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PROPERTIES OF THE PRIMARY IONIZATION OR PHOTO-ELECTRON DISTRIBUTION FOR AN ENERGY LOSS DETECTOR*

DIDIER BESSET

Princeton University

and

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

ABSTRACT

The distribution of the number of primary ion pairs is discussed for a gaseous detector measuring ionization energy loss. This distribution also applies to the number of photo-electrons emitted at the photo-cathode of a photo-multiplier. After examining the general properties of the distribution, explicit formulas are given for the Landau and Vavilov models of the energy loss. The numerical evaluation of the distribution is fast enough to allow the fitting of experimental data to yield the distribution parameters. The main parameter — called the *collection factor* — is the number of primary ions (or photo-electrons) per unit of deposited energy. It can be used for calibration purposes as well as to monitor the detector performance.

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Particle identification becomes increasingly difficult as the energy of the particle goes up. Momentum determination is still feasible because of the progress made in position resolution in drift chambers; however measurements of the particle velocity cannot be performed using time of flight techniques. There are two methods currently applied in high energy physics: Čerenkov ring imaging devices and energy loss measurement ($\frac{dE}{dx}$) in a thin detector. For particles heavier than the electron, the average value of the energy deposited in a thin medium is asymptotically proportional to $\ln \gamma$. It is therefore suited to the measurement of the velocity of high energy particles. However the $\frac{dE}{dx}$ values are subjected to large fluctuations and the estimation of the average requires several measurements. Therefore a typical $\frac{dE}{dx}$ detector consists of several cells.

In addition to the fluctuation of the energy deposited in the cells, one must take into account the variations associated to the detection of the deposited energy. If the medium is a gas (in a drift chamber for example), the ionization energy deposited is directly measured by the number of ion or electron-hole pairs induced in the medium. If the medium consists of a thin sheet of scintillator, the deposited energy may be collected as light onto the photo-cathode of a photo-multiplier tube. The number of primary ions (PI) or photo-electrons (PE) generated is rather small (of the order of 100). The fluctuation of this number can be modeled by a Poisson distribution for most practical purposes. The average number of primary ions or that of primary electrons is often used as a figure of merit of a detector; we will define instead an other parameter — the collection factor — which is useful to describe and monitor the detector's performance. For both types of detectors, the fluctuations caused by the subsequent amplification are negligible compared to these effects, due to the very large gain involved, unless the average number of photo-electrons is very small (of the order of 5 or less). Therefore the distribution of the number of primary ions or photo-electrons, which we will call PI/PE for short, is a convolution of the $\frac{dE}{dx}$ with a Poisson distribution. This paper points out the remarkable fact that, whereas the $\frac{dE}{dx}$ distribution is quite difficult to compute, the PI/PE is much easier to calculate and a lot of its properties can be inferred without even knowing the exact shape of the $\frac{dE}{dx}$ distribution.

After a brief review of the $\frac{dE}{dx}$ problem in the first section, the PI/PE distribution is introduced in the second section and its properties are discussed in the third section. The two next sections carry out some of the general formulas for the Landau[1] and Vavilov[2] approximation of the $\frac{dE}{dx}$ distribution. The results can be applied to the calibration and the monitoring of a $\frac{dE}{dx}$ detector.

1. Theory of ionization energy loss

Ionization energy loss occurs through collisions of an incident particle with the atomic electrons of the medium used for detection. Let $f(E, x, \Delta)$ be the probability for a particle of energy E going through a medium of thickness x to lose a energy between Δ and $\Delta + d\Delta$. This probability function must then obey the kinetic equation:[2]

$$\begin{aligned}\frac{\partial f(E,x,\Delta)}{\partial x} = &\int_0^b w(E+\epsilon,\epsilon)\, f(E+\epsilon,x,\Delta-\epsilon)\, d\epsilon \\ &- f(E,x,\Delta)\int_0^{\epsilon_{max}} w(E,\epsilon)\, d\epsilon \ ,\end{aligned} \tag{1.1}$$

which is a statement about conservation of particle flux and energy. In this equation, $w(E,\epsilon)$ is the probability per unit length for a particle of energy E to lose an amount ϵ of energy, ϵ_{max} is the maximum kinetic energy transferred during a single collision and b is equal to $\min(\Delta, \epsilon_{max})$. Usually, the incident particle's mass is large compared to that of the electron and the maximum energy transfer may be written as:

$$\epsilon_{max} = 2m_e \frac{\beta^2}{1-\beta^2} = 2m_e\left(\gamma^2 - 1\right) \ . \tag{1.2}$$

Since we are considering thin detectors, the energy lost by incident particles is negligible compared to their kinetic energy; thus the explicit energy dependence in f and w can be forgotten. Under these conditions, equation (1.1) can be solved by introducing the Laplace transform of the probability density function f defined by:

$$F(x,t) = \int_0^\infty f(x,\Delta)\, e^{-t\Delta}\, d\Delta \ . \tag{1.3}$$

Combined with boundary conditions, the Laplace transform of the energy loss distribution satisfying equation (1.1) can be written as:

$$\varphi(t) = \ln F(x,t) = -x \int_0^{\epsilon_{max}} w(\epsilon)\left(1 - e^{-t\epsilon}\right) d\epsilon \ . \tag{1.4}$$

Determining $w(\epsilon)$ allows one to calculate the distribution $f(x,\Delta)$. Of course, carrying out the inverse Laplace transform is not always possible and a simple analytical expression for the distribution cannot be obtained in most cases.

