eq. (2.4), introducing these expansions into the relations for the synchrotron integrals I_2 and I_4 and omitting the suffix "0" of h, k and r leads to

$$I_2 = \int h^2\,ds - 2\delta \int (h^2 + hkD_0)\,ds \tag{4.2}$$

and

$$I_4 = \int (h^3 - 2hk) D_0\,ds$$
$$+ 2\delta \int \left[k^2 D_0^2 - (h^3 - 2hk)D_0 - h^2 k D_0^2 - hr D_0^2 + (\frac{1}{2}h^3 - hk) D_1 \right] ds\,, \tag{4.3}$$

These formulations are completely general and give the contribution to the second and fourth synchrotron integral for any magnetic element, from simple dipoles and quadrupoles to combined function magnets with sextupole. The contribution of an element's edges to these integrals can then be deduced directly from the characteristics of this element.

4.3 Contribution of the Edges

For magnetic elements with inclined boundaries whose entrance and/or exit faces are not normal to the design trajectory, the local quadrupole $k(s)$ and sextupole $r(s)$ fields experienced by a particle traversing the element and it's fringe field along the design trajectory are given by[15]

$$\begin{aligned} k(s) &= k_{\rm m} + h' \tan\theta - hh' D_o \tan\theta\delta \,, \\ r(s) &= r_{\rm m} - 2k'_{\rm m} \tan\theta \mp h'' \tan^2\theta + hh' \tan^3\theta \,, \end{aligned} \tag{4.4}$$

with θ as the effective edge angle θ_e modified by the slope of the off–momentum particle trajectory,

$$\theta = \theta_e \pm \left(D_0' \pm \frac{1}{2} h D_0 \tan\theta \right) \delta \,. \tag{4.5}$$

The sign convention for the entrance/exit edge angles θ_e is defined in[12]; $'$ and $''$ denote the first and second derivatives with respect to the azimuth s, $k_{\rm m}$ and $r_{\rm m}$ are the quadrupole and sextupole components of the element. The upper sign corresponds to the entrance edge, the lower sign to an exit edge.

4.4 Variation of the Damping Partition Number with Energy

After integrating through the edges and applying hard edge approximation the variation of the damping partition numbers to first order with particle energy deviation becomes

$$\begin{aligned} \frac{dJ_\epsilon}{d\delta} = \frac{1}{I_{20}} \Bigg\{ & 2\frac{I_{40}}{I_{20}} \int hkD_0\,ds + 2\int (k^2 - h^2k - hr) D_0^2\,ds + \int (h^3 - 2hk) D_1\,ds \\ & + \sum \left(\frac{I_{40}}{I_{20}} hD_0 + \left[4k + \frac{5}{6} h^2 (1 - \tan^2\theta_e) \right] D_0^2 - hD_1 \right) h \tan\theta_e \\ & \mp \sum h^2 (1 + 3\tan^2\theta_e) D_0 D_0' \Bigg\} \,, \end{aligned}$$

where the upper sign corresponds to an entrance edge and the lower sign corresponds to an exit edge and

$$\frac{dJ_x}{d\delta} = -\frac{dJ_\epsilon}{d\delta} ,$$

$$\frac{dJ_y}{d\delta} = 0 .$$

The different contributions can be separated into four categories according to where they originate:

- the quadrupole magnets (terms in k);
- the bending magnets (terms in h);
- the combined function magnets including quadrupole (terms in hk) and/or sextupole components (terms in hr);
- the variation of the dispersion function with the azimuth (terms in D'_0) and/or with particle momentum (terms in D_1).

The contribution of the quadrupoles responsible for the term

$$\frac{2}{I_{20}} \int k^2 D_0^2 \, ds$$

is normally dominant in large machines and is therefore the only one which is calculated in various optics programs[16]. Nevertheless, for rings with small bending radius or with combined function magnets the other terms cannot be neglected, as shown in the following example.

5. Application

The LEP Electron Positron Accumulator (EPA)[17] provides an ideal application for the calculation of the dispersion function D_1 as well as for the variation of the momentum compaction factor and of the damping partition number to first order in energy deviation. In fact, in this machine all the driving terms of these functions are present because of the low bending radius ($\rho = 1.426$ m) and of its combined function bending magnets ($k = 0.5$ m^{-2}) with entrance and exit angles.