2. Generation of the PI/PE distribution

An energy loss detector measures Δ by detecting the amount of ionization deposited in its active region. We define $\bar{n}$ as the average number of primary ions or photo-electrons. Let η be the proportionality constant between $\bar{n}$ and Δ the amount of energy deposited in the cell. Thus by definition:

$$\bar{n} = \eta\Delta. \tag{2.1}$$

We shall refer to η as the *collection factor* of the cell. For a gaseous or a thin silicon detector, this is simply the average number of ions or electron-hole pairs induced in the medium per unit energy. For a detector with photo-multiplier read-out, this factor takes into account how much light is collected by the light guide and the quantum efficiency of the photo-cathode. Combining the energy loss and the Poisson distributions, the probability P_n of detecting n PI/PE will be given by:

$$P_n = \int_0^{\infty} \frac{(\eta\Delta)^n}{n!} e^{-\eta\Delta} f(x, \Delta)\, d\Delta \ . \tag{2.2}$$

Because of the Poisson term, this probability can simply be expressed as a high order derivative of the Laplace transform of the energy loss distribution:

$$P_n = \frac{(-\eta)^n}{n!} \left. \frac{\partial^n F(x,t)}{\partial t^n} \right|_{t=\eta} \ . \tag{2.3}$$

Thus, the inverse Laplace transformation does not need to be performed to calculate the PI/PE distribution. As an example, this equation can be used to calculate the inefficiency of the detector:

$$P_0 = F(x, \eta) \ , \tag{2.4}$$

when the function $F(x,t)$ is known. We will apply this technique to the Landau and Vavilov distributions. As we shall see, the computation of the PI/PE distribution is overwhelmingly simplified. Before doing so, we will deduce some general properties of the distribution using only the fact that f is a probability density function (*i.e.* it is continuous, its integral is 1, *etc...*).

3. Distribution's properties and numerical evaluation

A distribution obtained by the convolution of a probability function describing the occurrence of a primary event with a Poisson distribution is called a contagious distribution because it was first introduced by Neyman[3] to describe the propagation of the larvae of some vegetable pest. However most contagious distributions combine two discrete elementary processes,[4,5] whereas here the energy loss, which is described by a continuous distribution, is convoluted with a discrete one, the collection of the primary ions or photo-electrons.

Asymptotic estimation.

Using (2.2), it is possible to obtain an upper limit for the probability of having a signal exceeding the dynamic range of the apparatus. This is a parameter particularly important in choosing the amplification gain of the electronics and useful to retrieve information about particles whose energy loss falls above the normal range of detection. By definition, we have:

$$\mathrm{Prob}\,(n \geq N) = \sum_{n=N}^{\infty} \int_0^{\infty} df \frac{(\eta\Delta)^n}{n!} e^{-\eta\Delta} \,, \tag{3.1}$$

where we have used the short hand notation $df = f\,(x, \Delta)\, d\Delta$. Permuting the sum with the integral and using the formula to compute the remainder of a Taylor series yields:

$$\begin{aligned} \mathrm{Prob}\,(n \geq N) &= \int_0^{\infty} df \int_0^{\eta\Delta} \frac{s^{N-1}\, e^{-s}}{(N-1)!}\, ds \\ &\leq \int_0^{A} df \int_0^{\eta\Delta} \frac{s^{N-1}}{(N-1)!}\, ds + \int_A^{\infty} df \\ &\leq \int_0^{A} df \frac{(\eta\Delta)^N}{N!} + \int_A^{\infty} df, \end{aligned} \tag{3.2}$$

where A is an arbitrary number which will be chosen so as to minimize the above estimation. Now we know that:

$$\int_A^{\infty} df \leq \int_0^{\infty} \frac{h\,(\Delta)}{h\,(A)}\, df \,, \tag{3.3}$$

where $h\,(\Delta)$ is any function monotonically increasing with Δ. In particular, using $h\,(\Delta) = \Delta$ gives:

$$\mathrm{Prob}\,(n \geq N) \leq \int_0^{A} df \frac{(\eta\Delta)^N}{N!} + \frac{\langle\Delta\rangle}{A} \,, \tag{3.4}$$

where $\langle\Delta\rangle$ is the average energy loss. Using as upper limit to the first integral its end point value and minimizing over A, we get for large N:

$$\text{Prob}\,(n \geq N) \lesssim \frac{\eta e \langle \Delta \rangle}{N} \; . \tag{3.5}$$

Although this does not constitute a very tight upper bound, this formula allows to quickly obtain some rough design parameters. A somewhat better estimate will be obtained later when we will discuss energy loss according to the Landau model.

Generating function.

Equation (2.3) shows that $G\,(y) = F\,[x, \eta\,(1-y)]$ is the generating function of the PI/PE distribution, *i.e.*:

$$G\,(y) = \sum_{n=0}^{\infty} P_n y^n \; . \tag{3.6}$$

Using the property of the generating function, we must have:

$$F\,(x, 0) = 1 \; , \tag{3.7}$$

which is clearly satisfied by equation (1.4) for any function w, and:

$$\begin{aligned} \mu = \langle n \rangle &= -\eta \left. \frac{\partial F\,(x,t)}{\partial t} \right|_{t=0} , \\ s^2 = \langle n^2 \rangle - \langle n \rangle^2 &= \eta^2 \left. \frac{\partial^2 F\,(x,t)}{\partial t^2} \right|_{t=0} - \mu\,(\mu - 1) \, . \end{aligned} \tag{3.8}$$

Higher order moments of the distribution can also be obtained in a similar way. However the cumulants discussed in the next section provide a better method to compute the moments of a distribution.

Cumulant generating function.

The cumulant generating function is defined as:

$$\Psi(y) = \ln G(e^y) \ , \tag{3.9}$$

and the cumulants κ_n are defined by the series:

$$\Psi(y) = \sum_{m=0}^{\infty} \kappa_m \frac{y^m}{m!} \ . \tag{3.10}$$

In case the function $\varphi(t)$, defined at equation (1.4), can be developed in a power series around the origin, we can obtain all the cumulants of the distribution. Before showing this, we shall express all energies in units of ϵ_{max} in order to simplify the equations. Introducing the variable $z = -\epsilon_{max} t$, and the function

$$u(z) = \varphi\left(-\frac{z}{\epsilon_{max}}\right) \ , \tag{3.11}$$

equation (2.3) becomes:

$$P_n = \frac{(n_{max})^n}{n!} \left. \frac{d^n e^{u(z)}}{dz^n} \right|_{z=-n_{max}} \ , \tag{3.12}$$

where $n_{max} = \eta \epsilon_{max}$. Then the cumulant generating function can be written as:

$$\Psi(x) = \ln u\left[n_{max}(e^x - 1)\right] \ , \tag{3.13}$$

If the function $u(z)$ has a power series defined around the origin* :

$$u(z) = \sum_{k=1}^{\infty} a_k \frac{z^k}{k!} \ , \tag{3.14}$$

it is very easy to show[6] that the cumulants κ_m of the distribution are given by:

$$\kappa_m = \sum_{k=1}^{m} \left\{ {m \atop k} \right\} a_k (n_{max})^k \ , \tag{3.15}$$

where the $\left\{ {m \atop k} \right\}$ are the Stirling numbers of the second kind using the notation