Moreover, the strong sextupoles ($r \approx \pm 8.30$ m^{-3}) localized in the arcs do change the perturbed dispersion function D_1 in the long dispersion free sections ($D_0 = 0$), as shown in Fig. 1. This figure displays the functions $D_0(s)$ (*solid*) and $D_1(s)$ without (*dots*) and with(*dashes*) sextupoles as calculated by introducing the results of the optics program COMFORT[18] into the formulae listed in APPENDIX I and APPENDIX II.

Table 1 and Table 2 summarize the contributions of the different terms to the variation of the momentum compaction factor and the damping partition numbers with particle energy deviation respectively, both before and after the ring's natural chromaticities have been corrected. The variation of the longitudinal damping partition number and of the momentum compaction factor with energy after chromaticity correction are increased by a factor "2" and "–3" respectively due mainly to the change of the perturbed dispersion function $D_1(s)$ in the combined function magnets by the sextupoles.

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Table 1. Momentum compaction factor

	LEP EPA *sext. off (on)*
$\alpha_1 = \frac{1}{L_0} \int hD_0 \, ds$	0.03322
$\frac{1}{2L_0} \int {D_0'}^2 \, ds$	0.05021
$\frac{1}{L_0} \int hD_1 \, ds$	0.062 (–0.37)
$\alpha_1 \alpha_2$	0.11 (–0.32)

Table 2. Damping partition numbers

	LEP EPA *sext. off (on)*
I_2	4.406
I_4	−4.259
J_ϵ	1.033
$2/I_{20}\int k^2 D_0^2\,ds$	14.009
$2I_{40}/I_{20}^2\int hkD_0\,ds$	−0.916
$-2/I_{20}\int h^2 kD_0^2\,ds$	−0.844
$I_{40}/I_{20}^2\sum h^2 D_0\tan\theta$	−0.469
$4/I_{20}\sum hkD_0^2\tan\theta$	1.792
$5/(6I_{20})\sum h^3 D_0^2\tan\theta(1-\tan^2\theta)$	0.353
$\mp 1/I_{20}\sum h^2 D_0 D_0'(1+3\tan^2\theta)$	1.382
$1/I_{20}\int h^3 D_1\,ds$	0.88 (−5.18)
$-2/I_{20}\int hkD_1\,ds$	−1.78 (10.53)
$-1/I_{20}\sum h^2 D_1\tan\theta$	−0.89 (5.22)
$dJ_\epsilon/d\delta$	13.52 (25.88)

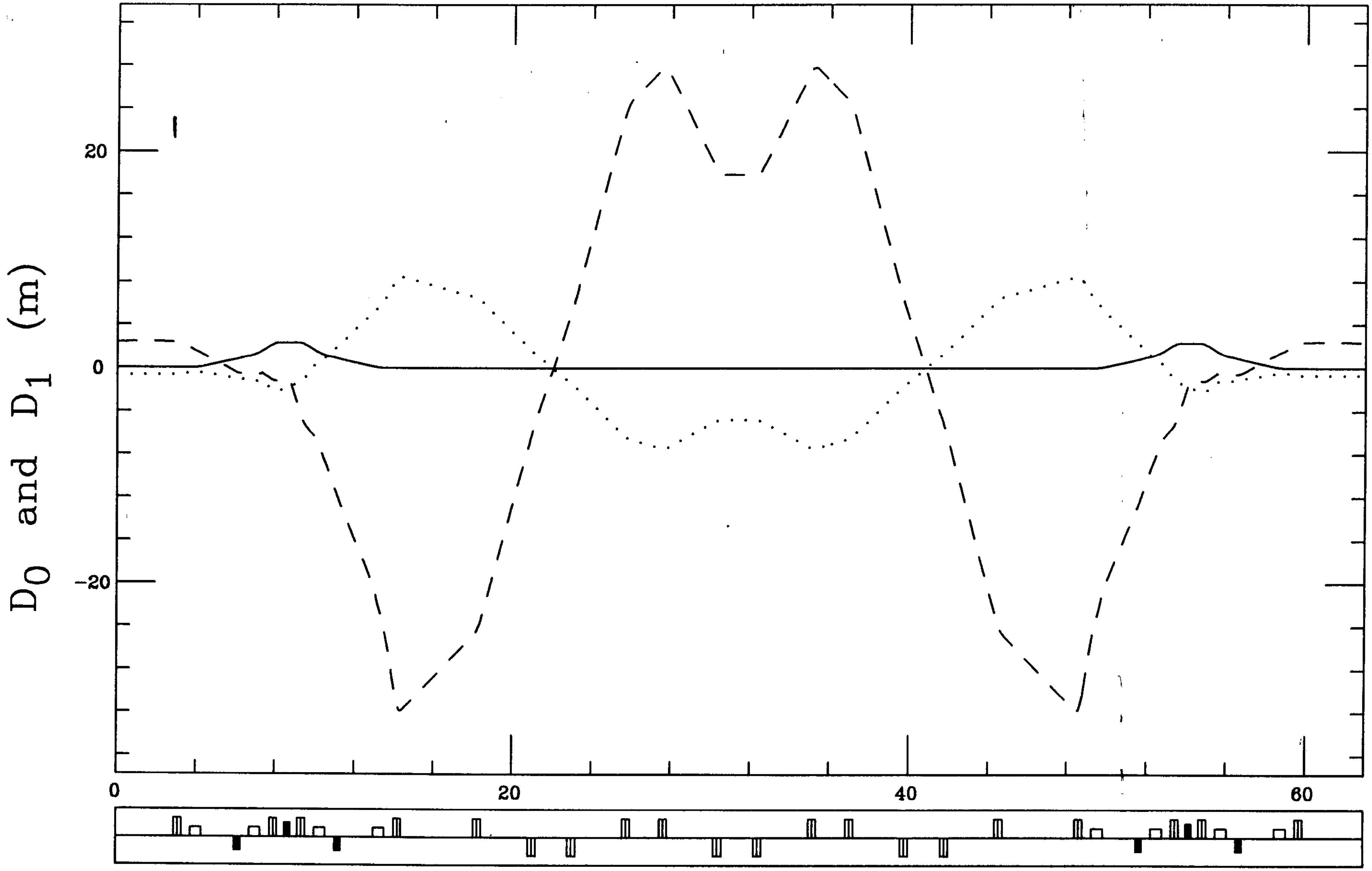

Fig. 1: Variation of the dispersion function D_0 (solid) and D_1 with sextupoles OFF (dots) and ON (dashes) [r_{sv}=8.25 m^{-3}, r_{sh}=−8.35 m^{-3}] along half of the LEP Electron Positron Accumulator(17)

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APPENDIX I

Contributions of the different elements to the periodic solution of the dispersion functions D_0 and D_1

According to the equations (2.17) the periodic solutions of the dispersion functions are completely determined by the two integrals

$$\int_0^{l_j} f_i(\sigma) R_{11}(\sigma)\, d\sigma \qquad \text{and} \qquad \int_0^{l_j} f_i(\sigma) R_{12}(\sigma)\, d\sigma$$

with f_i defined in eq. (2.9):

$$f_0 = h\,,$$

$$f_1 = -h + (2h^2 - k + h' D_0') D_0 + (2hk - h^3 + \frac{1}{2} r) D_0^2 + \frac{1}{2} h D_0'^{\,2}\,.$$

Assuming hard edge approximation, where the magnetic field rises abruptly from zero outside the magnet to a constant value inside it, the two integrals above can be computed analytically for different elements.

<u>COMBINED FUNCTION BENDING MAGNET</u>

In this case the corresponding transfer matrix elements are

$$\begin{aligned} &R_{11}(\sigma) = C(\sigma), && R_{12}(\sigma) = S(\sigma), && R_{13}(\sigma) = \tfrac{h}{K}[1 - C(\sigma)], \\ &R_{21}(\sigma) = -KS(\sigma), && R_{22}(\sigma) = C(\sigma), && R_{23}(\sigma) = hS(\sigma), \end{aligned}$$

with the abbreviations

$$K = h^2 - k > 0: \qquad C(\sigma) = \cos(\sqrt{K}\sigma)\,, \qquad S(\sigma) = \frac{\sin(\sqrt{K}\sigma)}{\sqrt{K}}\,,$$

$$K = h^2 - k < 0: \qquad C(\sigma) = \cosh(\sqrt{|K|}\sigma)\,, \qquad S(\sigma) = \frac{\sinh(\sqrt{|K|}\sigma)}{\sqrt{|K|}}\,.$$

The dispersion function D_0 is easily deduced from the first order transfer matrix but can also be determined analytically using eqs. (2.17), since

$$\int_0^l f_0(\sigma) R_{11}(\sigma)\ d\sigma = hS$$

$$\int_0^l f_0(\sigma) R_{12}(\sigma)\ d\sigma = -\frac{h}{K}(C-1)$$

with the notation $C \equiv C(l)$ and $S \equiv S(l)$.