* From equation (3.7): $u(0) = 0$.

of D. Knuth.[7] These numbers are discussed in the appendix. In particular, we have:

$$\begin{aligned} \mu &= \kappa_1 = a_1 n_{max}, \\ s^2 &= \kappa_2 = (a_1 + a_2 n_{max})\, n_{max}. \end{aligned} \tag{3.16}$$

Since the detector read-out is a value $\alpha_n = gn + p$, where g is an overall gain and p the pedestal, the cumulants of a distribution are very convenient for the quantities:

$$\gamma_m = \frac{\kappa_m}{\kappa_2^{\frac{m}{2}}}, \quad \text{for } m = 3, 4, \dots \quad , \tag{3.17}$$

are the same for both random variables α_n and n.

Numerical evaluation.

Finally let us rewrite equation (3.6) using a complex variable varying on the unit circle:

$$\sum_{n=0}^{\infty} P_n e^{in\vartheta} = F\left[x, \eta\left(1 - e^{i\vartheta}\right)\right] . \tag{3.18}$$

From the estimation of equation (3.5), we know the above sum may be approximated by a finite number of terms, say M, and let:

$$\vartheta = \frac{2\pi i k}{M}, \tag{3.19}$$

then we have:

$$\sum_{n=0}^{M-1} P_n e^{2\pi i \frac{nk}{M}} \simeq F\left[x, \eta\left(1 - e^{2\pi i \frac{k}{M}}\right)\right] . \tag{3.20}$$

This equation shows that the PI/PE probabilities are the discrete Fourier transform of the series $\left\{F\left[x, \eta\left(1 - e^{2\pi i \frac{k}{M}}\right)\right], \; k = 0, M-1\right\}$. The numerical evaluation of a discrete Fourier transformation is very fast so that the computation overhead will be minimal. In addition, since the transformation maps a set of complex numbers into a set of reals, looking at the magnitude of the imaginary parts of the result provides a fair evaluation of the numerical error in the determination of the P_n.

This method is applicable to the computation of any discrete distribution. If the expression of the generating function is simpler than that of the distribution, it is the fastest way to evaluate the probabilities. The application of this method to the number of photo-electrons generated on the photo-cathode of a Čerenkov detector is discussed in the appendix.

4. Application to the Landau case

In a classical paper,[1] Landau solves equation (1.4) by using the following collision probability:

$$xw(\epsilon) = \frac{\xi}{\epsilon^2}\ , \tag{4.1}$$

where

$$\xi = \frac{2\pi \mathcal{N} q^2 q_e^4 \rho x}{m_e \beta^2} \frac{\sum Z}{\sum A} = \frac{Dx}{m_e} \frac{q^2}{\beta^2} \tag{4.2}$$

In the definition of ξ, $\mathcal{N}$ is the Avogadro number, m_e the mass of the electron and q_e its charge, β is the velocity of the incident particle, q its charge, ρ the medium density and Z and A the atomic and mass numbers of the elements which the medium is made of. In order to simplify the result, Landau let the upper limit of the integral of equation (1.4) go to infinity, assuming highly relativistic particles. His final solution is:

$$\varphi(t) = -\xi t \left(1 + \ln \frac{\epsilon_{max}^2}{I^2} - \beta^2 - C - \ln \epsilon_{max} t\right)\ , \tag{4.3}$$

where $C = .577216\ldots$ is the Euler constant. A distribution generated with the above equations is shown in figure 1 together with the result of a simulation program. Using the fast Fourier transform, the computation of 1024 points of the distribution on a VAX 780 with floating point accelerator is performed in 600 milliseconds. In contrast, a program computing the inverse Laplace transform of the above equation and performing the convolution of equation (2.2) required nearly 3 hours of cpu time to obtain the same result.

Landau also gave the formula for the asymptotic expansion of the $\frac{dE}{dx}$ probability density function:

$$f(x,\Delta) \sim \frac{1}{\xi\omega(\omega+1)} , \tag{4.4}$$

for large Δ, where $\omega(\Delta)$ is defined by:

$$\omega + \ln\omega = \frac{\Delta}{\xi} - \ln\frac{\xi}{\epsilon'} . \tag{4.5}$$

He also shows that:

$$\int_A^\infty df = \frac{1}{\omega(A)} + O\left(\frac{1}{\omega^3}\right) . \tag{4.6}$$

Using this result back in equation (3.2) and using our previous estimation for the first integral, the smallest bound is obtained by minimizing respective to $\omega(A)$ and we get:

$$\text{Prob}\,(n \geq N) \lesssim \frac{1}{\eta\xi e(N-1)} , \tag{4.7}$$

which is a better bound than that of equation (3.5) for large values of the collection factor.

However, the function $\varphi(t)$ in equation (4.3) does not have a power series expansion around the origin. From equations (3.8), one can see that, in the Landau model, the PI/PE distribution does not have a finite mean nor a finite variance. This comes mainly from the fact that extending the integral of equation (1.4) to infinity allows a infinite energy transfer to the electrons of the medium and therefore introduces (non-physical) infinite contributions. When convoluted with the Poisson distribution, the weight of these overwhelms the average computation. Therefore the average number of PI/PE in the Landau case is not a meaningful quantity since it depends logarithmically upon the cut-off value. It is much preferable to use the collection factor η.

5. Application to the Vavilov case

Revising the work of Landau in order to obtain a theory which would also be valid for non-relativistic particles, Vavilov[2] uses the following collision probability:

$$xw(\epsilon) = \frac{\xi}{\epsilon^2}\left(1-\beta^2\frac{\epsilon}{\epsilon_{max}}\right) . \tag{5.1}$$

In order to subtract the non-essential singularity, equation (1.4) is rewritten as:

$$\varphi(t) = -At - x\int_0^{\epsilon_{max}} w(\epsilon)\left(1-e^{-t\epsilon}-t\epsilon\right)d\epsilon \tag{5.2}$$

where

$$A = x\int_0^{\infty} \epsilon w(\epsilon)\, d\epsilon = \xi\left(1+\ln\frac{\epsilon_{max}^2}{I^2}-\beta^2\right) \tag{5.3}$$

is the average collision energy loss as has been used by Vavilov. A detailed discussion of this quantity is given by Fano[8] who describes many correction terms to this expression.

Integrating by parts, one can rewrite the solution in the form:

$$\varphi(t) = -\xi t\left[1+\ln\frac{\epsilon_{max}^2}{I^2}-\beta^2-\frac{1-e^{-\epsilon_{max}t}}{\epsilon_{max}t}-\left(1+\frac{\beta^2}{\epsilon_{max}t}\right)\int_0^{\epsilon_{max}t}\frac{1-e^{-y}}{y}dy\right] \tag{5.4}$$

where the integral is a regular function:

$$E(s) = \int_0^{s}\frac{1-e^{-y}}{y}dy = -\sum_{n=1}^{\infty}\frac{(-s)^n}{n\,n!} \tag{5.5}$$

which converges on the entire complex plane and diverges at infinity. Since one can show:

$$E(s) \sim \ln s + C , \tag{5.6}$$

we see that, for large values of $\epsilon_{max}t$, the Vavilov and Landau solutions are equivalent. A distribution generated with the above equations is shown in figure 2 together with the result of a simulation program. Because the evaluation of the generating function requires the computation of an integral in the complex plane, the calculation of 1024 points on a VAX takes 3.2 seconds which is significantly more than in the Landau case. Using the variable $z = -\epsilon_{max}t$ and introducing the Vavilov parameter $\kappa = \frac{\xi}{\epsilon_{max}}$, the function $u(z)$ defined in equation (3.11) becomes:

$$u(z) = \kappa \left[\left(1 + \ln \frac{\epsilon_{max}^2}{I^2} - \beta^2 \right) z + 1 - e^z + \left(z - \beta^2 \right) \int_0^{-z} \frac{1 - e^{-y}}{y} dy \right] . \qquad (5.7)$$

Using the power series expansion for the integral and the exponential yields:

$$\begin{aligned} u(z) &= \kappa \left[2 \left(\ln \frac{\epsilon_{max}}{I} - \beta^2 \right) z + \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{\beta^2}{n} \right) \frac{z^n}{n!} \right] \\ &= \kappa \left[2 \left(\ln \frac{\epsilon_{max}}{I} - \beta^2 \right) z + \sum_{n=2}^{\infty} \left(\frac{1}{n-1} + \frac{1}{\gamma^2} \right) \frac{z^n}{n\, n!} \right] \end{aligned} \qquad (5.8)$$

Using this expression in equations (3.16) gives:

$$\begin{aligned} \mu &= 2\xi \left(\ln \frac{\epsilon_{max}}{I} - \beta^2 \right) \eta, \\ s^2 &= 2\xi \left(\ln \frac{\epsilon_{max}}{I} - \beta^2 \right) \eta + \xi \epsilon_{max} \left(1 + \frac{1}{\gamma^2} \right) \frac{\eta^2}{2}. \end{aligned} \qquad (5.9)$$

The first equation is simply our definition of the collection factor. From the expression of the average energy loss, the logarithmic divergence of the average number of PI/PE in the Landau case is obvious. A more accurate formula for the average energy loss can be found in reference 8 or in the Particle Data Book.[9] Combining the previous equations we find that the following statistic:

$$s^2 - \mu = \langle n^2 \rangle - \langle n \rangle \left(\langle n \rangle + 1 \right) = D q^2 \left(\gamma^2 + 1 \right) \eta^2 \qquad (5.10)$$

allows to compute the collection factor using particles of known type and momentum. Let us recall that q is the charge of the incident particle and that D, defined in equation (4.2), characterizes the properties of the medium. In practice, however, this equation cannot be used because the detector will not register large amount of energy loss. The cause may be due to the limitation of the electronics (*e.g.* saturation). The fundamental problem, however, is that a single collision which causes the particle to lose a lot of energy (*i.e.* close to ϵ_{max}) gives rise to electrons energetic enough to emerge as delta rays out of the detector cell. The loss of large signals will strongly bias the determination of μ and s^2 and the above equation will not be correct for a truncated distribution.

6. Truncated distribution

The problem of signal cut-off is a difficult one. When delta rays are produced, the energy loss will not be deposited into a single detector cell. An accurate treatment of this phenomenon requires a theory of truncated distribution. Unfortunately, such a theory does not yet exist and only a few simple distributions can fully be treated analytically[5] [10–12].

One can however make some reasonable assumption about the physical process leading to a delta ray leaving the detector cell. The energy lost by the particle in a single collision will not be deposited in the cell if it is larger than the minimum energy, ϵ_{cut}, which is needed for a delta ray to emerge from the cell. The treatment here can no longer be exact because the delta ray will give back part of its energy in the cell through regular $\frac{dE}{dx}$. In addition the presence of delta rays emerging from a cell will perturb the measurement of the next cell. In drift chambers the emerging delta ray can be separated after some distance and the net effect is a loss of information in a few cells which can be identified geometrically. Therefore ϵ_{cut} is a quantity which must be determined empirically because it strongly depends on the detector type and configuration, as well as on the property of the associated apparatus if the latter is needed to identify delta rays.

In case ϵ_{cut} can be defined in a reasonable way, it will replace ϵ_{max} in equation (1.4) since collisions yielding electrons with energies larger than ϵ_{cut} no longer deposit energy in the detector cell. In the definition of A and $w(\epsilon)$, ϵ_{max} is a normalization factor and it will also be replaced by ϵ_{cut} since we now deal with conditional probabilities. Thus, we can rewrite our results using ϵ_{cut} instead of ϵ_{max}. However, the number of PI/PE is no longer proportional to the energy lost by the particle crossing a detector cell, but to the amount of energy deposited in the cell, these two quantities being different. One must be wary that, whereas D is a parameter of the medium and η is a property of the detector cell, ϵ_{cut} is a function of the cell and the whole detector since it depends on how well delta rays are separated. In particular, it may well be dependent on the reconstruction algorithm. With this caveat equation (5.10) becomes:

$$\begin{aligned} s^2 - \mu &= \frac{Dxq^2}{2m_e}\frac{\gamma^2+1}{\gamma^2-1}\epsilon_{cut}\eta^2 \\ &\simeq \frac{Dxq^2}{2m_e}\epsilon_{cut}\eta^2 \quad \text{for } \gamma \gg 1. \end{aligned} \tag{6.1}$$

Thus, for sufficiently energetic particles, the above statistic is independent of the particle velocity.