To calculate the perturbation D_1, one replaces, in eq. (2.9), the function $D_0(\sigma)$ by its values at the beginning of that element (index b) and one obtains

$$\begin{aligned} f_1(\sigma) &= -h + (2h^2 - k) D_0(\sigma) + (2hk - h^3 + \frac{1}{2} r) D_0(\sigma)^2 + \frac{1}{2} h D_0'(\sigma)^2 \\ &= -h + (2h^2 - k)\left[R_{11} D_{0b} + R_{12} D_{0b}' + R_{13}\right] \\ &\quad + (2hk - h^3 + \frac{1}{2} r)\left[R_{11} D_{0b} + R_{12} D_{0b}' + R_{13}\right]^2 \\ &\quad + \frac{1}{2} h \left[R_{21} D_{0b} + R_{22} D_{0b}' + R_{23}\right]^2 \end{aligned}$$

Integration leads to

$$\begin{aligned} \int_0^l f_1(\sigma) R_{11}(\sigma)\ d\sigma = &- hS + \left[\left(D_{0b} - \frac{h}{K}\right)(l + SC) + S^2 D_{0b}'\right]\left(\frac{k}{2} + \frac{k^2}{K} + \frac{rh}{2K}\right) \\ &+ \frac{1}{3} h D_{0b}' \left(D_{0b} - \frac{h}{K}\right)(C^3 - 1)\left[1 - \frac{1}{K}\left(4k + \frac{r}{h} - 2h^2\right)\right] \\ &+ \frac{1}{3} hS \left(D_{0b} - \frac{h}{K}\right)^2 \left[\left(2k + \frac{r}{2h} - h^2\right)(2 + C^2) + \frac{1}{2} K^2 S^2\right] \\ &+ \frac{1}{2} hS {D_{0b}'}^2 \left[C^2 + \frac{1}{3} S^2 \left(2k + \frac{r}{h}\right)\right] + S\frac{h}{K}\left(h^2 + \frac{k^2}{K} + \frac{rh}{2K}\right) \end{aligned}$$

and

$$\int_0^l f_1(\sigma)R_{12}(\sigma)\, d\sigma = \frac{h}{K}(C-1) + \left[\left(D_{0b} - \frac{h}{K}\right) S^2 + \frac{1}{K}\left(l - SC\right) D'_{0b}\right]\left(\frac{k}{2} + \frac{k^2}{K} + \frac{rh}{2K}\right)$$
$$+ \frac{1}{3} h S^3 D'_{0b}\left(D_{0b} - \frac{h}{K}\right)\left(4k - 2h^2 - K + \frac{r}{h}\right) - \frac{h}{K^2}(C-1)\left(h^2 + \frac{k^2}{K} + \frac{rh}{2K}\right)$$
$$+ \frac{1}{3} h \left(D_{0b} - \frac{h}{K}\right)^2 \left[\frac{1}{K}\left(1 - C^3\right)\left(2k - h^2 + \frac{r}{2h}\right) + 1 + \frac{1}{2}C^3 - \frac{3}{2}C\right]$$
$$+ \frac{1}{3} {D'_{0b}}^2 \frac{h}{K}\left[\frac{1}{2}\left(1 - C^3\right) + \frac{1}{K}\left(2 + C^3 - 3C\right)\left(2k - h^2 + \frac{r}{2h}\right)\right] .$$

In the particular case of $K = h^2 - k \equiv 0$, these equations simplify to

$$\int_0^l f_1(\sigma)\, d\sigma = l\Bigg\{ -h + hD_{0b}\left[h + \frac{l^2}{3}\left(h^3 + \frac{1}{2}r\right)\right] + hlD'_{0b}\left[h + \frac{l^2}{4}\left(h^3 + \frac{1}{2}r\right)\right]$$
$$+ lD_{0b}D'_{0b}\left(h^3 + \frac{1}{2}r\right) + D_{0b}^2\left(h^3 + \frac{1}{2}r\right)$$
$$+ {D'_{0b}}^2\left[\frac{h}{2} + \frac{l^2}{3}\left(h^3 + \frac{1}{2}r\right)\right] + h^2 l^2\left[\frac{h}{3} + \frac{l^2}{20}\left(h^3 + \frac{1}{2}r\right)\right]\Bigg\},$$