7. Calibration and monitoring procedure

Because of the simplicity of equation (4.3), the numerical evaluation of the PI/PE distribution in the Landau case is fast. The Vavilov case is not as favorable. Equation (5.4) still contains an integral and the function given in equation (5.7) converges poorly, especially when, for a typical detector, z can be as large as 700. Some distribution shapes are plotted in figures 3a–c. However, for highly relativistic particles, the Landau and Vavilov PI/PE distributions are practically indistinguishable for $n \lesssim 1000$ as can be seen from figure 3c. It is thus possible to fit a $\frac{dE}{dx}$ spectrum of known particles using the Landau formula in order to determine independently the collection factor η and the cut-off parameter ϵ_{cut}.

Once these parameters are known, the stability of the product $\epsilon_{cut}\eta^2$ can be used to monitor the detector performance by selecting particles of sufficiently high momentum to compute the statistic of equation (6.1). These particles could be selected by determining the track curvature. The path length through the cell, x, can then be calculated and combined with the $\frac{dE}{dx}$ readout. One could also use this statistic to flag particles having a charge different from one if they have a sufficiently high momentum.

Programs to compute and fit the PI/PE distribution can be obtained from the author.

8. Conclusions

This paper shows that, when taken into account, the fluctuations caused by the Poisson statistic in detecting the primary ionization of an energy loss signal simplify the probabilistic treatment of the phenomenon. We have given analytical expressions for the generating functions of the primary ionization or the photo-electron distribution, for the Landau and Vavilov theory of energy loss and shown how one can use these expression for a fast numerical evaluation of the probability distributions.

Because of this, an experimental spectrum can be fitted to determine the detector parameters, in particular the collection factor which describes how well the detector cell is performing. This can be done over a large data sample; alternatively, with certain kinematic cuts, one can define subsamples to monitor possible changes in time of the collection factor. Clearly such monitoring is useful to diagnose any change (gas composition, ageing effects, *etc.*) that could be taking place in a detector.

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APPENDIX

Application to Čerenkov counters signals

The first example of a contagious distribution is the compound Poisson distribution, which was studied by Neyman.[8] We will show that this distribution describes the number of photo-electrons generated at the cathode of a photomultiplier of a Čerenkov counter.

The number of photons n_γ emitted in the radiator follows a Poisson distribution given by:

$$P_{n_\gamma} = \frac{\mu^{n_\gamma} e^{-\mu}}{n_\gamma!} \ , \tag{A1}$$

where μ is the average number of photons generated in the radiator given by:

$$\mu \approx 500 \sin^2 \theta_c \tag{A2}$$

for one centimeter of radiator (*cf.* reference 9 for the exact formula). Assuming a collection factor η between the number of generated photons μ and the number of photo-electrons emitted by the cathode n_e (*i.e.* $\langle n_e \rangle = \eta\mu$), we have:

$$P_{n_e} = \sum_{n_\gamma=0}^{\infty} \frac{(\eta n_\gamma)^{n_e} e^{-\eta n_\gamma}}{n_e!} \frac{\mu^{n_\gamma} e^{-\mu}}{n_\gamma!} \ . \tag{A3}$$

The similarity with the equations for the PI/PE distributions is obvious. The generating function of the P_{n_e} is:

$$G(z) = \exp\left(-\mu + \mu e^{-\eta + \eta z}\right) \ , \tag{A4}$$

which is much easier to evaluate than the expression of equation (A3).

If the average number of photo-electron is small (of the order of 5), the fluctuations caused by the multiplication of the signal at the first dynode of the photo-multiplier can no longer by ignored. Let n_d be the number of electrons emerging from the first dynode and let λ be the average multiplication factor of dynode. Then the probability must be computed as

$$P_{n_d} = \sum_{n_e=0}^{\infty} \frac{(\lambda n_e)^{n_d} e^{-\lambda n_e}}{n_d!} \sum_{n_\gamma=0}^{\infty} \frac{(\eta n_\gamma)^{n_e} e^{-\eta n_\gamma}}{n_e!} \frac{\mu^{n_\gamma} e^{-\mu}}{n_\gamma!} \quad , \qquad \text{(A5)}$$

whereas the generating function expression is still quite simple:

$$G(z) = \exp\left\{-\mu + \mu \exp\left(-\eta + \eta e^{-\lambda+\lambda z}\right)\right\} \ , \qquad \text{(A6)}$$

Stirling numbers of the second kind

The Stirling number of the second kind are defined by the relation of recurrence:

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\},$$

and the initial values:

$$\left\{ {0 \atop 0} \right\} = 1, \qquad \left\{ {n \atop 0} \right\} = 0 \text{ and } \left\{ {n \atop n+1} \right\} = 0 \qquad \text{for } n \neq 0.$$

The following table give the first of them

$\left\{ {n \atop k} \right\}$	$k=1$	2	3	4	5	6	7	8	9	10
$n=1$	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

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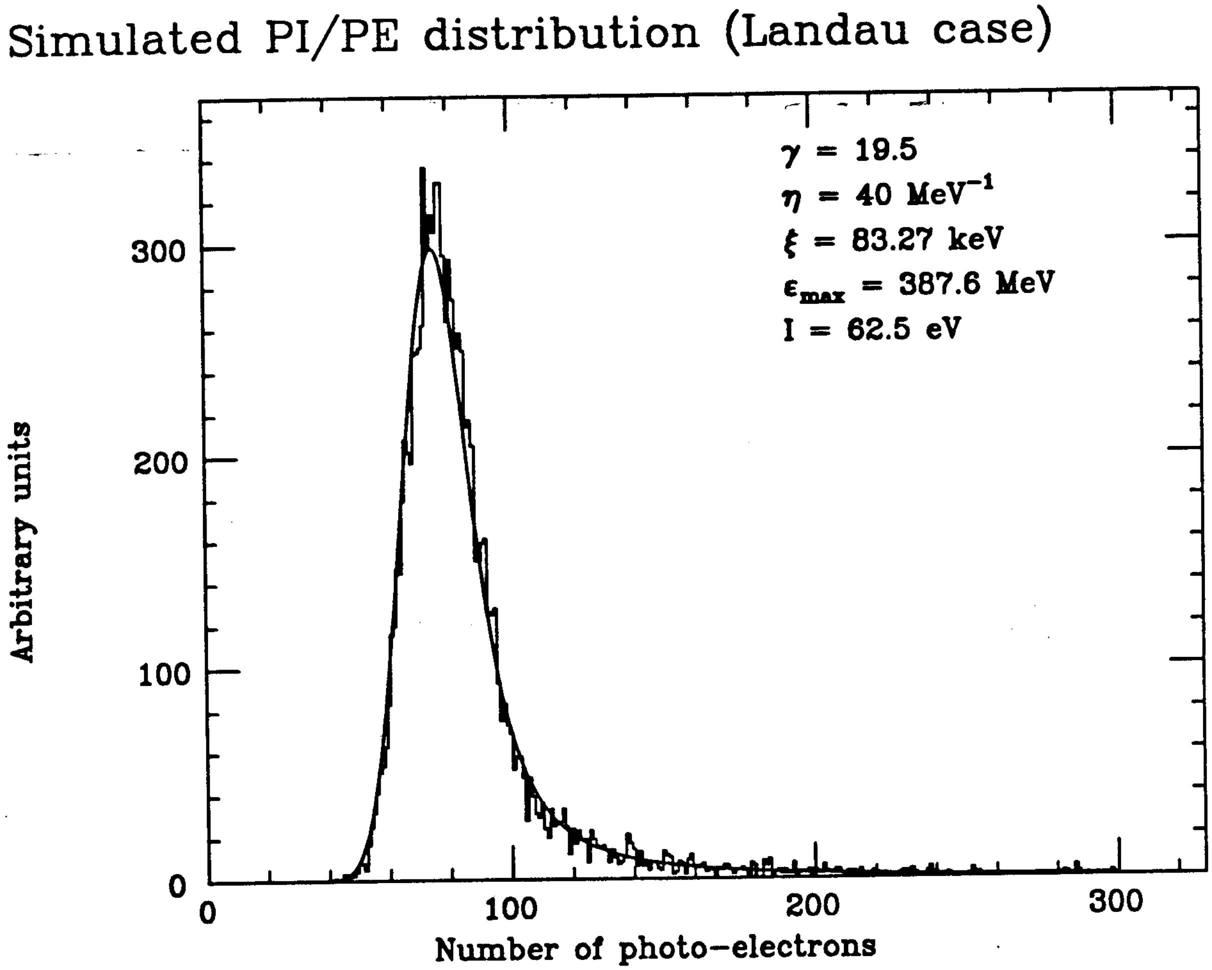
FIGURE CAPTIONS

1. Simulated PI/PE distribution for a thin plastic scintillator (Pilot B) for high relativistic particles. The solid line is the PI/PE distribution obtained by computing the generating function with equation (4.3) and using a fast Fourier transform as described in this paper. The value of the parameters is indicated on the figure. The histogram was generated using the routines DISLAN[13] and POISSN from the CERN computing library using the same parameters. The curve was normalized to the same number of events. No other adjustment was made.

2. Simulated PI/PE distribution for the same detector as figure 1 for non-relativistic particles. The solid line is the PI/PE distribution obtained by computing the generating function with equation (5.4) and using a fast Fourier transform. The CERN library routine DISVAV[13] was used instead of DISLAN.

3. Comparison of the PI/PE distributions for the Landau and Vavilov models for different particle velocity. When the two curves are discernable, the distribution is plotted with a solid line for the Vavilov case and a dotted line for the Landau case.



Simulated PI/PE distribution (Landau case)
γ = 19.5
η = 40 MeV^{-1}
ξ = 83.27 keV
ϵ_{max} = 387.6 MeV
I = 62.5 eV
Arbitrary units
0
100
200
300
0
100
200
300
Number of photo-electrons

Figure 1

Simulated PI/PE distribution (Vavilov case)