$$\int_0^l f_1(\sigma)\sigma\, d\sigma = l^2\Bigg\{ -\frac{h}{2} + \frac{1}{2}hD_{0b}\left[h + \frac{l^2}{2}\left(h^3 + \frac{1}{2}r\right)\right] + hlD'_{0b}\left[\frac{2}{3}h + \frac{l^2}{5}\left(h^3 + \frac{1}{2}r\right)\right]$$
$$+ \frac{2}{3}lD_{0b}D'_{0b}\left(h^3 + \frac{1}{2}r\right) + \frac{1}{2}D_{0b}^2\left(h^3 + \frac{1}{2}r\right)$$
$$+ \frac{1}{4}{D'_{0b}}^2\left[h + l^2\left(h^3 + \frac{1}{2}r\right)\right] + \frac{1}{4}h^2 l_b^2\left[h + \frac{l^2}{6}\left(h^3 + \frac{1}{2}r\right)\right]\Bigg\}.$$

The expressions corresponding to elementary elements can then easily be deduced by cancelling the relevant parameters in the general expressions of the integrals and one gets:

Pure drift :

$$\int_0^l f_1(\sigma)R_{11}(\sigma)\, d\sigma = 0$$

$$\int_0^l f_1(\sigma)R_{12}(\sigma)\, d\sigma = 0$$

Pure sextupole :

$$\int_0^l f_1(\sigma)R_{11}(\sigma)\,d\sigma = \frac{1}{2}rl\left[D_{0b}^2 + lD_{0b}D'_{0b} + \frac{1}{3}l^2 {D'_{0b}}^2\right]$$

$$\int_0^l f_1(\sigma)R_{12}(\sigma)\,d\sigma = \frac{1}{2}rl^2\left[\frac{1}{2}D_{0b}^2 + \frac{2}{3}lD_{0b}D'_{0b} + \frac{1}{4}l^2 {D'_{0b}}^2\right]$$

Pure quadrupole :

$$\int_0^l f_1(\sigma)R_{11}(\sigma)\,d\sigma = -\frac{1}{2}k\left[(l+SC)D_{0b} + S^2 D'_{0b}\right]$$

$$\int_0^l f_1(\sigma)R_{12}(\sigma)\,d\sigma = -\frac{1}{2}\left[kS^2 D_{0b} - (l-SC)D'_{0b}\right]$$

EDGES

With $\theta(\sigma)$ as the angle between the pole face of the element and the particle trajectory the transfer matrix elements are defined as

$$R_{11} = 1, \quad R_{12} = \sigma, \quad R_{13} = 0,$$
$$R_{21} = \int h' \tan\theta\, d\sigma, \quad R_{22} = 1, \quad R_{23} = 0,$$

since the variation h' of the curvature cannot be neglected. Moreover, for magnetic elements with inclined boundaries whose entrance or exit faces are not normal to the design trajectory, the local quadrupole $k(\sigma)$ and sextupole $r(\sigma)$ components acting on a particle passing through the fringe field of a magnetic element along the design trajectory are given by[15]:

$$k(\sigma) = h' \tan\theta \,,$$
$$r(\sigma) = -2k'_{\mathrm{m}} \tan\theta \mp h'' \tan^2\theta + hh' \tan^3\theta \,,$$

where

$$\theta = \theta_e - \int h \; d\sigma \,.$$

The sign convention for the entrance (exit) angle θ_e is the same as in[12]; the first and second derivatives with respect to the azimuth σ are denoted by $'$ and $''$ respectively and k_{m} is the quadrupole component of the element. The upper sign corresponds to the entrance edge, the lower sign to an exit edge. Introducing these relations for the quadrupole and sextupole fields into the equation for $f_1(\sigma)$ defined in eq. (2.9), integrating over the edges and applying the "hard edge" approximation ($s \to 0$) gives