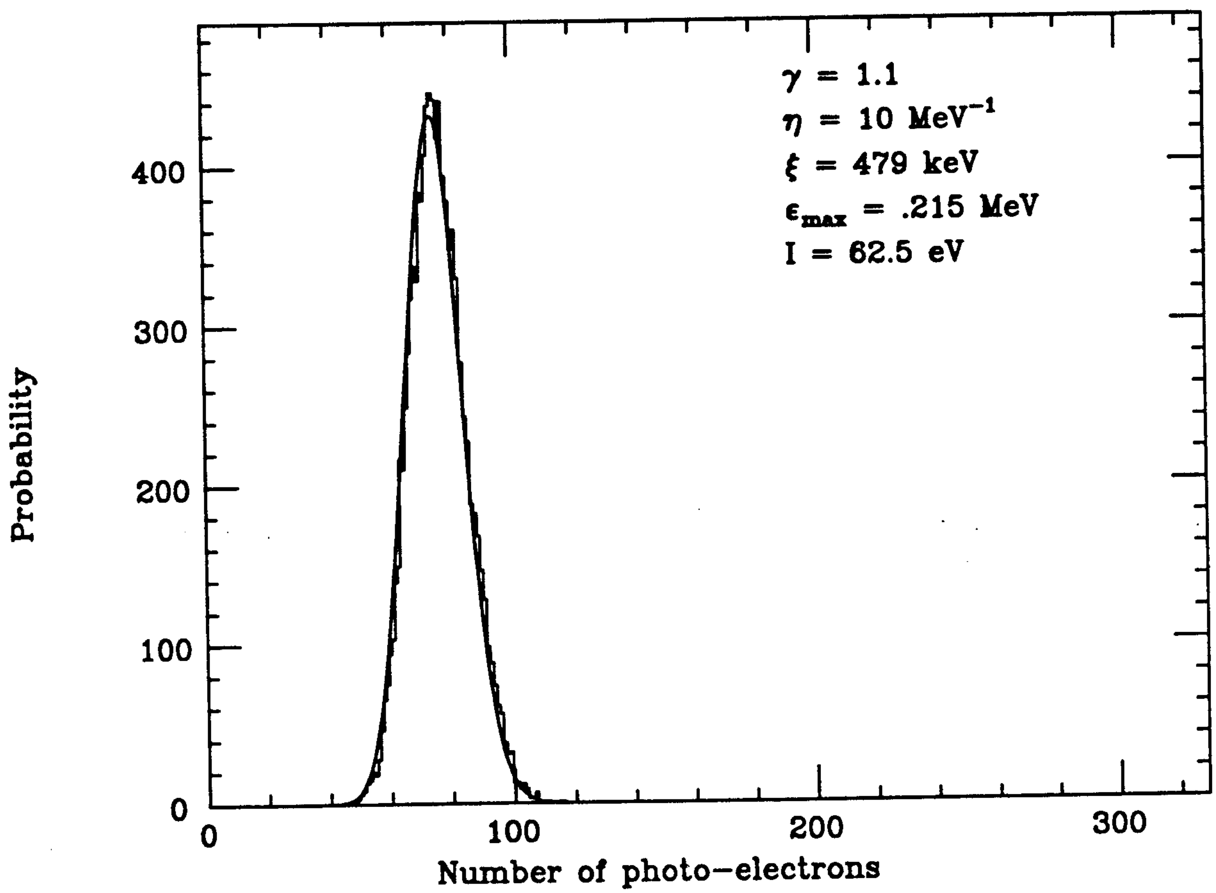


Figure 2

PI/PE distribution (non-relativistic case)

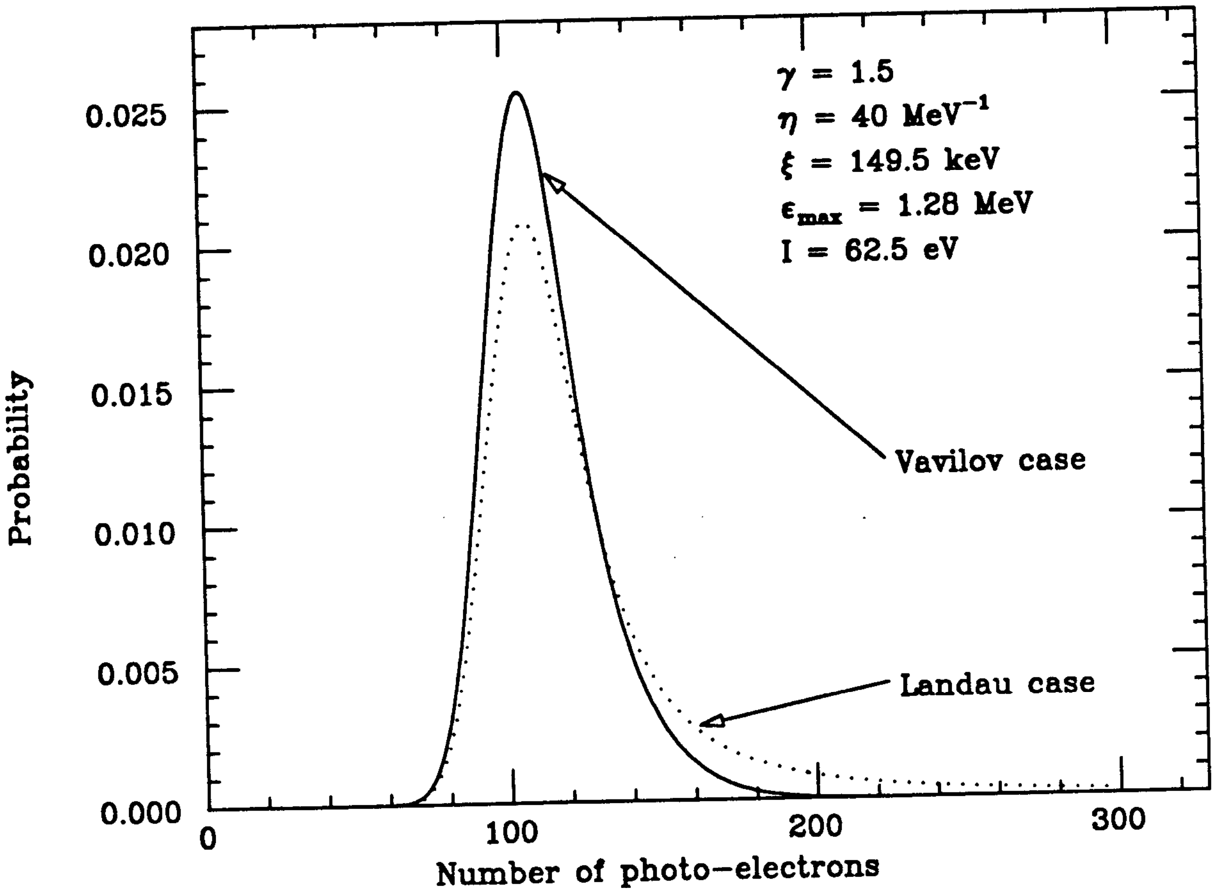


Figure 3.a

PI/PE distribution (relativistic case)