$$\begin{aligned}\lim_{s\to 0}\int_0^s f_1(\sigma)R_{11}(\sigma)\; d\sigma &= \lim_{s\to 0}\int_0^s f_1(\sigma)\; d\sigma \\ &= \pm h\langle D'_0\rangle_e D_{0e}(1+\tan^2\theta_e) - hD_{0e}(1-hD_{0e})\tan\theta_e \\ &\quad - kD_{0e}^2 \tan\theta_e - \frac{1}{2}\,h^2 D_{0e}^2 (1+\frac{1}{2}\,\tan^2\theta_e)\tan\theta_e \,,\end{aligned}$$

where the index "m" has been omitted. The index "e" denotes either the *entrance* of the entrance edge or the *exit* of the exit edge. After introducing

$$\langle D'_0\rangle_e = D'_{0e} \pm \frac{1}{2}\,hD_{0e}\,\tan\theta_e \,,$$

the relation for the entrance (upper sign) and for the exit edge (lower sign) is

$$\begin{aligned}\lim_{s\to 0}\int_0^s f_1(\sigma)R_{11}(\sigma)\; d\sigma &= \pm hD'_{0e}D_{0e}(1+\tan^2\theta_e) - hD_{0e}(1-hD_{0e})\tan\theta_e \\ &\quad - kD_{0e}^2\tan\theta_e + \frac{1}{4}\,h^2D_{0e}^2\tan^3\theta_e \,.\end{aligned}$$

Similarly, one obtains

$$\lim_{s\to 0}\int_0^s f_1(\sigma)R_{12}(\sigma)\;d\sigma = \lim_{s\to 0}\int_0^s \sigma f_1(\sigma)d\sigma = \pm\frac{1}{2}hD_{0e}^2\tan^2\theta_e\,.$$

APPENDIX II

Change of the dispersion function D_1 due to different elements

Introducing into eqs. (2.20) the expressions for the integrals derived in APPENDIX I and rearranging the different terms, one obtains the relations for the transport of the function D_1 through the different machine elements.

COMBINED FUNCTION BENDING MAGNET

$$\begin{aligned} D_1(l) = \; & D_{1b}C + D'_{1b}S - \frac{h}{K}(1-C) \\ & + \left[\left(D_{0b}-\frac{h}{K}\right)lS + \frac{1}{K}\left(S-lC\right)D'_{0b}\right]\left(\frac{k}{2}+\frac{k^2}{K}+\frac{rh}{2K}\right) \\ & + \frac{1}{3}SD'_{0b}\left(D_{0b}-\frac{h}{K}\right)(1-C)\left[\frac{1}{K}\left(4hk+r-2h^3\right)-h\right] \\ & + \frac{1}{3}\left(D_{0b}-\frac{h}{K}\right)^2\left[\frac{1}{K}\left(2hk+\frac{1}{2}r-h^3\right)\left(2-C-C^2\right)+\frac{1}{2}h\left(1-C\right)^2\right] \\ & + \frac{1}{3}{D'_{0b}}^2\frac{h}{K}\left\{C\left(\frac{3}{2}C-\frac{1}{2}-C^3\right)+\frac{1}{K}\left[k\left(1+4C^2-4C-C^4\right)+h^2C\left(2+C^3-3C\right)\right]\right. \\ & \left. \qquad + \frac{r}{2hK}\left(1-C\right)^2\right\} + \frac{h}{K^2}\left(h^2+\frac{k^2}{K}+\frac{rh}{2K}\right)(1-C) \end{aligned}$$

$$
\begin{aligned}
D'_1(l) = & - D_{1b}KS + D'_{1b}C - hS \\
& + \left[\left(D_{0b} - \frac{h}{K}\right)(lC + S) + SlD'_{0b}\right]\left(\frac{k}{2} + \frac{k^2}{K} + \frac{rh}{2K}\right) \\
& + \frac{1}{3}D'_{0b}\left(D_{0b} - \frac{h}{K}\right)(C + 1 - 2C^2)\left[\frac{1}{K}(4hk + r - 2h^3) - h\right] \\
& + \frac{1}{3}hS\left(D_{0b} - \frac{h}{K}\right)^2\left[\left(2k + \frac{r}{2h} - h^2\right)(2C + 1) + K(1 - C)\right] \\
& + \frac{1}{3}hSD'^{\,2}_{0b}\left\{C^3 + \frac{1}{2} + \frac{1}{K}[k(4 + C^3 - 5C) - h^2(2 + C^3 - 3C)] + \frac{r}{hK}(1 - C)\right\} \\
& + S\frac{h}{K}\left(h^2 + \frac{k^2}{K} + \frac{rh}{2K}\right)
\end{aligned}
$$