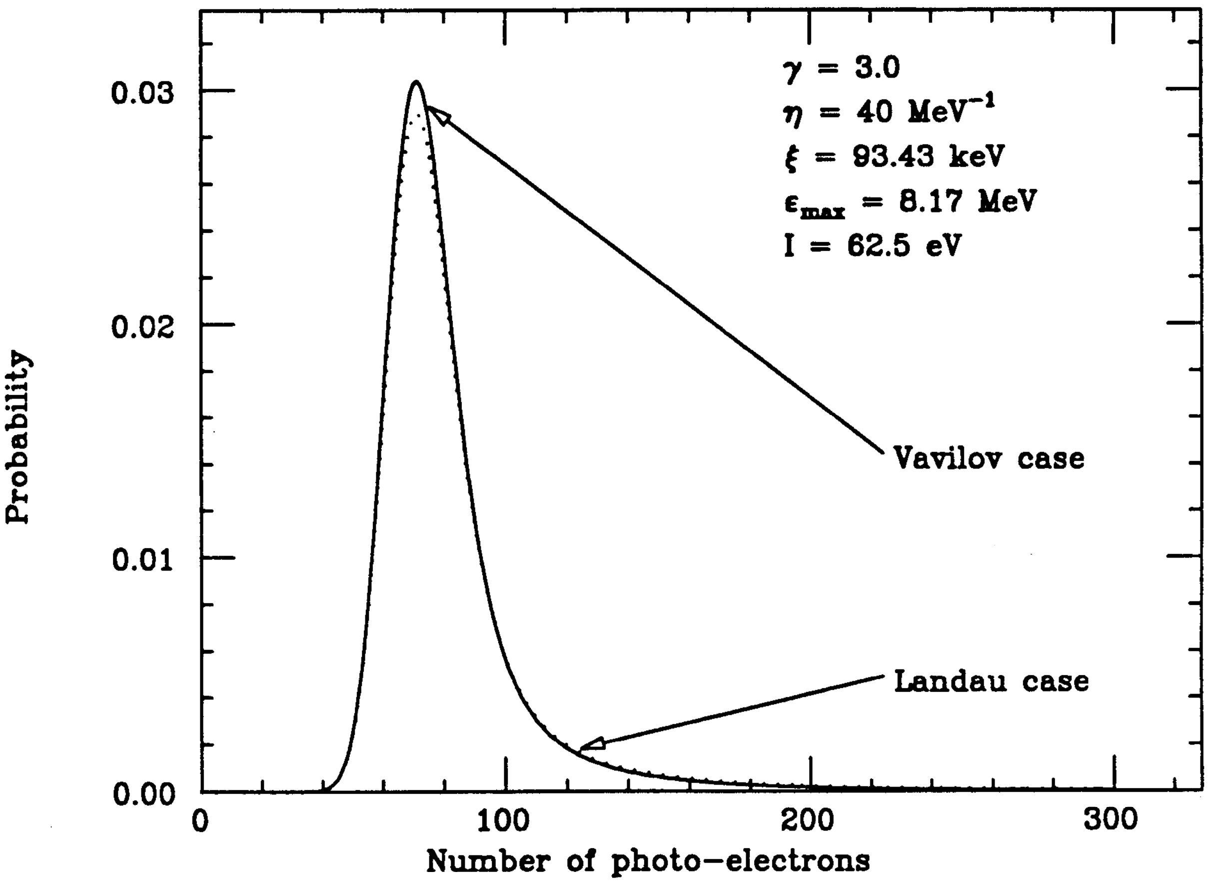


Figure 3.b

PI/PE distribution (high relativistic case)

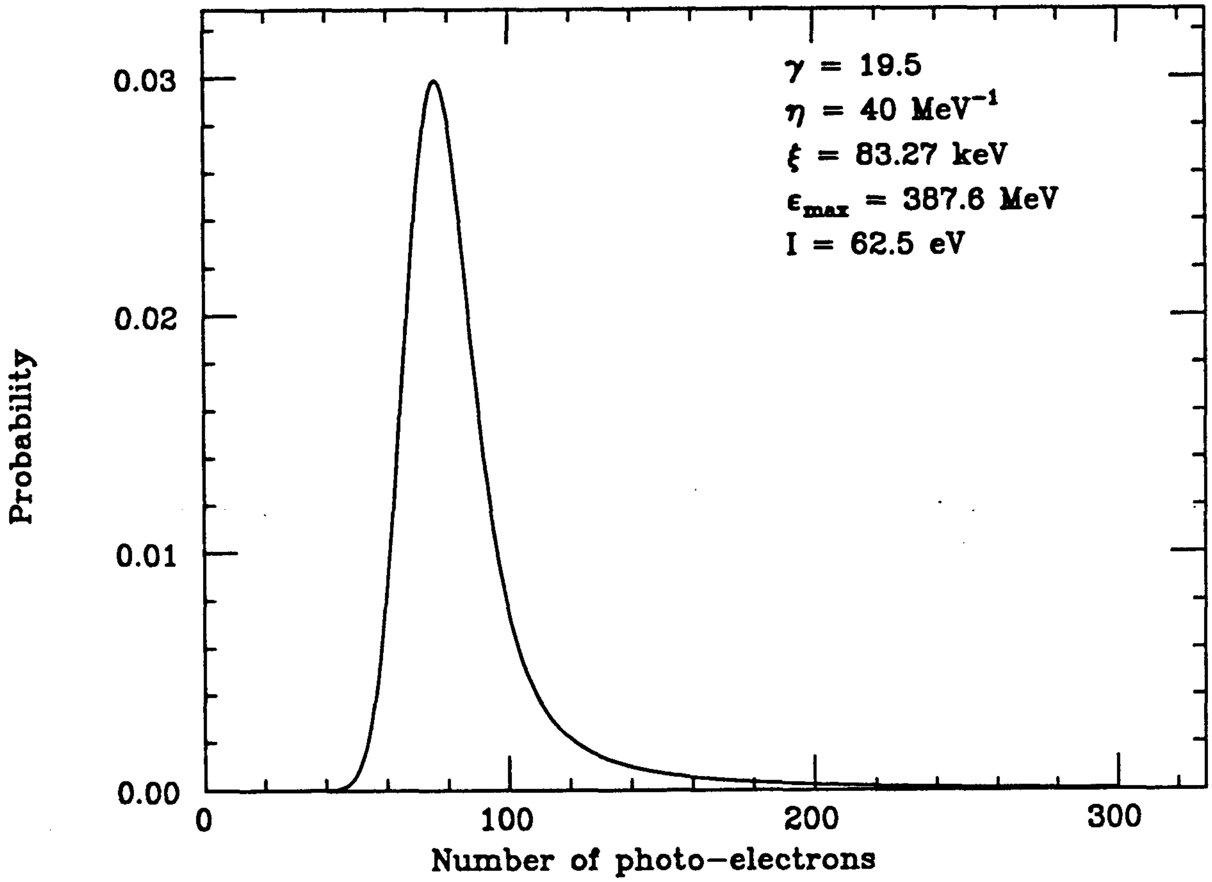


Figure 3.c