In the particular case of $K = h^2 - k \equiv 0$, these equations simplify to

$$
\begin{aligned}
D_1(l) = & D_{1b} + lD'_{1b} - \frac{1}{2}hl^2 \\
& + l^2\left\{\frac{1}{2}hD_{0b}\left[h + \frac{l^2}{6}\left(h^3 + \frac{1}{2}r\right)\right] + hlD'_{0b}\left[\frac{h}{3} + \frac{l^2}{20}\left(h^3 + \frac{1}{2}r\right)\right]\right. \\
& \quad + \frac{1}{3}lD_{0b}D'_{0b}\left(h^3 + \frac{1}{2}r\right) + \frac{1}{2}D^2_{0b}\left(h^3 + \frac{1}{2}r\right) \\
& \quad \left. + \frac{1}{4}D'^{\,2}_{0b}\left[h + \frac{l^2}{3}\left(h^3 + \frac{1}{2}r\right)\right] + \frac{1}{12}h^2l^2\left[h + \frac{l^2}{10}\left(h^3 + \frac{1}{2}r\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
D'_1(l) = & D'_{1b} - hl \\
& + l\left\{hD_{0b}\left[h + \frac{l^2}{3}\left(h^3 + \frac{1}{2}r\right)\right] + hlD'_{0b}\left[h + \frac{l^2}{4}\left(h^3 + \frac{1}{2}r\right)\right]\right. \\
& \quad + lD_{0b}D'_{0b}\left(h^3 + \frac{1}{2}r\right) + D^2_{0b}\left(h^3 + \frac{1}{2}r\right) \\
& \quad \left. + D'^{\,2}_{0b}\left[\frac{h}{2} + \frac{l^2}{3}\left(h^3 + \frac{1}{2}r\right)\right] + h^2l^2\left[\frac{h}{3} + \frac{l^2}{20}\left(h^3 + \frac{1}{2}r\right)\right]\right\}
\end{aligned}
$$

One can easily deduce the expressions corresponding to elementary elements by cancelling the relevant parameters:

Pure drift :

$$
\begin{aligned}
D_1(l) &= D_{1b} + D'_{1b}l \\
D'_1(l) &= D'_{1b}
\end{aligned}
$$

Pure sextupole :

$$D_1(l) = D_{1b} + D'_{1b} l + \frac{1}{2} r l^2 \left(\frac{1}{2} D_{0b}^2 + \frac{1}{3} D_{0b} D'_{0b} l + \frac{1}{12} {D'_{0b}}^2 l^2 \right)$$

$$D'_1(l) = D'_{1b} + \frac{1}{2} r l \left(D_{0b}^2 + D_{0b} D'_{0b} l + \frac{1}{3} {D'_{0b}}^2 l^2 \right)$$

Pure quadrupole :

$$D_1(l) = D_{1b} C + D'_{1b} S - \frac{1}{2} \left[k D_{0b} l S - (S - lC) D'_{0b} \right]$$

$$D'_1(l) = D_{1b} k S + D'_{1b} C - \frac{1}{2} k \left[(S + lC) D_{0b} + D'_{0b} l S \right]$$

EDGES

Entrance edge : entrance angle θ_1

(D_1, D'_1 at the end of the entrance edge defined by the values at the beginning, index "b")

$$D_1 = D_{1b} - \frac{1}{2} h D_{0b}^2 \tan^2 \theta_1$$

$$D'_1 = D_{1b} h \tan \theta_1 + D'_{1b} + h D'_{0b} D_{0b} (1 + \tan^2 \theta_1) - h D_{0b} (1 - h D_{0b}) \tan \theta_1 - k D_{0b}^2 \tan \theta_1$$

Exit edge : exit angle θ_2

(D_1, D'_1 at the end of the exit edge defined by the values at the beginning, index "b")

$$D_1 = D_{1b} + \frac{1}{2} h D_{0b}^2 \tan^2 \theta_2$$

$$D'_1 = D_{1b} h \tan \theta_2 + D'_{1b} - h D'_{0b} D_{0b} (1 + \tan^2 \theta_2) - h D_{0b} \tan \theta_2 - k D_{0b}^2 \tan \theta_2 - \frac{1}{2} h^2 D_{0b}^2 \tan^3 \theta_2$$